# A Universal Recursive Analytical Solution of the Nonlinear Storage Equation for Flood Control Reservoir Sizing

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## Abstract

The nonlinear storage equation to arbitrary inflow rates is solved for the outflow time-series by recursively employing the analytical solution of the linear storage equation evaluated at each time-increment with an updated storage coefficient value. As the procedure is fully analytical it avoids any problems a numerical solution may present.

## Keywords

nonlinear storage equation, universal recursive analytical solution, discrete linear cascade model, recursion, flood control reservoir

## 1 Introduction

With the emergence of climate change, the occurrence of flashfloods has significantly increased across Europe ([1] – [2]). Flood mitigation by flood control reservoirs is a common practice in civil engineering ([3] – [8]). Among the different approaches to reservoir sizing (for a concise review, see [4]), one objective is common: finding a solution to the nonlinear storage equation (NLSE) in terms of the outflow rates provided inflow rates (in the form of a flood hydrograph or a complete time series of discharge values) and basic physical characteristics of the reservoir and control structure are known. In special cases, when the inflow hydrograph follows a certain well-defined shape, there exist analytical solutions to the NLSE (see [4] for a list of such studies), otherwise it is solved numerically. Numerical solutions however create their own problems in terms of stability, accuracy and time of execution.

In this study we offer a recursive approach to the NLSE made up of piece-wise analytical solutions ([9] – [14]) of the linear storage equation (LSE). The advantage of such an approach over a numerical one is that it is simple (no need for learning and meeting the requirements of a numerical solver), always stable, and its accuracy depends simply on the applied time increment, while the time of execution is a non-issue. It is hoped that such a solution will aid future domestic as well as international flood control reservoir sizing efforts as an adequate engineered response to increased flashflood risks reported globally.

## 2 Methods

Following the recent study of Pirone et al. [4], the outlet discharge ($Q$) of the flood control reservoir can be described by a permanent rating curve in the form of

$$Q(t) = c[h(t)]^n \; . \qquad (1)$$

Here $h$ is the water level (stage) in the reservoir relative to the spillway crest elevation, $t$ is a time reference, and $c$, $n$ are constants, determined by the outlet type and its geometry. For a rectangular spillway $n$ =1.5, $c = \mu L(2g)^{0.5}$ where $\mu$ is the discharge coefficient, $L$ the width of the spillway, and $g$ the gravitational acceleration [4]. The water volume ($S$) stored in the reservoir at any given time $t$ is given by the stage-storage relationship [4] as

$$S(t) = a[h(t)]^m \qquad (2)$$

where $a$ and $m$ are another constants, determined predominantly by the topography of the land in the vicinity and upstream of the spillway. $m$ has a typical range of 1 – 4.5 [4].

When Eqs. (1) and (2) are inserted into the lumped continuity (i.e., storage) equation

$$\frac{dS(t)}{dt} = I(t) - Q(t) \qquad (3)$$

where $I(t)$ is the inflow time series to the reservoir, one obtains an inhomogeneous, nonlinear, first-order ordinary differential equation [4] in the form of

$$\upsilon\kappa[Q(t)]^{\nu-1}\frac{dQ(t)}{dt} + Q(t) = I(t) \qquad (4)$$

where $\upsilon = m / n$ and $\kappa = a / c^{\upsilon}$. Eq. (4) when subjected to arbitrary inflows can generally be solved numerically as done below by the MATLAB (Mathworks Inc.) ‘ode45’ solver ([15] – [16]) for ‘verification’ of the recursive analytical solution described next.

Eq. (4) can be linearized over arbitrary $dt$ time-steps by treating the last value of $\upsilon\kappa[Q(t)]^{\upsilon-1}$ constant, $K$, for the $dt$ interval, which after rearrangement becomes

$$\frac{dQ(t)}{dt} = -kQ(t) + kI(t) \qquad (5)$$

where $k = K^{-1}$ now. Note that $K$ must have a dimension of time [T], therefore $k$ a dimension of [$T^{-1}$]. Eq. (5) over an interval of $dt$ is the linear storage equation [i.e., Eq. (3)] with storage coefficient $k$, so that $Q = kS$, but it is also the continuous, spatially discrete form of the linear kinematic wave equation

$$\frac{\partial Q(x,t)}{\partial t} + C\frac{\partial Q(x,t)}{\partial x} = 0 \qquad (6)$$

