Intrinsic Kinematics: Acceleration Decomposition and Path Orientation via Curvature Integration

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Abstract

We present a concise, coordinate-free (intrinsic) derivation of the standard tangential–normal decomposition of acceleration for a particle moving along a smooth planar curve [1, 2]. While the decomposition itself is classical, our approach emphasizes the conceptual distinction between the local acceleration tilt ϕ (intrinsic, measurable without coordinates) and the global path orientation $\Theta(s)$ (extrinsic, relative to a fixed reference). Using the Frenet relations and curvature κ [3, 4], we derive an exact expression for ϕ and show that Θ satisfies the differential relation $d\Theta/ds = \kappa(s)$ [5]. Integrating this relation, we obtain a compact, intrinsic–extrinsic formula for the particle's absolute acceleration direction θ_a entirely in terms of curvature and kinematic rates $(\kappa, \dot{s}, \ddot{s})$:

$$\theta_a(t) = \Theta_0 + \int_{s_0}^{s(t)} \kappa(\sigma)\, d\sigma + \operatorname{atan2}\big(\kappa(s(t))\dot{s}(t)^2,\ \ddot{s}(t)\big).$$

A worked example for motion along a circular path confirms consistency with the standard Cartesian formulation. This intrinsic perspective clarifies subtle geometric distinctions often overlooked in classical treatments of planar motion.

1 Introduction and Intrinsic Motivation

The analysis of particle motion often requires decomposing the acceleration vector $\mathbf{a}$ into components parallel and perpendicular to the direction of motion [2]. While standard treatments typically rely on coordinate-dependent Cartesian formulations $\mathbf{r}(t) = (x(t), y(t))$, this paper develops a **coordinate-free (intrinsic)** framework using the Frenet frame and arc length s [1, 3].

The primary conceptual contribution of this work is a clear separation between two distinct angles characterizing motion:

1. **Acceleration Tilt (ϕ):** The angle between the acceleration $\mathbf{a}$ and the tangent vector $\mathbf{T}$ at a given instant. This angle represents the instantaneous balance between speed change ($\mathbf{a}_T$) and turning ($\mathbf{a}_N$) and depends only on the intrinsic scalar quantities $\ddot{s}$ and κ, making it measurable without reference to any external coordinate system [5].

2. **Path Orientation (Θ):** The absolute angle of the tangent $\mathbf{T}$ relative to a fixed global reference (e.g., the x-axis). This angle satisfies $d\Theta/ds = \kappa(s)$ but requires an imposed initial condition Θ_0, making it an extrinsic quantity [3, 4].

Intrinsic Perspective (Thought Experiment): Consider a particle moving in a coordinate-free environment, where no external reference axes exist. The particle can measure only its speed change ($\ddot{s}$) and its instantaneous turning rate (κ). The local acceleration tilt ϕ can be determined from these measurements and is therefore fully **intrinsic**. In contrast, the path orientation Θ is **extrinsic**, depending on an arbitrarily chosen initial reference Θ_0 [1]. This paper demonstrates that both ϕ and Θ are fully determined by the intrinsic geometric property of curvature, and their combination yields the absolute acceleration direction $\theta_a = \Theta + \phi$, providing a complete intrinsic–extrinsic characterization of planar motion.

2 Definitions and Notation

Let $\mathbf{r} : \mathcal{I} \to \mathbb{R}^2$ be a $\mathcal{C}^2$ planar curve parameterized by time t, with arc length $s(t)$ [2, 3].

- The unit tangent vector is $\mathbf{T}(s) = d\mathbf{r}/ds$.
- The unit normal vector $\mathbf{N}(s)$ is obtained by a counterclockwise rotation of $\mathbf{T}$ by $\pi/2$.
- The time derivatives are $\dot{s} = ds/dt$ (speed) and $\ddot{s} = d^2s/dt^2$ (tangential acceleration) [4].

