Intrinsic Kinematics: Coordinate-Free Acceleration Direction in Two and Three Dimensions

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December 2025

Abstract

Standard treatments of curvilinear motion decompose acceleration into tangential and normal components but do not address the global direction of the acceleration vector in coordinate-free terms. We fill this gap for both planar curves and space curves in $\mathbb{R}^3$. Using the Frenet–Serret frame, curvature κ, and torsion τ [1, 5, 4], we establish four results. First, for a smooth planar curve, the absolute direction of acceleration is given by the exact, coordinate-free formula

$$\theta_a(s) = \Theta_0 + \int_{s_0}^{s} \kappa(\sigma)\, d\sigma + \operatorname{atan2}\!\big(\kappa \dot{s}^2,\ \ddot{s}\big),$$

separating an intrinsic tilt $\phi = \operatorname{atan2}(\kappa\dot{s}^2, \ddot{s})$ from an extrinsic path orientation $\Theta(s)$. Second (Theorem 3.2), for any $\mathcal{C}^2$ space curve, the acceleration vector is confined to the osculating plane: the binormal component $a_B = \mathbf{a} \cdot \mathbf{B}$ vanishes identically. Third, the acceleration tilt ϕ is τ-independent and remains fully intrinsic in three dimensions; the absolute spatial direction of acceleration is

$$\hat{a} = \cos\phi\, \mathbf{T}(s) + \sin\phi\, \mathbf{N}(s),$$

where $\mathbf{T}, \mathbf{N}$ are determined by integrating both κ and τ. Fourth (Theorem 8.1), setting $\tau \equiv 0$ reduces the 3D formula to the planar formula exactly—the planar result is a strict special case, not an independent result. Worked examples on a circular path and a circular helix confirm consistency with direct Cartesian computation.

Contents

1 Introduction

Students in mechanics and multivariable calculus courses routinely encounter the decomposition of acceleration into tangential and normal components [2, 3]:

$$\mathbf{a} = \ddot{s}\,\mathbf{T} + \kappa\dot{s}^2\,\mathbf{N}.$$

This decomposition is presented in virtually every undergraduate treatment of curvilinear motion and is well understood. What is *not* addressed in standard treatments—neither in calculus textbooks nor in classical mechanics courses—is the following natural question:

> *What is the absolute direction of* $\mathbf{a}$*—the angle or unit vector it makes in space—expressed purely in terms of the intrinsic geometric and kinematic quantities* κ*,* τ*,* $\dot{s}$*, and* $\ddot{s}$*, without introducing a coordinate system?*

This gap is not merely aesthetic. Students who have mastered the Frenet frame and the tangential–normal decomposition are left without a complete coordinate-free picture of where the acceleration vector actually points. The formula $\mathbf{a} = \ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N}$ tells them the *components* of $\mathbf{a}$ in the moving frame, but not the *global direction* of $\mathbf{a}$—a distinction that becomes especially important when the curve is defined intrinsically by its curvature and torsion rather than by an explicit Cartesian parameterization.

This pedagogical paper fills that gap completely for both planar and spatial motion. In doing so it clarifies a conceptual distinction that standard treatments leave implicit: some properties of acceleration are **intrinsic** (determinable without any reference frame) while others are **extrinsic** (requiring a global reference). Making this separation explicit is both geometrically illuminating and pedagogically valuable.

Intrinsic vs. extrinsic: the void analogy

Consider a particle moving through a featureless void—no coordinate axes, no fixed stars, no external reference of any kind. The particle carries only a local instrument package: a speedometer measuring $\dot{s}$, an accelerometer measuring $\ddot{s}$, and a curvature sensor measuring κ and τ. From these readings alone it can compute the **acceleration tilt**

$$\phi = \operatorname{atan2}(\kappa\dot{s}^2,\ \ddot{s}),$$

the angle between $\mathbf{a}$ and $\mathbf{T}$ within the osculating plane. This is **intrinsic**: no external reference is needed, and any two particles with identical $(\kappa, \dot{s}, \ddot{s})$ histories will agree on ϕ regardless of where or how they are oriented in space.