written for a single characteristic channel reach (of no lateral inflow) of length $\Delta x$ when employing a backward difference scheme ([9] – [11]). Here $x$ is the spatial coordinate along the flow direction and $C$ the constant wave celerity. Note that $k$ in Eq. (5) equals $C / \Delta x$ when derived from Eq. (6). When the outflow $Q$ is the inflow to the next characteristic reach, one obtains a cascade of characteristic reaches, or equivalently, a cascade of linear reservoirs. A temporally and spatially discrete solution of such a system of ordinary differential equations (made up of Eq. (5) for each element of the cascade) given in a state-space form was first published by Szöllősi-Nagy [9]. When applied for the linear reservoir it becomes [14]

$$Q(t+dt) = e^{-kdt}Q(t) + (1 - e^{-kdt})I(t) \qquad (7)$$

provided *I* measured at *t* is kept constant (this is the so-called classical pulsed data system framework [10]) over the *dt* interval. When *I* is let to change linearly from its value measured at *t* to the one measured at *t* + *dt*, Eq. (7) becomes [11]

$$Q(t+dt) = e^{-kdt}Q(t) + (1-e^{-kdt})\left[\frac{1}{kdt} - \frac{e^{-kdt}}{1-e^{-kdt}}\right]I(t) + (1-e^{-kdt})\left[1 - \frac{1}{kdt} + \frac{e^{-kdt}}{1-e^{-kdt}}\right]I(t+dt) \quad (8)$$

Eq. (8) reduces to Eq. (7) with the $I(t + dt) = I(t)$ choice. Eq. (7) is appropriate in a forecasting situation when the future value of the inflow rate is not yet known at the time of the forecast. It however is not the case for reservoir design purposes.

With the help of Eq. (8) the recursive solution of Eq. (4) becomes:

$t = 0$; $k = \{\upsilon\kappa[Q(t)]^{\upsilon-1}\}^{-1}$; $I(t) = 0$;

Start

$Q(t + dt)$ from Eq. (8); $k = \{\upsilon\kappa[Q(t + dt)]^{\upsilon-1}\}^{-1}$; $t = t + dt$;

Return

This simple recursive solution of the NLSE is compared to a numerical one obtained by MATLAB next.

## 3 Results and Discussion

For testing the above recursive solution, the following parameter values were prescribed: $c = 6$; $n = 1.5$; and $m$ taken from the range of 1 – 4.5 (Fig. 1).

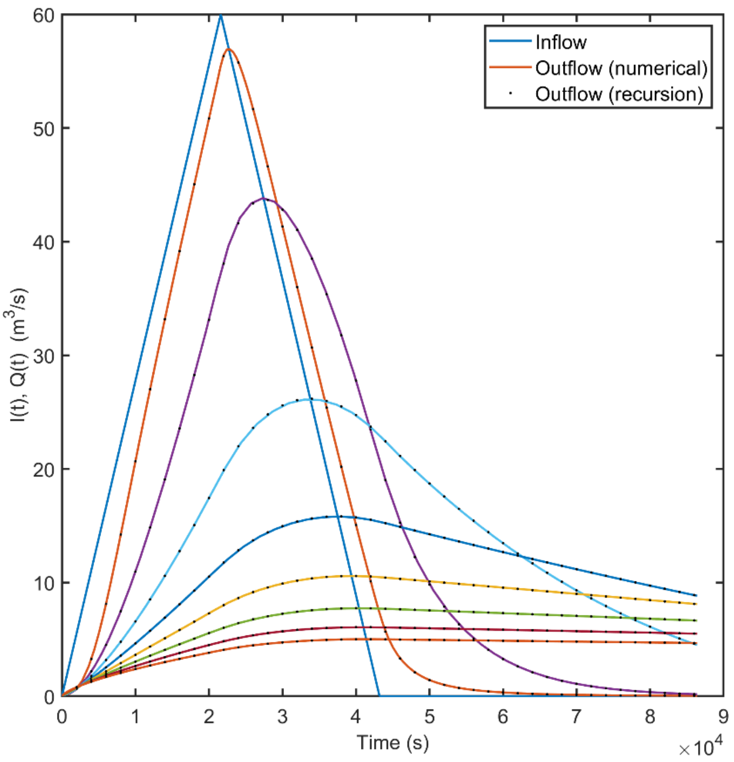


Figure 1. Comparison of the recursive solution [Eq. (8) plotted at 200·*dt* s time increments] of Eq. (4), to one obtained by numerical integration for different values of *m* incremented by 0.5. The largest peak outflow value

belongs to $m$ = 1 (representing a canyon with vertical walls), the smallest to $m$ = 4.5 (wide valley with gentle slopes), $a$ = 30,000. For either calculations inflow is specified at $dt$ increments of 10 s. $Q(0)$ = 0.1 $m^3/s$.

