Intrinsic Kinematics: Coordinate-Free Acceleration Direction in Two and Three Dimensions

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Abstract. Standard treatments of curvilinear motion decompose acceleration into tangential and normal components, but the resulting components are seldom assembled into an explicit expression for the *global* direction of the acceleration vector in coordinate-free terms. This tutorial collects, organizes, and works through such an expression for both planar curves and space curves in $\mathbb{R}^3$, with worked exercises intended for classroom use. Using the Frenet–Serret frame, curvature κ, and torsion τ [1–5], we present four results. First, for a smooth planar curve, the absolute direction of acceleration is assembled into the exact, coordinate-free formula $\theta_a(s) = \Theta_0 + \int_{s_0}^{s} \kappa\, d\sigma + \operatorname{atan2}(\kappa \dot{s}^2, \ddot{s})$, whose two terms separate cleanly by what they require of the observer: the tilt $\phi = \operatorname{atan2}(a_N, a_T)$ is read directly from an onboard accelerometer triad and needs no external input at all, whereas the path orientation $\Theta(s)$ requires both an external axis Θ_0 and an external initial speed $\dot{s}_0$. Following Galileo and Newton, speed itself is not an intrinsic quantity even in Newtonian mechanics, and we make the resulting three-level hierarchy explicit throughout. Second, for any $\mathscr{C}^2$ space curve, the acceleration vector is confined to the osculating plane: the binormal component $a_B = \mathbf{a} \cdot \mathbf{B}$ vanishes identically. Third, the acceleration tilt ϕ is τ-independent and remains onboard-measurable in three dimensions; the absolute spatial direction of acceleration is $\hat{\mathbf{a}} = \cos\phi\, \mathbf{T}(s) + \sin\phi\, \mathbf{N}(s)$, where $\mathbf{T}, \mathbf{N}$ are determined by integrating both κ and τ. Fourth, setting $\tau \equiv 0$ reduces the three-dimensional formula to the planar formula exactly—the planar result is a strict special case, not an independent result. Worked examples on a circular path and a circular helix confirm consistency with direct Cartesian computation.

Keywords: intrinsic kinematics; Frenet–Serret frame; acceleration decomposition; coordinate-free mechanics; curvature and torsion; curvilinear motion; classical mechanics pedagogy.

Contents

1 Introduction

Students in mechanics and multivariable calculus courses routinely encounter the decomposition of acceleration into tangential and normal components [6–9]:

$$\mathbf{a} = \ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N}. \tag{1}$$

This decomposition is presented in virtually every undergraduate treatment of curvilinear motion and is well understood. What is *not* addressed in standard treatments—neither in calculus textbooks nor in classical mechanics courses—is the following natural question:

> *What is the absolute direction of* **a***—the angle or unit vector it makes in space—expressed purely in terms of the intrinsic geometric and kinematic quantities* κ*,* τ*,* $\dot{s}$*, and* $\ddot{s}$*, without introducing a coordinate system?*

This question is not merely aesthetic. Students who have mastered the Frenet frame and the tangential–normal decomposition are left without a complete coordinate-free picture of where the acceleration vector actually points. The formula $\mathbf{a} = \ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N}$ tells them the *components* of **a** in the moving frame, but not the *global direction* of **a**—a distinction that becomes especially important when the curve is defined intrinsically by its curvature and torsion rather than by an explicit Cartesian parameterization.

This situation is not hypothetical. It is precisely the situation faced by an inertial measurement unit (IMU): a small sensor package, now found in every smartphone, aircraft, and autonomous vehicle, that measures acceleration and turning rate locally but has no direct access to a global coordinate grid. The framework developed here describes exactly how such a device can reconstruct the direction of its own acceleration from local measurements alone—the principle underlying dead-reckoning navigation when a global position reference (such as GPS) is unavailable. The Frenet–Serret apparatus that we use is itself widely applied in autonomous-vehicle path planning [10], robotics, computer-graphics camera motion and tube geometry [11], and the design of roads and roller-coaster tracks [3, 5].

We emphasize at the outset that the expression we assemble is built entirely from classical, well-established ingredients—the Frenet–Serret relations of 1852 [1] and 1851 [2], and the standard tangential–normal decomposition. Our aim is not to claim a new theorem of differential geometry, but to collect these ingredients into a single explicit, coordinate-free expression for the acceleration direction and to demonstrate, through worked exercises, how that expression is used. This pedagogical paper presents that expression for both planar and spatial motion. In doing so it makes explicit a conceptual distinction that standard treatments often leave implicit: some properties of acceleration are **intrinsic** (readable from instruments carried with the particle) while others are **extrinsic** (obtainable only with information handed in from outside). The dividing line is sharper, and falls in a less obvious place, than one might expect — it is fixed by Galilean relativity, and it puts *speed* on the extrinsic side. We make the distinction operational in the next subsection and use it as an organising principle throughout.

Intrinsic vs. extrinsic: the moving-frame viewpoint

Terminology, and a caution inherited from Galileo. Throughout this paper a quantity is called *intrinsic* if an observer travelling with the particle can determine it from onboard instruments alone, with no reference to any externally chosen axis, origin, or velocity; and *extrinsic* if its determination requires such external information. This is the operational counterpart of the differential-geometric usage [3, 5], in which a property is intrinsic when it can be determined by an observer confined to the curve itself.

It is essential to state at the outset what this does *not* include. Newtonian mechanics rests on Galilean relativity: the laws of mechanics take the same form in every inertial frame, and consequently *no mechanical* *experiment performed onboard can reveal the observer's own velocity*. This is the content of Galileo's ship argument [12] and of the Scholium to the Definitions in Newton's *Principia* [13]; it is not a refinement introduced by Einstein, and it remains a notorious conceptual hurdle for students [14–16]. **Speed is therefore not an intrinsic quantity—already in Newtonian mechanics, and not merely in relativity.** An inertial measurement unit contains accelerometers and rate gyroscopes; it contains no speedometer, and no arrangement of such instruments can supply one. This restriction governs everything below, and we have found that making it explicit sharpens rather than weakens the intrinsic/extrinsic separation.