The curvature $\kappa(s)$ (signed magnitude) is defined by the fundamental Frenet relation:

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s)[1].$$

The velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ are given by

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \dot{s}\mathbf{T}(s(t)), \qquad \mathbf{a}(t) = \frac{d\mathbf{v}}{dt}[2].$$

3 Tangential–Normal Decomposition (Coordinate-Free)

Applying the product and chain rules to differentiate the velocity vector yields

$$\mathbf{a}(t) = \frac{d}{dt}\big(\dot{s}\mathbf{T}(s(t))\big) = \ddot{s}\mathbf{T} + \dot{s}\frac{d\mathbf{T}}{dt}[5].$$

Using the Frenet relation and the chain rule on the second term gives

$$\frac{d\mathbf{T}}{dt} = \frac{ds}{dt}\frac{d\mathbf{T}}{ds} = \dot{s}\big(\kappa\mathbf{N}\big) = \kappa\dot{s}\mathbf{N}.$$

Substituting back, the standard intrinsic decomposition emerges:

$$\boxed{\mathbf{a}(t) = \underbrace{\ddot{s}(t)}_{a_T}\,\mathbf{T}(s(t)) + \underbrace{\kappa(s(t))\dot{s}(t)^2}_{a_N}\,\mathbf{N}(s(t))}\,[2].$$

Thus, the acceleration decomposes naturally into:

- **Tangential Acceleration:** $a_T = \ddot{s}(t)$, the rate of change of speed.
- **Normal Acceleration:** $a_N = \kappa(s(t))\dot{s}(t)^2$, the curvature multiplied by the squared speed.

Notably, this decomposition is fully intrinsic: it depends only on the scalar quantities κ, $\dot{s}$, and $\ddot{s}$, independent of any external coordinate system.

4 Angle Between Acceleration and Tangent (Acceleration Tilt ϕ)

Let $\phi(t)$ denote the directed angle from the tangent $\mathbf{T}$ to the acceleration $\mathbf{a}$, measured in the local Frenet $(\mathbf{T}, \mathbf{N})$ frame [3]. The components of $\mathbf{a}$ in this frame are $(a_T, a_N) = (\ddot{s}, \kappa\dot{s}^2)$.

The angle ϕ is computed using the two-argument arctangent function to preserve quadrant information and handle degenerate cases [6]:

$$\phi(t) = \operatorname{atan2}(a_N, a_T)$$

$$\boxed{\phi(t) = \operatorname{atan2}\big(\kappa(s(t))\dot{s}(t)^2, \ddot{s}(t)\big)}$$

The acute angle satisfies

$$|\phi| = \arctan\left(\frac{|\kappa|\dot{s}^2}{|\ddot{s}|}\right).$$

- If $\ddot{s} = 0$ (constant speed), then $\phi = \operatorname{sign}(\kappa)\pi/2$ (purely normal acceleration).
- If $\kappa = 0$ (straight path), then $\phi = 0$ or π (purely tangential acceleration).

Notably, ϕ is entirely intrinsic: it depends solely on the local kinematic scalars $\dot{s}, \ddot{s}$ and the curvature κ, independent of any global coordinate system.

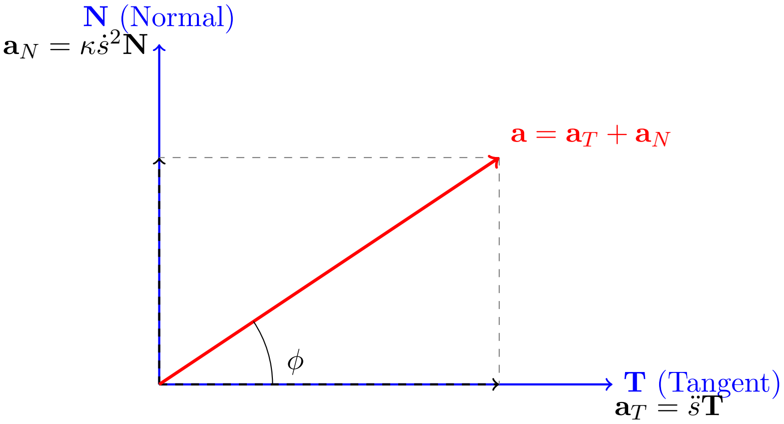

Figure 1: Intrinsic decomposition of acceleration in the moving $(\mathbf{T}, \mathbf{N})$ frame. The total acceleration $\mathbf{a}$ is the sum of the tangential component $\mathbf{a}_T$ and normal component $\mathbf{a}_N$, forming the acceleration tilt angle ϕ [1].