What the particle *cannot* determine from its instrument package alone is **path orientation** $\Theta(s)$—the direction $\mathbf{T}$ makes relative to any axis in the surrounding void. Determining Θ requires the particle to remember where it started (Θ_0) and integrate its turning history. Without that initial anchor, the global orientation of the path is fundamentally underdetermined. This is **extrinsic**: it requires information from outside the particle's local experience.

The absolute acceleration direction θ_a (in 2D) or $\hat{a}$ (in 3D) is the combination of these two contributions. The void analogy makes the separation vivid: ϕ is what the particle *feels*; Θ is where it *is* relative to the world.

Organization

Section 2 establishes notation for both 2D and 3D. Sections 3–5 develop the planar theory and derive the explicit formula for θ_a. Sections 6–7 extend the framework to $\mathbb{R}^3$. Section 8 proves the planar formula is a strict special case of the 3D formula. Sections 9–10 present worked examples and Cartesian verifications. Section 11 discusses geometric implications and open problems.

Teaching context and intended audience

This paper is aimed at undergraduate students who have completed a first course in multivariable calculus and are encountering the Frenet–Serret frame for the first time—typically second-year students in a mechanics, mathematical physics, or vector calculus course.

The primary **learning objective** is to give students a complete coordinate-free picture of acceleration direction: not just how to decompose $\mathbf{a}$ into components, but where $\mathbf{a}$ actually points in space, and why that question separates naturally into an intrinsic part (computable from local measurements alone) and an extrinsic part (requiring a global reference). This distinction is rarely made explicit at the undergraduate level, despite being conceptually fundamental.

In a **classroom setting**, the planar formula (4) can be introduced immediately after the Frenet frame and curvature are taught, as a direct answer to the question students naturally ask: "we know the components of $\mathbf{a}$—so what direction does it point?" The 3D extension (Section 7) and the reduction theorem (Section 8) serve as natural follow-on material, showing students how the 2D result generalises and how torsion enters the picture purely through frame orientation. The worked examples (Section 9) and Cartesian verifications (Section 10) are written to be suitable for tutorial or homework use.

2 Definitions and Notation

Let $\mathbf{r} : \mathcal{I} \to \mathbb{R}^n$ ($n = 2$ or 3) be a $\mathcal{C}^2$ curve parameterized by time t, with arc length $s(t)$ satisfying $\dot{s} > 0$.

- **Unit tangent:** $\mathbf{T}(s) = d\mathbf{r}/ds$.
- **Unit principal normal:** $\mathbf{N}(s) = (d\mathbf{T}/ds) \,/\, |d\mathbf{T}/ds|$.
- **Unit binormal** (3D only): $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$.
- **Curvature** $\kappa(s) \geq 0$ and **torsion** $\tau(s) \in \mathbb{R}$ are defined by the Frenet–Serret equations [1, 5, 4]:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

In 2D, $\tau \equiv 0$ and the $\mathbf{B}$ equation is absent. In 3D, writing $R(s) = [\mathbf{T} \mid \mathbf{N} \mid \mathbf{B}]$ for the matrix whose columns are the three frame vectors, the Frenet–Serret equations take the compact form

$$\frac{dR}{ds} = R(s)\,K(s)^{\top}, \qquad K(s) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}. \tag{1}$$

Because K has the special structure $K^{\top} = -K$ (called skew-symmetric), equation (1) automatically preserves the orthonormality of $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ at every point along the curve—the frame never distorts, it only rotates.

- **Frame rotation vector** (3D): $\boldsymbol{\Omega} = \tau\mathbf{T} + \kappa\mathbf{B}$, with $|\boldsymbol{\Omega}| = \sqrt{\kappa^2 + \tau^2}$ the total rate at which the Frenet frame rotates per unit arc length [1]. This vector points along the instantaneous axis of frame rotation.
- **Speed:** $\dot{s} = ds/dt$; **tangential acceleration:** $\ddot{s} = d^2s/dt^2$.