What the onboard instruments actually deliver. Consider a particle carrying an accelerometer triad and rate gyroscopes, with one instrument axis mounted along the particle's forward direction. We stress that "forward" is a *structural* choice — the direction the vehicle is built to face, which for a particle constrained to a curve is **T** by definition of the motion — and not a measured velocity: knowing which way one's own nose points requires no speedometer. The instruments then read the components of the acceleration vector in the moving frame *directly*,

$$a_T = \ddot{s}, \qquad a_N = \kappa \dot{s}^2, \qquad a_B = 0, \tag{2}$$

the last by Theorem 3.2. The gyroscopes read the turning rate $\omega = \kappa \dot{s}$, *not* κ itself. Three levels of determination follow, and keeping them apart is the central conceptual point of this tutorial.

Level 1 — onboard alone. From (2) the particle obtains a_T, a_N, the magnitude $|\mathbf{a}| = \sqrt{a_T^2 + a_N^2}$, and the **acceleration tilt**

$$\phi = \operatorname{atan2}(a_N,\, a_T),$$

the angle between **a** and **T** within the osculating plane. This is **intrinsic** in the strict operational sense: it is formed from two readings of the same instrument, and needs no speed, no external axis, and no initial data whatever. Two particles with identical accelerometer histories agree on ϕ however they are oriented and however fast they travel.

Level 2 — onboard + the initial speed $\dot{s}_0$. To split the single reading a_N into a geometric factor κ and a kinematic factor $\dot{s}^2$, one must know $\dot{s}$. The particle may integrate its own accelerometer,

$$\dot{s}(t) = \dot{s}_0 + \int_0^t \ddot{s}(t')\, dt',$$

but this leaves the constant $\dot{s}_0$ undetermined, and by the Galilean argument above no onboard experiment can supply it. **The initial speed is extrinsic.** Given $\dot{s}_0$ from an external reference, the particle recovers $\dot{s}(t)$; a second integration gives $s(t)$ at the cost of a second extrinsic constant s_0; and only then do $\kappa = a_N/\dot{s}^2$ and $\tau = \omega_T/\dot{s}$ become available separately.

Level 3 — onboard + $\dot{s}_0$ + the initial orientation Θ_0. The **path orientation** $\Theta(s)$ — the direction **T** makes relative to a chosen axis in the surrounding space — requires in addition a starting orientation Θ_0, obtained by integrating the turning history from an external anchor. It is extrinsic on a second and independent count: it needs external *axes*, not merely an external *scalar*.

The absolute acceleration direction θ_a (in 2D) or $\hat{\mathbf{a}}$ (in 3D) sits at Level 3. Writing the formula explicitly, as we do in Section 5, is valuable precisely because it makes the mixing of levels visible:

$$\underbrace{\theta_a}_{\text{Level 3}} = \underbrace{\Theta_0 + \int \kappa\, d\sigma}_{\text{needs } \Theta_0 \text{ and } \dot{s}_0} + \underbrace{\phi}_{\text{Level 1: onboard}}.$$

One caution deserves emphasis, and it is the reason we have laboured the point: expressing ϕ in the *path* variables $(\kappa, \dot{s}, \ddot{s})$ rather than in the *instrument* variables (a_N, a_T) silently imports $\dot{s}$, and with it $\dot{s}_0$. The tilt itself is Level 1; the customary way of writing it is not. In the language of observers: ϕ is what an observer *riding on the particle* reads off the accelerometer; Θ is what a *fixed external observer* measures for the tangent direction; and the external observer's answer is the sum of the two.

Organization

Section 2 establishes notation for both 2D and 3D. Sections 3–5 develop the planar theory and derive the explicit formula for θ_a. Sections 6–7 extend the framework to $\mathbb{R}^3$. Section 8 proves the planar formula is a strict special case of the 3D formula. Sections 9–10 present worked examples and Cartesian verifications. Section 11 discusses geometric implications and open problems.

Teaching context and intended audience

This paper is aimed at undergraduate students who have completed a first course in multivariable calculus and are encountering the Frenet–Serret frame for the first time—typically second-year students in a mechanics, mathematical physics, or vector calculus course.

There are two **learning objectives**, and the second is the one we would most like to see taken up.

The first is to give students a complete coordinate-free picture of acceleration direction: not just how to decompose **a** into components, but where **a** actually points in space, and why that question separates naturally into a part readable onboard and a part that must be handed in from outside. This distinction is rarely made explicit at the undergraduate level, despite being conceptually fundamental.

The second is to use that separation as a concrete vehicle for **the relativity of motion**. Students routinely emerge from a mechanics course able to manipulate Galilean transformations while still believing, at some level, that a body has a speed in the way it has a mass [14, 17]. The hierarchy of Section 2 makes the correction unavoidable and quantitative rather than merely stated: the tilt ϕ is measured, the speed $\dot{s}$ is not, and the formula (11) displays both facts in a single line. An instructor can introduce Galileo's ship [12] and Newton's Scholium [13] at exactly the moment the students ask where $\dot{s}_0$ came from — a question the worked examples of Section 9 are designed to provoke. The Discussion (Section 11) develops this note further.

In a **classroom setting**, the planar formula (11) can be introduced immediately after the Frenet frame and curvature are taught, as a direct answer to the question students naturally ask: "we know the components of **a**—so what direction does it point?" The 3D extension (Section 7) and the reduction theorem (Section 8) serve as natural follow-on material, showing students how the 2D result generalises and how torsion enters the picture purely through frame orientation. The worked examples (Section 9) and Cartesian verifications (Section 10) are written to be suitable for tutorial or homework use.

2 Definitions and Notation

Let $\mathbf{r} : \mathscr{I} \to \mathbb{R}^n$ ($n = 2$ or 3) be a $\mathscr{C}^2$ curve parameterized by time t, with arc length $s(t)$ satisfying $\dot{s} > 0$.

- **Unit tangent:** $\mathbf{T}(s) = d\mathbf{r}/ds$.
- **Unit principal normal:** $\mathbf{N}(s) = (d\mathbf{T}/ds)\,/\,|d\mathbf{T}/ds|$.
- **Unit binormal** (3D only): $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$.
- **Curvature** $\kappa(s) \geq 0$ and **torsion** $\tau(s) \in \mathbb{R}$ are defined by the Frenet–Serret equations, derived independently by Frenet [1] and Serret [2] and given in modern vector form by standard texts [3–5, 18, 19]:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}. \tag{3}$$

Frenet stated six of these nine scalar component relations in 1852 and Serret supplied the remaining three in 1851; the compact vector notation used here is due to the later vector-analysis tradition of Gibbs and Heaviside [20]. In 2D, $\tau \equiv 0$ and the **B** equation is absent. In 3D, writing $R(s) = [\mathbf{T} \mid \mathbf{N} \mid \mathbf{B}]$ for the matrix whose columns are the three frame vectors, the Frenet–Serret equations take the compact form

$$\frac{dR}{ds} = R(s)\,K(s)^\top, \qquad K(s) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}. \tag{4}$$

Because K has the special structure $K^\top = -K$ (called skew-symmetric), equation (4) automatically preserves the orthonormality of $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ at every point along the curve—the frame never distorts, it only rotates.