5 Intrinsic Computation of Path Orientation (Θ)

Let $\Theta(s)$ denote the absolute orientation of the unit tangent $\mathbf{T}$ relative to a fixed global reference direction $\mathbf{e}$ (e.g., the positive x-axis) [3]. By definition in differential geometry, the curvature $\kappa(s)$ measures the instantaneous rate of rotation of the tangent vector with respect to arc length s:

$$\frac{d\Theta}{ds} = \kappa(s)[1].$$

Integrating from a base point s_0 where $\Theta(s_0) = \Theta_0$ yields

$$\boxed{\Theta(s) = \Theta_0 + \int_{s_0}^{s} \kappa(\sigma)\, d\sigma}\,[5].$$

This fundamental result shows that the absolute path orientation is extrinsic: it depends on both the intrinsic curvature $\kappa(s)$ and the choice of the initial reference orientation Θ_0 [3]. Unlike the acceleration tilt ϕ, which is purely intrinsic, Θ requires an external reference, highlighting the separation between intrinsic and extrinsic contributions to the acceleration vector.

6 Intrinsic–Extrinsic Decomposition of Acceleration Direction

The absolute direction of the total acceleration vector $\mathbf{a}$, measured from the same global reference $\mathbf{e}$, can be expressed as the sum of the extrinsic path orientation $\Theta(s)$ and the intrinsic acceleration tilt $\phi(t)$:

$$\theta_a(t) = \Theta(s(t)) + \phi(t)[2].$$

Substituting the explicit formulas derived above for $\Theta(s)$ and $\phi(t)$ gives a compact, fully intrinsic expression for the global acceleration direction:

$$\boxed{\theta_a(t) = \Theta_0 + \int_{s_0}^{s(t)} \kappa(\sigma)\, d\sigma + \operatorname{atan2}\big(\kappa(s(t))\dot{s}(t)^2,\ \ddot{s}(t)\big)}\,[6].$$

Here, the contributions are clearly distinguished:

- **Intrinsic Component:** The tilt $\phi(t) = \operatorname{atan2}(a_N, a_T)$ depends only on the local kinematic quantities $(\dot{s}, \ddot{s})$ and the curvature κ, measurable without reference to any global coordinate system.
- **Extrinsic Component:** The path orientation $\Theta(s) = \Theta_0 + \int_{s_0}^{s} \kappa(\sigma) d\sigma$ requires an initial reference angle Θ_0 and integrates the intrinsic curvature to yield a global direction.

This decomposition emphasizes a clear conceptual separation: the intrinsic tilt ϕ captures how the particle accelerates relative to its instantaneous motion, while the extrinsic path orientation Θ captures how the particle's trajectory is oriented in the larger reference frame [1]. Together, they provide a complete, coordinate-free description of the absolute acceleration direction.

7 Worked Example: Particle on a Circular Path

Setup

Let the curve be a circle of radius $R = 5$ m. The curvature is constant: $\kappa = \frac{1}{R} = 0.2\ \text{m}^{-1}$. Let the speed be $v(t) = \dot{s}(t) = 4t^2$ m/s for $t \geq 0$. We compute values at $t = 1$ s.

Compute derivatives and values

- Speed: $\dot{s}(1) = 4(1)^2 = 4$ m/s.
- Tangential acceleration: $\ddot{s}(t) = \frac{d}{dt}(4t^2) = 8t$, so $\ddot{s}(1) = 8\ \text{m/s}^2$.
- Normal acceleration: $a_N = \kappa \dot{s}^2 = 0.2 \cdot 4^2 = 3.2\ \text{m/s}^2$.
- Magnitude of acceleration: $|\mathbf{a}| = \sqrt{\ddot{s}^2 + (\kappa \dot{s}^2)^2} = \sqrt{8^2 + 3.2^2} = \sqrt{74.24} \approx 8.613\ \text{m/s}^2$.