3 Acceleration Decomposition

Proposition 3.1. *For a $\mathcal{C}^2$ curve parameterized by time,*

$$\mathbf{a}(t) = \ddot{s}(t)\,\mathbf{T}(s(t)) + \kappa(s(t))\dot{s}(t)^2\,\mathbf{N}(s(t)).$$

Proof. Differentiate $\mathbf{v} = \dot{s}\mathbf{T}$ with respect to t:

$$\mathbf{a} = \frac{d}{dt}(\dot{s}\mathbf{T}) = \ddot{s}\mathbf{T} + \dot{s}\frac{d\mathbf{T}}{dt}.$$

By the chain rule and the first Frenet–Serret equation, $d\mathbf{T}/dt = \dot{s}\,(d\mathbf{T}/ds) = \dot{s}\,\kappa\mathbf{N}$. Substituting gives the result. □

The decomposition names the two scalar components:

- **Tangential acceleration:** $a_T = \ddot{s}$, the rate of speed change.
- **Normal acceleration:** $a_N = \kappa\dot{s}^2$, curvature times squared speed.

Theorem 3.2 (Osculating plane confinement)**.** *For any $\mathcal{C}^2$ space curve, $\mathbf{a}\cdot\mathbf{B} \equiv 0$. The acceleration vector lies entirely in the osculating plane* $\mathrm{span}\{\mathbf{T},\mathbf{N}\}$.

Proof. $\mathbf{a}\cdot\mathbf{B} = \ddot{s}(\mathbf{T}\cdot\mathbf{B}) + \kappa\dot{s}^2(\mathbf{N}\cdot\mathbf{B}) = 0$, since $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is orthonormal. □

Remark. Theorem 3.2 corrects a common misconception: the 3D acceleration decomposition does *not* include a binormal term $a_B\mathbf{B}$. The binormal component is identically zero regardless of path complexity or torsion. Some treatments write the 3D decomposition as $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} + a_B\mathbf{B}$ and attribute a_B to "out-of-plane curvature"; this is incorrect. The correct statement is $a_B = 0$ for every $\mathcal{C}^2$ curve.

4 Acceleration Tilt ϕ

Definition 4.1. The **acceleration tilt** $\phi(t)$ is the directed angle from $\mathbf{T}$ to $\mathbf{a}$, measured in the osculating plane:

$$\phi(t) = \mathrm{atan2}\big(\kappa(s(t))\dot{s}(t)^2,\ \ddot{s}(t)\big). \tag{2}$$

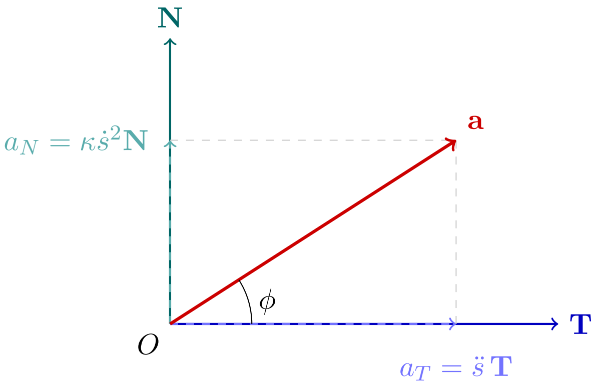
Proposition 4.2 (ϕ is intrinsic and τ-independent)**.** *The tilt ϕ depends only on κ, $\dot{s}$, $\ddot{s}$. It is independent of τ, of the initial frame orientation, and of any coordinate system.*

Proof. Equation (2) involves only κ, $\dot{s}$, $\ddot{s}$. Torsion τ does not appear in a_T or a_N (Proposition 3.1), hence not in their ratio. The atan2 function is a scalar operation on two scalars; no frame vectors are referenced. □

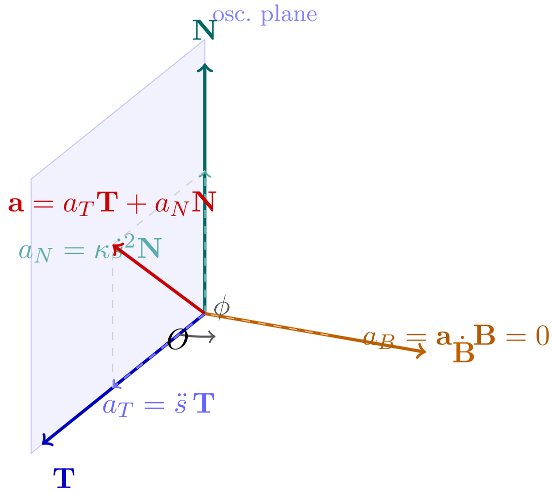
Corollary 4.3. *The magnitude $|\mathbf{a}| = \sqrt{\ddot{s}^2 + \kappa^2\dot{s}^4}$ is also τ-independent and fully intrinsic.*