- **Frame rotation vector** (3D): $\Omega = \tau\mathbf{T} + \kappa\mathbf{B}$, with $|\Omega| = \sqrt{\kappa^2+\tau^2}$ the total rate at which the Frenet frame rotates per unit arc length [3, 5]. This vector, classically known as the Darboux vector, points along the instantaneous axis of frame rotation.
- **Speed:** $\dot{s} = ds/dt$ [4, 6]; **tangential acceleration:** $\ddot{s} = d^2s/dt^2$ [6, 7].

3 Acceleration Decomposition

A note on attribution. The numbered results in this and the following sections are not claimed as new theorems of differential geometry. Proposition 3.1 (the tangential–normal decomposition) is entirely classical and appears in every vector-calculus text [4, 6, 7]. The content of Theorem 3.2 (that acceleration has no binormal component) is likewise a standard consequence of the Frenet relations, though it is rarely stated as an explicit result for the benefit of students. We present these as labelled statements purely for pedagogical clarity—so that each step has a name students can refer to—and we indicate in each case whether the result is standard or is simply our own organization of standard material. The contribution of this tutorial is the explicit assembly of the direction formulas (11) and (13) and the framing of the intrinsic/extrinsic separation at the undergraduate level; these rest entirely on classical ingredients, and we make no claim of priority for them (see the attribution note following equation (11)).

Proposition 3.1 (Tangential–normal decomposition; classical, see 6)**.** *For a $\mathscr{C}^2$ curve parameterized by time,*

$$\mathbf{a}(t) = \ddot{s}(t)\,\mathbf{T}(s(t)) + \kappa(s(t))\dot{s}(t)^2\,\mathbf{N}(s(t)). \tag{5}$$

Proof. Differentiate $\mathbf{v} = \dot{s}\mathbf{T}$ with respect to t, using the product rule [6]:

$$\mathbf{a} = \frac{d}{dt}(\dot{s}\mathbf{T}) = \ddot{s}\mathbf{T} + \dot{s}\frac{d\mathbf{T}}{dt}.$$

By the chain rule and the first Frenet–Serret equation (3), $d\mathbf{T}/dt = \dot{s}\,(d\mathbf{T}/ds) = \dot{s}\,\kappa\mathbf{N}$. Substituting gives the result. *(end of proof)*

The decomposition names the two scalar components:

- **Tangential acceleration:** $a_T = \ddot{s}$, the rate of speed change.
- **Normal acceleration:** $a_N = \kappa\dot{s}^2$, curvature times squared speed.

Theorem 3.2 (Osculating plane confinement; standard consequence of the Frenet relations)**.** *For any $\mathscr{C}^2$ space curve,* $\mathbf{a}\cdot\mathbf{B} \equiv 0$. *The acceleration vector lies entirely in the osculating plane* $\operatorname{span}\{\mathbf{T},\mathbf{N}\}$.

Proof. $\mathbf{a}\cdot\mathbf{B} = \ddot{s}(\mathbf{T}\cdot\mathbf{B}) + \kappa\dot{s}^2(\mathbf{N}\cdot\mathbf{B}) = 0$, since $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is orthonormal. *(end of proof)*

Physical interpretation. No matter how sharply a path twists out of a plane, the acceleration of a particle following it never has a component along the binormal. All of the acceleration lives in the osculating plane—the plane that momentarily contains the velocity and the centripetal turn. Torsion reorients this plane in space but does not push acceleration out of it.

Remark. Although Theorem 3.2 is mathematically elementary, it addresses a genuine and widespread *student* intuition trap rather than an error in the standard literature. Students frequently assume that because a space curve twists (nonzero torsion), the acceleration must acquire a binormal component to “follow the twist.” The standard textbooks are correct on this point—they derive the acceleration strictly within the osculating plane—but they seldom flag the misconception explicitly. The decomposition is and remains $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, with $a_B = 0$ identically, for every $\mathscr{C}^2$ curve. Stating this as a named result gives instructors a direct way to confront the misconception.

4 Acceleration Tilt ϕ

Definition 4.1 (The authors introduce the following expression)**.** *The* ***acceleration tilt*** *$\phi(t)$ is the directed angle from* **T** *to* **a***, measured in the osculating plane. In* instrument *variables — the two channels of the onboard accelerometer triad — it is*

$$\phi(t) = \operatorname{atan2}\big(a_N(t),\ a_T(t)\big), \tag{6}$$

and in path *variables, using (2), it is*

$$\phi(t) = \operatorname{atan2}\big(\kappa(s(t))\dot{s}(t)^2,\ \ddot{s}(t)\big). \tag{7}$$

The two expressions are numerically identical, but they are not epistemically identical: (6) is Level 1 (onboard alone), whereas (7) presupposes that $\dot{s}$ has already been reconstructed, and hence sits at Level 2. We use (7) throughout because it is the form that connects to the path geometry, but the reader should keep (6) in mind as the operational definition.

Proposition 4.2 (ϕ is τ-independent and axis-free; observation by the authors, proof uses only Definition 4.1)**.** *The tilt ϕ is independent of τ, of the initial frame orientation Θ_0, and of any choice of coordinate axes. It is determined by the accelerometer readings (a_N, a_T) alone, and is therefore intrinsic at Level 1; in particular it requires neither Θ_0 nor $\dot{s}_0$.*

Proof. By (6), ϕ is a function of a_N and a_T only. Torsion τ does not appear in a_T or a_N (Proposition 3.1), hence not in their ratio. The atan2 function is a scalar operation on two scalars; no frame vectors and no external axes are referenced. Neither reading depends on $\dot{s}_0$: the accelerometer measures a_N directly rather than inferring it from a reconstructed speed. *(end of proof)*

Corollary 4.3 (Immediate consequence of Proposition 4.2)**.** *The magnitude $|\mathbf{a}| = \sqrt{a_T^2 + a_N^2} = \sqrt{\ddot{s}^2 + \kappa^2\dot{s}^4}$ is likewise τ-independent and intrinsic at Level 1.*

Remark. What is not *Level 1.* The individual factors κ and $\dot{s}$ are not separately available onboard, since the instruments deliver only their combination $a_N = \kappa\dot{s}^2$ (and $\omega = \kappa\dot{s}$). Splitting them requires $\dot{s}_0$, which is extrinsic (Section 2). Consequently any statement of the form "κ is known" or "$\dot{s}(t)$ is given" — including the data supplied in the worked examples of Section 9 — is a Level 2 statement, and we flag it as such wherever it occurs.