1. Acceleration Tilt (ϕ)

$$\phi(1) = \operatorname{atan2}(3.2, 8) = \arctan\left(\frac{3.2}{8}\right) = \arctan(0.4) \approx 0.3805 \text{ rad} \approx \mathbf{21.8}^\circ$$

2. Path Orientation (Θ)

First, find the arc length traveled from $t = 0$ to $t = 1$:

$$s(1) = \int_0^1 v(t)\, dt = \int_0^1 4t^2\, dt = \frac{4}{3} \text{ m} \approx 1.3333 \text{ m}$$

Now, integrate curvature with $\Theta_0 = 0$ at $s = 0$:

$$\Theta(s(1)) = \int_0^{4/3} \kappa\, d\sigma = \kappa s(1) = 0.2 \cdot \frac{4}{3} = \frac{4}{15} \text{ rad} \approx 0.2667 \text{ rad} \approx \mathbf{15.28}^\circ$$

3. Absolute Direction of Acceleration (θ_a)

$$\theta_a(1) = \Theta(s(1)) + \phi(1) \approx 15.28^\circ + 21.80^\circ = \mathbf{37.08}^\circ$$

All numerical results are consistent with the coordinate-dependent Cartesian computation, validating the intrinsic framework.

8 Cartesian Verification (Reverse Solution)

We now solve the worked example using explicit Cartesian coordinates to verify the intrinsic computation "backwards." This also clarifies the role of the chosen reference direction $\mathbf{e}$ (the choice $\Theta_0 = 0$ in the intrinsic solution).

1. Parameterization

Parameterize the circle of radius R by the polar angle θ:

$$\mathbf{r}(\theta) = R(\cos\theta, \sin\theta).$$

If arc length is $s = R\theta$, then for the motion in the worked example we have

$$\theta(t) = \frac{s(t)}{R}, \qquad s(1) = \frac{4}{3}\text{ m},$$

so at $t = 1$,

$$\theta(1) = \frac{s(1)}{R} = \frac{4/3}{5} = \frac{4}{15}\text{ rad} \approx 0.2667\text{ rad}.$$

2. Tangent and normal in Cartesian coordinates

Differentiate $\mathbf{r}(\theta)$ with respect to θ to obtain the unit tangent direction (after normalization). For this standard circle parameterization the unit tangent and chosen unit normal are

$$\mathbf{T}(\theta) = (-\sin\theta,\ \cos\theta), \qquad \mathbf{N}(\theta) = (-\cos\theta,\ -\sin\theta),$$

with curvature $\kappa = 1/R = 0.2\text{ m}^{-1}$. At $\theta(1) = 4/15$ we compute numerically

$$\sin\theta(1) \approx 0.2636, \qquad \cos\theta(1) \approx 0.9646,$$

so

$$\mathbf{T}(1) \approx (-0.2636,\ 0.9646), \qquad \mathbf{N}(1) \approx (-0.9646,\ -0.2636).$$

3. Cartesian acceleration components

Using the intrinsic decomposition $\mathbf{a} = \ddot{s}\,\mathbf{T} + (\kappa\dot{s}^2)\mathbf{N}$ with the values from the worked example ($\ddot{s}(1) = 8$, $\kappa\dot{s}(1)^2 = 3.2$), we compute the Cartesian acceleration vector at $t = 1$:

$$\mathbf{a}(1) = 8\mathbf{T}(1) + 3.2\mathbf{N}(1).$$

Substituting the numerical vectors gives

$$8\mathbf{T}(1) \approx 8(-0.2636,\ 0.9646) = (-2.1088,\ 7.7168),$$

$$3.2\mathbf{N}(1) \approx 3.2(-0.9646,\ -0.2636) = (-3.0867,\ -0.8435),$$

and therefore

$$\mathbf{a}(1) \approx (-2.1088 - 3.0867,\ 7.7168 - 0.8435) \approx (-5.1955,\ 6.8733).$$