Special cases of ϕ:

- $\ddot{s} = 0$ (constant speed): $\phi = \mathrm{sign}(\kappa)\,\pi/2$ (purely centripetal).
- $\kappa = 0$ (instantaneously straight): $\phi = 0$ or π (purely tangential).



(a) Planar case ($n = 2$).



(b) Spatial case ($n = 3$). Acceleration lies in the shaded osculating plane; the binormal component vanishes identically (Theorem 3.2).

Figure 1: Intrinsic decomposition of acceleration in the Frenet frame. In both cases the tilt ϕ depends only on κ, $\dot{s}$, $\ddot{s}$; torsion τ plays no role.

5 Absolute Acceleration Direction: Planar Case

Path orientation

Let $\Theta(s)$ denote the angle of $\mathbf{T}$ relative to a fixed reference direction $\mathbf{e}$ (e.g., the positive x-axis). By definition of curvature in the plane [1, 5, 4],

$$\frac{d\Theta}{ds} = \kappa(s).$$

Integrating from s_0 with $\Theta(s_0) = \Theta_0$:

$$\Theta(s) = \Theta_0 + \int_{s_0}^{s} \kappa(\sigma)\, d\sigma. \tag{3}$$

This is **extrinsic**: it depends on the intrinsic curvature κ and the externally chosen reference Θ_0.

Main formula (planar)

The absolute direction of $\mathbf{a}$ is $\theta_a = \Theta(s) + \phi$, giving:

$$\boxed{\theta_a(s) = \Theta_0 + \int_{s_0}^{s} \kappa(\sigma)\, d\sigma + \operatorname{atan2}\big(\kappa \dot{s}^2,\ \ddot{s}\big).} \tag{4}$$

The contributions are cleanly separated:

- **Intrinsic component:** $\phi = \operatorname{atan2}(\kappa \dot{s}^2, \ddot{s})$ depends only on local scalars.
- **Extrinsic component:** $\Theta_0 + \int \kappa\, d\sigma$ requires an initial reference angle.

Remark. Formula (4) does not appear in standard references [1, 2, 5, 3]. Textbook treatments decompose $\mathbf{a}$ into a_T and a_N but do not address the global direction of $\mathbf{a}$ in coordinate-free terms.

6 Frame Orientation in Three Dimensions

In 3D, path orientation is no longer a single angle but the full spatial orientation of the Frenet frame—a continuously rotating orthonormal triple $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ governed by (1).

Distinct roles of κ and τ

The Frenet–Serret equations reveal a structural asymmetry:

- **Curvature** κ drives $d\mathbf{T}/ds = \kappa\mathbf{N}$ (bending), rotating $\mathbf{T}$ toward $\mathbf{N}$ within the osculating plane.
- **Torsion** τ drives the $\tau\mathbf{B}$ term in $d\mathbf{N}/ds$ and $-\tau\mathbf{N}$ in $d\mathbf{B}/ds$ (twisting), rotating the **N-B** half of the frame about the tangent axis $\mathbf{T}$. This rotates the osculating plane itself in $\mathbb{R}^3$ without altering anything *within* the plane.

Consequence. τ governs where in space the osculating plane is oriented, but has no effect on a_T, a_N, $|\mathbf{a}|$, or ϕ. This deepens the intrinsic/extrinsic separation: the extrinsic component now requires integrating both κ and τ, while ϕ remains a purely local scalar.