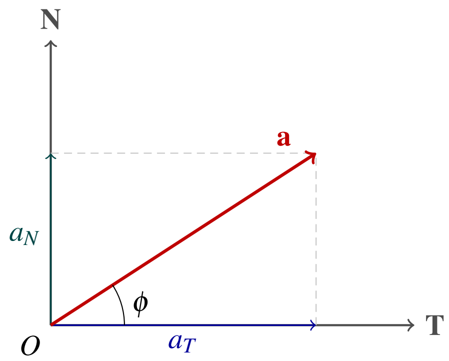
Special cases of ϕ:

- $\ddot{s} = 0$ (constant speed): $\phi = \operatorname{sign}(\kappa)\,\pi/2$ (purely centripetal).
- $\kappa = 0$ (instantaneously straight): $\phi = 0$ or π (purely tangential).

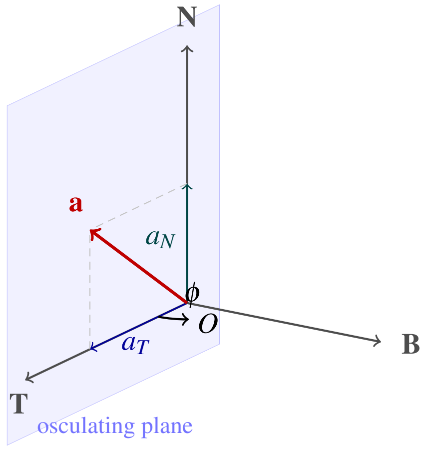
The straight-line case ($\kappa = 0$). The second special case deserves care, because the Frenet frame itself is built on the assumption $\kappa \neq 0$: when curvature vanishes, $d\mathbf{T}/ds = \mathbf{0}$, the principal normal $\mathbf{N} = (d\mathbf{T}/ds)/|d\mathbf{T}/ds|$ becomes $0/0$ and is undefined, and the binormal **B** with it. The osculating plane is then no longer determined by the geometry. At first glance this looks like a breakdown of the framework. It is not, for a simple physical reason: when $\kappa = 0$ the normal acceleration $a_N = \kappa\dot{s}^2$ vanishes as well, so by the decomposition (5) the acceleration reduces to

$$\mathbf{a} = \ddot{s}\mathbf{T}, \tag{8}$$

which is purely tangential and *perfectly well defined*—it requires only **T**, which exists for any regular curve, and not **N** or **B**. In other words, exactly when the frame loses its normal direction, the acceleration stops needing one. The tilt ϕ is then 0 (if $\ddot{s} > 0$) or π (if $\ddot{s} < 0$), recording that the acceleration is aligned with, or opposite to, the direction of motion. The only quantity that becomes ambiguous—the orientation of the osculating plane about the tangent axis—is precisely the quantity that no longer affects the acceleration. The formulas that follow should therefore be understood to apply on intervals where $\kappa > 0$, with the



(a) Planar case ($n = 2$).



(b) Spatial case ($n = 3$).

Figure 1. Intrinsic decomposition of the acceleration vector in the Frenet frame. In both panels the tangential component is $a_T\mathbf{T} = \ddot{s}\mathbf{T}$ and the normal component is $a_N\mathbf{N} = \kappa\dot{s}^2\mathbf{N}$, so that $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, and the tilt angle is $\phi = \operatorname{atan2}(\kappa\dot{s}^2, \ddot{s})$. **(a)** In the plane, **a** makes angle ϕ with the tangent. **(b)** For a space curve, **a** still lies entirely in the osculating plane span$\{\mathbf{T},\mathbf{N}\}$: the binormal component is identically zero, $a_B = \mathbf{a}\cdot\mathbf{B} = 0$ (Theorem 3.2). The tilt ϕ depends only on κ, $\dot{s}$, $\ddot{s}$; torsion τ reorients the osculating plane in space but does not enter the decomposition.

straight-line case handled by this limiting argument. Readers interested in frames that remain well defined even where $\kappa = 0$—at the cost of giving up the adaptation to the principal normal—may consult the rotation-minimizing (Bishop) frame [11]; we retain the Frenet frame here because it is the one students meet first and because the acceleration decomposition is most transparent in it.

5 Absolute Acceleration Direction: Planar Case

Path orientation

Let $\Theta(s)$ denote the angle of $\mathbf{T}$ relative to a fixed reference direction $\mathbf{e}$ (e.g., the positive x-axis). For a plane curve the (signed) curvature is, by definition, the rate at which this tangent angle turns with arc length [3–5]:

$$\frac{d\Theta}{ds} = \kappa(s). \tag{9}$$

Relation (9) is the standard planar definition of curvature as the turning rate of the tangent; we use it directly. Integrating (9) from s_0 with $\Theta(s_0) = \Theta_0$ gives the classical fundamental theorem of plane curves [5]:

$$\Theta(s) = \Theta_0 + \int_{s_0}^{s} \kappa(\sigma)\,d\sigma. \tag{10}$$

This is **extrinsic**: it depends on the intrinsic curvature κ and the externally chosen reference Θ_0.