4. Global acceleration direction (Cartesian atan2)

Compute the global direction of $\mathbf{a}(1)$ measured from the standard positive x-axis:

$$\theta_{\text{global}} = \operatorname{atan2}\big(a_y(1),\ a_x(1)\big) = \operatorname{atan2}(6.8733,\ -5.1955) \approx 2.2186\text{ rad} \approx 127.08^\circ.$$

5. Consistency with the intrinsic result

In the intrinsic computation we chose $\Theta_0 = 0$ at $s = 0$, which corresponds (for the circle parameterization used here) to taking the external reference direction $\mathbf{e}$ to be the *positive y-axis.* Consequently the intrinsic orientation $\Theta(s)$ in the worked example was the angle measured from $\mathbf{e}$ (not from the positive x-axis), yielding

$$\Theta(s(1)) \approx 15.28^\circ, \qquad \phi(1) \approx 21.80^\circ,$$

and thus the intrinsic acceleration direction relative to $\mathbf{e}$ was

$$\theta_a^{(\mathbf{e})} = \Theta(s(1)) + \phi(1) \approx 37.08^\circ.$$

To convert this to the standard x-axis reference, add the angle of $\mathbf{e}$ relative to the x-axis, which is 90°. Hence

$$\theta_{\text{global}} = 90^\circ + \theta_a^{(\mathbf{e})} \approx 90^\circ + 37.08^\circ = 127.08^\circ,$$

in agreement (within rounding) with the Cartesian atan2 result above. This completes the reverse (Cartesian) verification and shows the intrinsic method works consistently when translated back into coordinates.

9 Discussion and Remarks

1. Intrinsic vs. Extrinsic Properties

The Frenet–Serret framework provides the formal mathematical structure, but the primary conceptual contribution of this work is the clear separation between the **acceleration tilt** ϕ and the **path orientation** Θ. The angle ϕ is strictly *intrinsic*, depending only on the local scalar data (a_T, a_N), and thus on $(\dot{s}, \ddot{s}, \kappa)$. In contrast, Θ is *extrinsic*, encoding the integrated geometric turning of the curve relative to a chosen global reference:

$$\Theta(s) = \Theta(s_0) + \int_{s_0}^{s} \kappa(\sigma)\, d\sigma.$$

This distinction clarifies that while the global direction of motion depends on the chosen reference frame, the decomposition of acceleration into tangential and normal components is entirely reference-independent.

2. Importance of atan2

The two-argument arctangent function $\operatorname{atan2}(a_N, a_T)$ is essential for defining ϕ unambiguously. It preserves quadrant information and correctly handles degenerate cases such as $a_T = 0$ (constant-speed turning) or $a_N = 0$ (straight-line acceleration), including signed curvature ($\kappa \gtrless 0$) and negative tangential acceleration ($\ddot{s} < 0$). This guarantees that ϕ remains a globally continuous and intrinsically valid measure of the acceleration's deviation from the tangent.

3. Generalizations to Space Curves

The intrinsic decomposition extends naturally to three-dimensional motion. In this case, the Frenet frame includes the binormal vector $\mathbf{B}$ and the torsion τ. The acceleration can be decomposed as

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} + a_B\mathbf{B},$$

where a_B accounts for the out-of-plane curvature. The core argument remains unchanged: the decomposition into Frenet components is intrinsic, while the global spatial orientation of the path must be recovered by extrinsic integration of frame evolution along the curve.

10 Conclusion

This work provides a complete intrinsic characterization of planar acceleration while distinguishing it from the extrinsic geometric orientation of the path. The resulting expression

$$\theta_a(t) = \Theta(t) + \phi(t)$$

gives a fully coordinate-free representation of the absolute acceleration direction in terms of curvature and the scalar kinematic rates of change. Beyond clarifying foundational concepts in planar kinematics, this framework can inform teaching, simulations, and research in both planar and spatial dynamics, providing a rigorous and intuitive tool for understanding motion in any reference frame.

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