Frame rotation vector and the rate of change of $\hat{a}$

The vector $\mathbf{\Omega} = \tau\mathbf{T} + \kappa\mathbf{B}$ points along the instantaneous axis about which the Frenet frame rotates as arc length increases, with magnitude $|\mathbf{\Omega}| = \sqrt{\kappa^2 + \tau^2}$ equal to the total rotation rate per unit arc length. It governs how quickly $\hat{a}$ changes direction in space:

$$\frac{d\hat{a}}{ds} = \mathbf{\Omega} \times \hat{a} + \frac{d\phi}{ds}(-\sin\phi\,\mathbf{T} + \cos\phi\,\mathbf{N}).$$

The first term captures how the osculating plane itself sweeps through space due to bending (κ) and twisting (τ); the second captures how the tilt angle ϕ changes due to varying speed and curvature.

7 Absolute Acceleration Direction: Spatial Case

Definition 7.1. The **absolute acceleration direction** in $\mathbb{R}^3$ is the unit vector

$$\boxed{\hat{a}(t) = \frac{\mathbf{a}(t)}{|\mathbf{a}(t)|} = \cos\phi(t)\;\mathbf{T}(s(t)) + \sin\phi(t)\;\mathbf{N}(s(t)).} \tag{5}$$

The formula separates cleanly:

- **Intrinsic factor:** $\cos\phi$, $\sin\phi$ depend only on $(\kappa, \dot{s}, \ddot{s})$.
- **Extrinsic factor:** $\mathbf{T}(s)$, $\mathbf{N}(s)$ require integrating the Frenet–Serret system (1) forward from a known starting orientation, accumulating both the bending (κ) and twisting (τ) of the path.

This is the 3D analogue of (4). In 2D the extrinsic component was one scalar Θ (integral of κ only); in 3D it is the full rotational history of the frame, accumulating both κ and τ as the path bends and twists through space.

8 Reduction: 3D Formula Contains the Planar Formula

Theorem 8.1 (Recovery of the planar formula)**.** *Suppose* $\tau \equiv 0$ *and the initial frame satisfies* $\mathbf{B}_0 = \hat{\mathbf{e}}_z$*, so the curve lies in the xy-plane. Then* (5) *reduces to* $\hat{a} = (\cos\theta_a, \sin\theta_a, 0)$*, where* θ_a *is given by* (4)*.*

Proof. **Step 1.** With $\tau = 0$, the Frenet–Serret system reduces to $d\mathbf{T}/ds = \kappa\mathbf{N}$, $d\mathbf{N}/ds = -\kappa\mathbf{T}$, $d\mathbf{B}/ds = \mathbf{0}$. The last equation gives $\mathbf{B}(s) = \mathbf{B}_0 = \hat{\mathbf{e}}_z$ for all s.

Step 2. With $\mathbf{B} = \hat{\mathbf{e}}_z$, parameterize by the orientation angle $\Theta(s)$:

$$\mathbf{T} = (\cos\Theta, \sin\Theta, 0), \qquad \mathbf{N} = (-\sin\Theta, \cos\Theta, 0).$$

Differentiating $\mathbf{T}$ gives $d\mathbf{T}/ds = \dot{\Theta}\mathbf{N}$. Comparing with $d\mathbf{T}/ds = \kappa\mathbf{N}$ yields $d\Theta/ds = \kappa(s)$, which integrates to (3).

Step 3. Substituting into (5):

$$\begin{aligned}\hat{a} &= \cos\phi(\cos\Theta, \sin\Theta, 0) + \sin\phi(-\sin\Theta, \cos\Theta, 0)\\ &= (\cos\phi\cos\Theta - \sin\phi\sin\Theta,\ \cos\phi\sin\Theta + \sin\phi\cos\Theta,\ 0)\\ &= (\cos(\Theta+\phi),\ \sin(\Theta+\phi),\ 0),\end{aligned}$$

where the last step uses the angle-addition identities. Setting $\theta_a = \Theta(s) + \phi$ and substituting (3) recovers (4). □

Corollary 8.2. *Formula* (4) *is not an independent result. It is the unique planar restriction of* (5)*, obtained by* $\tau = 0$*.*

Remark. The proof identifies each ingredient: (i) ϕ is unchanged—it was always τ-independent; (ii) the scalar equation $d\Theta/ds = \kappa$ emerges from the full 3D Frenet–Serret system collapsing to a simple in-plane rotation when $\tau = 0$; (iii) the angle-addition step collapses the 3D unit vector $\hat{a}$ to the scalar angle θ_a; (iv) $\mathbf{B}$ freezes to $\hat{\mathbf{e}}_z$ and plays no further role, reducing three-dimensional frame rotation to a single planar angle.