Main formula (planar)

The absolute direction of **a** is $\theta_a = \Theta(s) + \phi$, giving:

$$\boxed{\theta_a(s) = \Theta_0 + \int_{s_0}^{s} \kappa(\sigma)\,d\sigma + \operatorname{atan2}\big(\kappa\dot{s}^2,\ \ddot{s}\big).} \tag{11}$$

What θ_a means, concretely. Fix an axis in the laboratory — say the positive x-axis — and let a stationary observer watch the particle go by. At the instant in question the particle's acceleration vector points somewhere in the plane; θ_a is the angle that vector makes with the chosen axis, measured anticlockwise. It is the answer to the question "if I stood here and drew the acceleration arrow on my graph paper, which way would it point?" Physically it is the direction in which a passenger riding the particle is pushed at

that instant, expressed in the laboratory's terms rather than the passenger's. The formula says that this direction is the sum of two independent pieces of information: how far the track has turned since the start (Θ), and how far the acceleration leans off the direction of travel (ϕ). A passenger feels only the second; the observer on the ground needs both.
The two contributions separate by what each demands of the observer:

- **Level 1 — onboard:** $\phi = \mathrm{atan2}(\kappa\dot{s}^2, \ddot{s}) = \mathrm{atan2}(a_N, a_T)$ is read directly off the accelerometer triad, and requires neither an external axis nor an external speed.

- **Levels 2–3 — external:** $\Theta_0 + \int \kappa\, d\sigma$ requires an initial reference angle Θ_0 (Level 3) and, since the integral runs over arc length, the initial speed $\dot{s}_0$ and position s_0 needed to know how much arc length has accumulated (Level 2).

It is worth pausing on the asymmetry. Written in the path variables $(\kappa, \dot{s}, \ddot{s})$, as in (11), both terms appear to need $\dot{s}$ and so both look extrinsic. Written in the instrument variables, the first term collapses to $\mathrm{atan2}(a_N, a_T)$ and needs nothing at all. The formula is the same; the epistemic reading is not. This is precisely the point emphasised in Section 2, and it is easy to miss.

Attribution of equation (11). The individual ingredients of (11) are all classical: the decomposition (5), the tilt ϕ as an atan2 of two known components, and the fundamental theorem of plane curves (10). We make no claim of priority for the assembled expression. Closely related constructions appear in the relativistic Frenet–Serret literature, where kinematic quantities are routinely built from the Frenet–Serret curvature invariants of a worldline [21, 22]; the orientation of the acceleration relative to the moving frame arises naturally in that setting. The contribution of the present tutorial is purely pedagogical: to assemble the expression explicitly at the level of a second-year Newtonian course, to read its two terms as an intrinsic tilt plus an extrinsic orientation, and to work through its use in concrete exercises.

6 Frame Orientation in Three Dimensions

In 3D, path orientation is no longer a single angle but the full spatial orientation of the Frenet frame—a continuously rotating orthonormal triple $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ governed by (4).

Distinct roles of κ and τ

The Frenet–Serret equations reveal a structural asymmetry:

- **Curvature** κ drives $d\mathbf{T}/ds = \kappa\mathbf{N}$ (bending), rotating $\mathbf{T}$ toward $\mathbf{N}$ within the osculating plane.

- **Torsion** τ drives the $\tau\mathbf{B}$ term in $d\mathbf{N}/ds$ and $-\tau\mathbf{N}$ in $d\mathbf{B}/ds$ (twisting), rotating the **N**-**B** half of the frame about the tangent axis **T**. This rotates the osculating plane itself in $\mathbb{R}^3$ without altering anything *within* the plane.

Consequence. τ governs where in space the osculating plane is oriented, but has no effect on a_T, a_N, $|\mathbf{a}|$, or ϕ. This deepens the intrinsic/extrinsic separation: the extrinsic component now requires integrating both κ and τ, while ϕ remains a purely local scalar.

Frame rotation vector and the rate of change of $\hat{\mathbf{a}}$

The vector $\Omega = \tau\mathbf{T} + \kappa\mathbf{B}$ points along the instantaneous axis about which the Frenet frame rotates as arc length increases, with magnitude $|\Omega| = \sqrt{\kappa^2 + \tau^2}$ equal to the total rotation rate per unit arc length. It governs how quickly $\hat{\mathbf{a}}$ changes direction in space; the authors obtain the following relation by differentiating (13) along the curve and applying the Frenet–Serret equations (3):

$$\frac{d\hat{\mathbf{a}}}{ds} = \Omega \times \hat{\mathbf{a}} + \frac{d\phi}{ds}(-\sin\phi\,\mathbf{T} + \cos\phi\,\mathbf{N}). \tag{12}$$

The first term captures how the osculating plane itself sweeps through space due to bending (κ) and twisting (τ); the second captures how the tilt angle ϕ changes due to varying speed and curvature.

7 Absolute Acceleration Direction: Spatial Case

Definition 7.1 (The authors assemble the following expression from Proposition 3.1 and the Frenet frame)**.** *The* ***absolute acceleration direction*** *in $\mathbb{R}^3$ is the unit vector*

$$\boxed{\hat{\mathbf{a}}(t) = \frac{\mathbf{a}(t)}{|\mathbf{a}(t)|} = \cos\phi(t)\,\mathbf{T}(s(t)) + \sin\phi(t)\,\mathbf{N}(s(t)).} \tag{13}$$

The formula separates cleanly:

- **Intrinsic factor:** $\cos\phi$, $\sin\phi$ depend only on $(\kappa, \dot{s}, \ddot{s})$.
- **Extrinsic factor:** $\mathbf{T}(s)$, $\mathbf{N}(s)$ require integrating the Frenet–Serret system (4) forward from a known starting orientation, accumulating both the bending (κ) and twisting (τ) of the path.

This is the 3D analogue of (11). In 2D the extrinsic component was one scalar Θ (integral of κ only); in 3D it is the full rotational history of the frame, accumulating both κ and τ as the path bends and twists through space.

8 Reduction: 3D Formula Contains the Planar Formula

Theorem 8.1 (Recovery of the planar formula; proof by the authors using classical ingredients)**.** *Suppose $\tau \equiv 0$ and the initial frame satisfies $\mathbf{B}_0 = \hat{\mathbf{e}}_z$, so the curve lies in the xy-plane. Then* (13) *reduces to $\hat{\mathbf{a}} = (\cos\theta_a, \sin\theta_a, 0)$, where θ_a is given by* (11)*.*

Proof. **Step 1.** With $\tau = 0$, the Frenet–Serret system reduces to $d\mathbf{T}/ds = \kappa\mathbf{N}$, $d\mathbf{N}/ds = -\kappa\mathbf{T}$, $d\mathbf{B}/ds = \mathbf{0}$. The last equation gives $\mathbf{B}(s) = \mathbf{B}_0 = \hat{\mathbf{e}}_z$ for all s.