9 Worked Examples

9.1 Circular path (planar formula)

Let $\kappa = 1/R = 0.2\ \text{m}^{-1}$ ($R = 5$ m) and $\dot{s}(t) = 4t^2$ m/s. At $t = 1$ s:

$$\dot{s}(1) = 4\ \text{m/s}, \quad \ddot{s}(1) = 8\ \text{m/s}^2, \quad s(1) = \tfrac{4}{3}\ \text{m}, \quad a_N = 3.2\ \text{m/s}^2.$$

Acceleration tilt: $\phi(1) = \arctan(3.2/8) = \arctan(0.4) \approx 21.80°$.

Path orientation ($\Theta_0 = 0$): $\Theta(s(1)) = 0.2 \cdot \frac{4}{3} = \frac{4}{15}$ rad $\approx 15.28°$.

Absolute direction: $\theta_a(1) \approx 15.28° + 21.80° = 37.08°$.

9.2 Circular helix (spatial formula)

Let $\mathbf{r}(\theta) = (R\cos\theta, R\sin\theta, b\theta)$ with $R = 3$ m, $b = 4$ m. The arc-length element is $\sqrt{R^2 + b^2} = 5$ m, so $s = 5\theta$. The standard helix invariants are [5]:

$$\kappa = \frac{R}{R^2 + b^2} = \frac{3}{25}\ \text{m}^{-1}, \qquad \tau = \frac{b}{R^2 + b^2} = \frac{4}{25}\ \text{m}^{-1}, \qquad |\mathbf{\Omega}| = \frac{1}{5}\ \text{m}^{-1}.$$

Speed profile $\dot{s}(t) = 2t$ m/s; at $t = 2$ s: $\dot{s} = 4$ m/s, $\ddot{s} = 2$ m/s^2, $s = 4$ m.

Kinematic quantities: $a_T = 2$ m/s^2, $a_N = 1.92$ m/s^2, $a_B = 0$ (Theorem 3.2), $|\mathbf{a}| = \sqrt{7.6864} \approx 2.773$ m/s^2.

Acceleration tilt: $\phi = \arctan(1.92/2) = \arctan(0.96) \approx 43.83°$.

Frenet frame at $s = 4$ **m** ($\theta = 0.8$ rad; $\sin\theta \approx 0.7174$, $\cos\theta \approx 0.6967$):

$$\mathbf{T} \approx (-0.4304,\ 0.4180,\ 0.8000),$$
$$\mathbf{N} \approx (-0.6967,\ -0.7174,\ 0),$$
$$\mathbf{B} \approx (0.5739,\ -0.5574,\ 0.6000).$$

Verification: $|\mathbf{T}| = |\mathbf{N}| = |\mathbf{B}| = 1$, $\mathbf{T}\cdot\mathbf{N} = 0$, $\mathbf{T}\times\mathbf{N} = \mathbf{B}$. ✓

Absolute acceleration direction:

$$\hat{a} = \frac{2}{2.773}\mathbf{T} + \frac{1.92}{2.773}\mathbf{N} = 0.7214\,\mathbf{T} + 0.6925\,\mathbf{N}$$
$$\approx (-0.7929,\ -0.1952,\ 0.5771).$$

Check: $|\hat{a}|^2 \approx 0.6287 + 0.0381 + 0.3331 = 1.0000$. ✓

10 Cartesian Verification

10.1 Circular path

Parameterize the circle by polar angle ϑ: $\mathbf{r}(\vartheta) = R(\cos\vartheta, \sin\vartheta)$. At $t = 1$, $\vartheta(1) = s(1)/R = 4/15$ rad, giving $\mathbf{T} \approx (-0.2636, 0.9646)$, $\mathbf{N} \approx (-0.9646, -0.2636)$. Then

$$\mathbf{a}(1) = 8\mathbf{T} + 3.2\mathbf{N} \approx (-5.1955,\ 6.8733).$$

Global direction: $\theta_{\text{global}} = \text{atan2}(6.8733, -5.1955) \approx 127.08°$.