Step 2. With $\mathbf{B} = \hat{\mathbf{e}}_z$, parameterize by the orientation angle $\Theta(s)$:

$$\mathbf{T} = (\cos\Theta, \sin\Theta, 0), \qquad \mathbf{N} = (-\sin\Theta, \cos\Theta, 0).$$

Differentiating $\mathbf{T}$ gives $d\mathbf{T}/ds = \dot{\Theta}\mathbf{N}$. Comparing with $d\mathbf{T}/ds = \kappa\mathbf{N}$ yields $d\Theta/ds = \kappa(s)$, which integrates to (10).

Step 3. Substituting into (13):

$$\begin{aligned}\hat{\mathbf{a}} &= \cos\phi(\cos\Theta, \sin\Theta, 0) + \sin\phi(-\sin\Theta, \cos\Theta, 0)\\ &= (\cos\phi\cos\Theta - \sin\phi\sin\Theta,\ \cos\phi\sin\Theta + \sin\phi\cos\Theta,\ 0)\\ &= (\cos(\Theta+\phi),\ \sin(\Theta+\phi),\ 0),\end{aligned}$$

where the last step uses the angle-addition identities (standard trigonometry; see e.g. 6). Setting $\theta_a = \Theta(s) + \phi$ and substituting (10) recovers (11). *(end of proof)*

Corollary 8.2 (Consequence of Theorem 8.1)**.** *Formula* (11) *is not an independent result. It is the unique planar restriction of* (13)*, obtained by $\tau = 0$.*

Remark. The proof identifies each ingredient: (i) ϕ is unchanged—it was always τ-independent; (ii) the scalar equation $d\Theta/ds = \kappa$ emerges from the full 3D Frenet–Serret system collapsing to a simple in-plane rotation when $\tau = 0$; (iii) the angle-addition step collapses the 3D unit vector $\hat{\mathbf{a}}$ to the scalar angle θ_a; (iv) $\mathbf{B}$ freezes to $\hat{\mathbf{e}}_z$ and plays no further role, reducing three-dimensional frame rotation to a single planar angle.

9 Worked Examples

9.1 Circular path (planar formula)

The problem. A particle travels along a circular track of radius $R = 5$ m. At the instant $t = 1$ s we ask: in what absolute direction (measured from a fixed axis in the laboratory) does the particle's acceleration point? This is the kind of question one would ask when, for example, deciding which way a passenger on the track is thrown at that instant.

The data, sorted by level. Following the discussion of Section 2, we state explicitly what is measured onboard and what must be supplied externally.

- **Level 1 (onboard).** The tangential accelerometer channel reads $\ddot{s}(t) = 8t$ m/s^2.
- **Level 2 (external scalars).** The initial speed $\dot{s}_0 = 0$ and the initial arc length $s_0 = 0$ are supplied by an external reference. Neither can be obtained onboard.
- **Level 3 (external axes).** The initial orientation $\Theta_0 = 0$ is supplied by a choice of laboratory axis.

Integrating the accelerometer reading with these constants gives

$$\dot{s}(t) = \dot{s}_0 + \int_0^t 8t'\,dt' = 4t^2 \text{ m/s}, \qquad s(t) = s_0 + \tfrac{4}{3}t^3 \text{ m}.$$

We emphasise that the familiar starting point "$\dot{s}(t) = 4t^2$" is *not* intrinsic data: it is the result of combining a Level-1 measurement with the extrinsic constant $\dot{s}_0$.

The track geometry gives $\kappa = 1/R = 0.2$ m^{-1}. At $t = 1$ s:

$$\dot{s}(1) = 4 \text{ m/s}, \quad \ddot{s}(1) = 8 \text{ m/s}^2, \quad s(1) = \tfrac{4}{3} \text{ m}, \quad a_N = \kappa\dot{s}^2 = 3.2 \text{ m/s}^2.$$

Acceleration tilt (Level 1)**:** $\phi(1) = \arctan(3.2/8) = \arctan(0.4) \approx 21.80°$. This is the angle between the acceleration and the forward direction of travel: the acceleration leans 21.80° off the tangent, toward the centre of the circle, reflecting the mixture of speeding up (a_T) and turning (a_N). The onboard observer reads this angle straight off the accelerometer triad as atan2(3.2, 8), without needing to know that the speed happens to be 4 m/s.

Path orientation (Level 3, with $\Theta_0 = 0$)**:** $\Theta(s(1)) = \kappa s(1) = 0.2 \cdot \frac{4}{3} = \frac{4}{15}$ rad $\approx 15.28°$. This is the direction the particle is currently heading, measured from the chosen laboratory axis.

Absolute direction: $\theta_a(1) = \Theta + \phi \approx 15.28° + 21.80° = 37.08°$. Interpretation: relative to the fixed laboratory axis, the acceleration vector points at 37.08°. A passenger at this instant is pushed in that absolute direction. We verify this number independently in Section 10. Note where each external input entered: Θ_0 fixed the axis against which θ_a is quoted, while $\dot{s}_0$ and s_0 fixed how far along the track the particle has travelled — and hence, through $\Theta(s)$, how much the tangent has turned. The tilt ϕ alone survives without either.

9.2 Circular helix (spatial formula)

The problem. A particle climbs a helical path—think of a bead on a spring, or a vehicle on a spiral ramp—of radius $R = 3$ m and pitch parameter $b = 4$ m, speeding up according to $\dot{s}(t) = 2t$ m/s. At $t = 2$ s we ask for the absolute direction in space of its acceleration, now a unit vector $\hat{\mathbf{a}}$ in $\mathbb{R}^3$ rather than a single angle.

Let $\mathbf{r}(\theta) = (R\cos\theta, R\sin\theta, b\theta)$ with $R = 3$ m, $b = 4$ m. The arc-length element is $\sqrt{R^2 + b^2} = 5$ m, so $s = 5\theta$. The standard helix invariants are [4, 5, 23]:

$$\kappa = \frac{R}{R^2 + b^2} = \frac{3}{25} \text{ m}^{-1}, \qquad \tau = \frac{b}{R^2 + b^2} = \frac{4}{25} \text{ m}^{-1}, \qquad |\Omega| = \frac{1}{5} \text{ m}^{-1}.$$

The data, sorted by level. The onboard accelerometer reads a constant tangential value $\ddot{s} = 2$ m/s^2 (Level 1). An external reference supplies $\dot{s}_0$ and $s_0 = 0$ (Level 2). Integration then gives

$$\dot{s}(t) = \dot{s}_0 + 2t \text{ m/s}, \qquad s(t) = s_0 + \dot{s}_0 t + t^2 \text{ m}. \tag{14}$$

The speed profile is therefore *not* given data: it is inferred, and the inference carries the extrinsic constant $\dot{s}_0$ with it.

Case $\dot{s}_0 = 0$. Taking the external reference to report that the bead started from rest, (14) gives $\dot{s}(t) = 2t$ m/s; at $t = 2$ s: $\dot{s} = 4$ m/s, $\ddot{s} = 2$ m/s^2, $s = 4$ m.

Kinematic quantities: $a_T = 2$ m/s^2, $a_N = 1.92$ m/s^2, $a_B = 0$ (Theorem 3.2), $|\mathbf{a}| = \sqrt{7.6864} \approx 2.773$ m/s^2.

Acceleration tilt: $\phi = \arctan(1.92/2) = \arctan(0.96) \approx 43.83°$.

Case $\dot{s}_0 \neq 0$: why the initial speed matters. Retaining $\dot{s}_0$ as a free external parameter, the normal channel at $t = 2$ s reads $a_N = \kappa(\dot{s}_0 + 4)^2 = \frac{3}{25}(\dot{s}_0 + 4)^2$, while the tangential channel still reads $a_T = 2$ m/s^2. Hence

$$\phi\big|_{t=2} = \operatorname{atan2}\left(\tfrac{3}{25}(\dot{s}_0 + 4)^2,\ 2\right), \tag{15}$$

which for $\dot{s}_0 = 0, 1, 2$ m/s gives $\phi \approx 43.83°,\ 56.31°,\ 65.77°$ respectively. The three cases are genuinely different motions along the *same* helix, and the onboard accelerometer distinguishes them without difficulty — it simply reports a different a_N in each case. What the onboard observer cannot do is run the argument backwards: from a_N alone, and without $\dot{s}_0$, the split of a_N into κ and $\dot{s}^2$ is not determined. This is the practical content of the Galilean restriction, and it is worth putting to students directly: the tilt is measured, the speed is not.

Frenet frame at $s = 4$ m ($\theta = 0.8$ rad; $\sin\theta \approx 0.7174$, $\cos\theta \approx 0.6967$):

$$\begin{aligned}
\mathbf{T} &\approx (-0.4304,\ 0.4180,\ 0.8000),\\
\mathbf{N} &\approx (-0.6967,\ -0.7174,\ 0),\\
\mathbf{B} &\approx (0.5739,\ -0.5574,\ 0.6000).
\end{aligned}$$

Verification: $|\mathbf{T}| = |\mathbf{N}| = |\mathbf{B}| = 1$, $\mathbf{T}\cdot\mathbf{N} = 0$, $\mathbf{T}\times\mathbf{N} = \mathbf{B}$. ✓

Absolute acceleration direction:

$$\begin{aligned}
\hat{\mathbf{a}} &= \frac{2}{2.773}\mathbf{T} + \frac{1.92}{2.773}\mathbf{N} = 0.7214\,\mathbf{T} + 0.6925\,\mathbf{N}\\
&\approx (-0.7929,\ -0.1952,\ 0.5771).
\end{aligned}$$

Check: $|\hat{\mathbf{a}}|^2 \approx 0.6287 + 0.0381 + 0.3331 = 1.0000$. ✓Interpretation: the unit vector $\hat{\mathbf{a}} \approx (-0.79, -0.20, 0.58)$ is the actual direction in laboratory space in which the bead is being accelerated at $t = 2$ s. Its positive z-component reflects the fact that the bead is still gaining speed as it climbs; its horizontal part points inward and around, reflecting the centripetal turning. We confirm this vector by direct Cartesian computation in Section 10.

10 Cartesian Verification

10.1 Circular path

Parameterize the circle by polar angle ϑ: $\mathbf{r}(\vartheta) = R(\cos\vartheta, \sin\vartheta)$. At $t = 1$, $\vartheta(1) = s(1)/R = 4/15$ rad, giving $\mathbf{T} \approx (-0.2636, 0.9646)$, $\mathbf{N} \approx (-0.9646, -0.2636)$. Then

$$\mathbf{a}(1) = 8\mathbf{T} + 3.2\mathbf{N} \approx (-5.1955,\ 6.8733).$$

Global direction: $\theta_{\text{global}} = \operatorname{atan2}(6.8733, -5.1955) \approx 127.08°$.

The intrinsic result used $\Theta_0 = 0$ at $s = 0$, which corresponds to taking $\mathbf{e}$ as the positive y-axis. Converting: $90° + 37.08° = 127.08°$. ✓

10.2 Circular helix

$$\mathbf{a} = 2\,\mathbf{T} + 1.92\,\mathbf{N} \approx (-2.1985,\ -0.5414,\ 1.6000), \quad |\mathbf{a}| \approx 2.773 \text{ m/s}^2.$$

Unit vector: $\hat{\mathbf{a}}_{\text{Cart.}} \approx (-0.7928, -0.1953, 0.5771)$.

Method	$\hat{\mathbf{a}}_x$	$\hat{\mathbf{a}}_y$	$\hat{\mathbf{a}}_z$
Intrinsic formula	-0.7929	-0.1952	0.5771
Cartesian verification	-0.7928	-0.1953	0.5771
Discrepancy	< 0.0002	< 0.0002	0.0000

Table 1. Component-wise comparison for the helix example. Discrepancies are rounding only.

11 Discussion

1. The intrinsic/extrinsic separation

The central conceptual contribution of both formulas is the clean separation between what is locally measurable (intrinsic) and what requires global history (extrinsic). In 2D, the separation is one-dimensional: scalar ϕ intrinsic, scalar Θ extrinsic. In 3D it deepens structurally: ϕ and $|\mathbf{a}|$ remain scalars computed from local data, while the extrinsic component becomes the full rotational history of the Frenet frame—a continuously updated orientation in space that encodes both κ and τ. This asymmetry—*scalar intrinsic, rotating frame extrinsic*—is the 3D manifestation of the same principle.

2. The atan2 function

The two-argument arctangent $\operatorname{atan2}(a_N, a_T)$ is essential for defining ϕ unambiguously. It preserves quadrant information and handles degenerate cases ($a_T = 0$ or $a_N = 0$), signed curvature ($\kappa \gtrless 0$), and negative tangential acceleration ($\ddot{s} < 0$).