The intrinsic result used $\Theta_0 = 0$ at $s = 0$, which corresponds to taking $\mathbf{e}$ as the positive y-axis. Converting: $90° + 37.08° = 127.08°$. ✓

10.2 Circular helix

$$\mathbf{a} = 2\,\mathbf{T} + 1.92\,\mathbf{N} \approx (-2.1985,\ -0.5414,\ 1.6000), \quad |\mathbf{a}| \approx 2.773\ \text{m/s}^2.$$

Unit vector: $\hat{a}_{\text{Cart.}} \approx (-0.7928, -0.1953, 0.5771)$.

Method	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
Intrinsic formula	-0.7929	-0.1952	0.5771
Cartesian verification	-0.7928	-0.1953	0.5771
Discrepancy	< 0.0002	< 0.0002	0.0000

Table 1: Component-wise comparison for the helix example. Discrepancies are rounding only.

11 Discussion

1. The intrinsic/extrinsic separation

The central conceptual contribution of both formulas is the clean separation between what is locally measurable (intrinsic) and what requires global history (extrinsic). In 2D, the separation is one-dimensional: scalar ϕ intrinsic, scalar Θ extrinsic. In 3D it deepens structurally: ϕ and $|\mathbf{a}|$ remain scalars computed from local data, while the extrinsic component becomes the full rotational history of the Frenet frame—a continuously updated orientation in space that encodes both κ and τ. This asymmetry—*scalar intrinsic, rotating frame extrinsic*—is the 3D manifestation of the same principle.

2. The atan2 function

The two-argument arctangent $\operatorname{atan2}(a_N, a_T)$ is essential for defining ϕ unambiguously. It preserves quadrant information and handles degenerate cases ($a_T = 0$ or $a_N = 0$), signed curvature ($\kappa \gtrless 0$), and negative tangential acceleration ($\ddot{s} < 0$).

3. Practical implications

The formulas are naturally aligned with how physical sensors work. Inertial measurement units (IMUs) measure $\dot{s}$, $\ddot{s}$, and κ directly—the precise inputs to ϕ. Reconstructing the global direction then requires integrating κ (and τ in 3D) forward from a known initial orientation, exactly the structure of (4) and (5). This has applications in dead-reckoning navigation and robot path following on curves that are defined by curvature profiles rather than Cartesian parameterizations.

4. Open problems

1. **Intrinsic Newton's second law.** Can $\mathbf{F} = m\mathbf{a}$ be written entirely in terms of κ, τ, $\dot{s}$, $\ddot{s}$, and their arc-length derivatives, with no coordinate system appearing?

2. **Conservation laws.** Which mechanical invariants (energy, momentum, angular momentum) admit purely intrinsic expressions?

3. **Sign change of τ.** For curves where τ changes sign, how does $\hat{a}$ behave at the torsion inflection point?

4. **Relativistic extension.** Does the framework connect naturally to the four-acceleration formalism when arc length is replaced by proper time?

12 Conclusion

Standard treatments decompose acceleration but leave the global direction of $\mathbf{a}$ unaddressed in coordinate-free terms. This pedagogical paper fills that gap. The principal results are:

1. The exact planar formula (4), separating the intrinsic tilt ϕ from the extrinsic path orientation Θ.

2. The confinement theorem (Theorem 3.2): $\mathbf{a}$ lies in the osculating plane for every $\mathcal{C}^2$ space curve; the binormal component is identically zero.

3. The spatial formula (5), extending the intrinsic/extrinsic separation to $\mathbb{R}^3$ with torsion entering only through frame orientation.

4. The reduction theorem (Theorem 8.1): the planar formula is the unique $\tau = 0$ restriction of the spatial formula.

Together these results provide a rigorous, coordinate-free characterization of acceleration direction for smooth curves in both two and three dimensions, with direct applications in teaching, simulation, and sensor-based navigation.

Acknowledgements

[Acknowledgements removed for anonymous review.]

The author discloses the use of Claude (Anthropic) for LaTeX typesetting and language refinement. All conceptual and mathematical derivations are the original work of the author.

Conflict of Interest

The author has no conflicts to disclose.

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