3. Practical implications and the wider use of the Frenet frame

The formulas are naturally aligned with how physical sensors work. Inertial measurement units measure $\ddot{s}$, κ, and τ directly—the precise inputs to ϕ. Reconstructing the global direction then requires integrating κ (and τ in 3D) forward from a known initial orientation, exactly the structure of (11) and (13). This has applications in dead-reckoning navigation and robot path following on curves that are defined by curvature profiles rather than Cartesian parameterizations. The Frenet–Serret frame on which all of this rests is far from a historical curiosity: since its introduction by Frenet [1] and Serret [2], and its translation into modern vector language by Gibbs and Heaviside [20], it has become a standard tool in autonomous-vehicle trajectory planning (the "Frenet frame" path representation [10]), computer-graphics camera and tube geometry [11], the engineering of roads and roller-coaster tracks, and the modelling of DNA supercoiling in biophysics. The present tutorial adds to this body of use a simple, explicit reading of one quantity—the absolute direction of acceleration—that these applications all implicitly rely on.

4. A pedagogical note: why speed was never intrinsic

The restriction that shaped Section 2 — that no onboard instrument reports the observer's own speed — is not a technicality about instrumentation, and it is not a refinement contributed by Einstein. It is one of the founding insights of mechanics, and it is worth presenting to students as such.

Galileo argued the point in 1632, in the celebrated ship passage of the *Dialogue Concerning the Two Chief World Systems* [12]. Shut yourself below decks, he wrote, with flies, butterflies, a bowl of fish, and a bottle dripping into a vessel beneath. With the ship at rest, observe how everything behaves; then have it proceed at any uniform speed you like. *You will discover not the least change in any of the effects*—the fish swim no more readily toward one end of the bowl than the other, the drops fall straight down, jumping toward the stern is no harder than toward the bow. From nothing that happens inside can you infer whether the ship moves or stands still.

Newton drew the same boundary in the Scholium to the Definitions of the *Principia* [13]: despite his commitment to absolute space, he conceded that the relative motions of bodies are what we actually observe, and that absolute rest cannot be identified by any experiment on those bodies. Modern treatments state the conclusion flatly: no experiment can measure a uniform velocity, so we can never determine how fast we are moving, only how fast we are moving *relative to something else* [16, 24]. What accelerometers detect—the whole reason our Level-1 quantities exist—is exactly what Galileo's cabin could also detect: *changes* in motion, not motion itself.

That this remains difficult is well documented. The relativity of motion is routinely under-emphasised in curricula, and students carry a stubborn intuition that motion is something a body possesses in itself rather than a relation between bodies [14, 17]. Classroom demonstrations of the Galilean principle —

the modern descendants of Galileo's cabin, conducted in moving vans rather than ships — are effective precisely because they confront that intuition head-on, and because they make vivid the distinction between uniform motion (undetectable from inside) and accelerated motion (immediately detectable) [15, 25].

We suggest that the intrinsic/extrinsic hierarchy of Section 2 is a natural place to make this concrete in a mechanics or vector-calculus course. The three levels are not an abstract taxonomy: Level 1 is exactly what Galileo's sealed cabin permits, Level 2 is what it forbids, and the formula (11) exhibits both in a single line. A student who understands why $\dot{s}_0$ must be handed in from outside — and why $\ddot{s}$ need not be — has understood Galilean relativity, whether or not the phrase is used.

Beyond the Newtonian setting. The same questions can be asked in Minkowski spacetime, where the natural construction replaces arc length by proper time and the Frenet frame by a Fermi–Walker transported frame, yielding a tilt built from the four-acceleration in the instantaneously comoving frame [26, 27]; kinematic quantities assembled from the Frenet–Serret invariants of a worldline are developed in [21, 22]. That treatment requires apparatus well beyond the prerequisites assumed here, and we do not pursue it; we note only that the Level-1/Level-2 boundary drawn above is the Galilean shadow of a distinction that persists, in sharper form, in relativity.

5. Open problems

1. **Intrinsic Newton's second law.** Can $\mathbf{F} = m\mathbf{a}$ be written entirely in terms of κ, τ, $\dot{s}$, $\ddot{s}$, and their arc-length derivatives, with no coordinate system appearing?

2. **Conservation laws.** Which mechanical invariants (energy, momentum, angular momentum) admit purely intrinsic expressions? The natural setting for this question is the Hamiltonian and symmetry framework of analytical mechanics [28].

3. **Sign change of τ.** For curves where τ changes sign, how does $\hat{\mathbf{a}}$ behave at the torsion inflection point?

4. **How far can Level 1 be pushed?** The tilt ϕ and the magnitude $|\mathbf{a}|$ are available onboard without $\dot{s}_0$. Which other kinematic quantities share this property, and is there a systematic characterisation of the Level-1 algebra generated by the accelerometer and gyroscope readings? A negative counterpart is equally interesting: is there a clean statement of exactly what an onboard observer can *never* recover, beyond the two constants $\dot{s}_0$ and Θ_0 identified here?

5. **Relativistic extension.** Replacing arc length by proper time and the Frenet frame by a Fermi–Walker transported frame should yield a tilt that is intrinsic in the stronger, Lorentz sense. Carrying this out for a concrete worldline and exhibiting its departure from the Newtonian ϕ as v/c grows would make a natural sequel, though one requiring apparatus beyond the present scope [21, 22, 26].

12 Conclusion

Standard treatments decompose acceleration but seldom assemble the result into an explicit, coordinate-free expression for the global direction of $\mathbf{a}$. This tutorial supplies that expression and works through its use. The principal points are:

1. The exact planar formula (11), separating the intrinsic tilt ϕ from the extrinsic path orientation Θ.

2. The confinement theorem (Theorem 3.2): $\mathbf{a}$ lies in the osculating plane for every $\mathscr{C}^2$ space curve; the binormal component is identically zero.

3. The spatial formula (13), extending the intrinsic/extrinsic separation to $\mathbb{R}^3$ with torsion entering only through frame orientation.

4. The reduction theorem (Theorem 8.1): the planar formula is the unique $\tau = 0$ restriction of the spatial formula.

Together these results provide a rigorous, coordinate-free characterization of acceleration direction for smooth curves in both two and three dimensions, with direct applications in teaching, simulation, and sensor-based navigation.

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