It is seen that the two solutions remain close together throughout the modeling period. Solutions of the NLSE are also displayed in Fig. 2 for a more realistic looking inflow hydrograph.

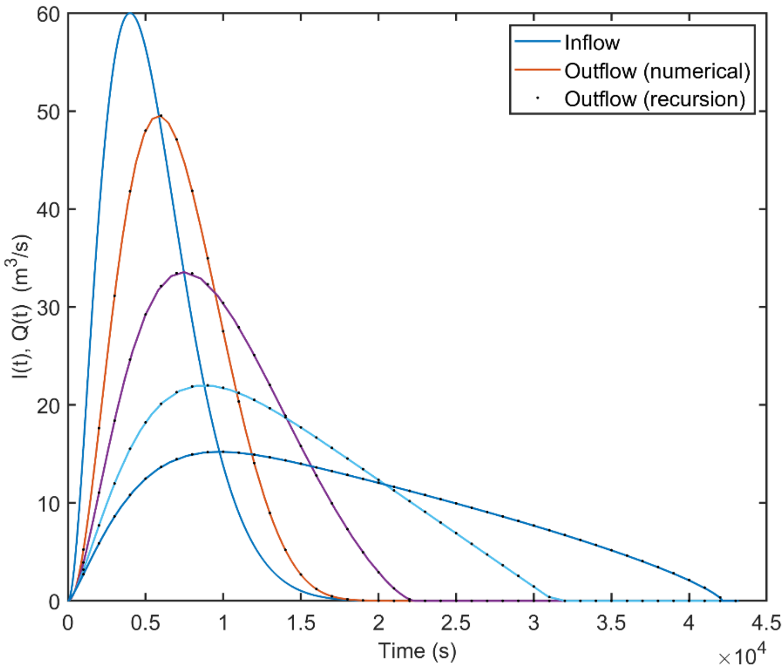


Figure 2. Comparison of the recursive solution [Eq. (8) plotted at 100·$dt$ s time increments] of Eq. (4) to one obtained by numerical integration for different values of $m$ incremented by 0.5. The largest peak outflow value belongs to $m$ = 2, while the smallest to $m$ = 3.5. Here $a$ = 5,000. $dt$ and $Q(0)$ are the same as in Fig. 1. The inflow hydrograph is defined [12] as: $I(t) = I_{max} e^{\sigma(D - t)}[1 - \cos(\omega t)] / [1 - \cos(\omega D)]$ with $I_{max}$ = 60 $m^3/s$; $D$ = 4,000 s; $\omega = 2\pi / T$ where $T$ = 21,600 s; and $\sigma = \omega \cot(\omega D / 2)$.

The numerical solver obtains complex values occasionally during its execution for three out of the four cases in Fig. 2, as well as it slows down considerably in its execution. Of course, this can never happen with the analytical recursion.

## 4 Conclusions

The proposed recursion employs piecewise (i.e., over $dt$ intervals) analytical solutions of the linear storage equation written in a state-space formulation and explicitly accounts for data sampling. The storage coefficient, $k$, is reevaluated in each $dt$ step of the calculations and kept constant over the $dt$ interval, making possible the application of the analytical solution of the linear storage equation. The method is simple, unconditionally stable, and naturally very fast as no iterations are involved characteristic of numerical integrations. Its simple structure and stability makes it an ideal tool for the inclusion of any kind of real-life flood control reservoir sizing procedure where the nonlinear storage equation is required to be solved repeatedly with trial values of the reservoir and/or control structure parameters. It is also recommended for teaching and/or demonstration purposes in institutes of higher education for civil/structural/infrastructural engineering where the students can spend more time on the engineering aspect of the problem rather than on the computational side of how to set up a numerical solver properly.


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## References


[1] Kahraman, A., Kendon, E. J., Chan, S. C., Fowler, H. J. "Quasi-stationary intense rainstorms spread across Europe under climate change". *Geophysical Research Letters*. 48, e2020GL092361. 2021. https://doi.org/10.1029/2020GL092361

[2] Fang, B., Bevacqua, E., Rakovec, O., Zscheischler, J. "An increase in the spatial extent of European floods over the last 70 years". *Hydrology and Earth System Science.* 28, pp. 3755–3775. 2024. https://doi.org/10.5194/hess-28-3755-2024

[3] Marone, V. "Calcolo di massima dell'effetto di laminazione di un serbatoio sulle piene". *L'Energia Elettr*. 46, pp. 1–8. 1964.

[4] Pirone, D., Cimorelli, L., D'Aniello, A., Pianese, D. "A novel generalized semi-analytical approach for flood control reservoir design". *Water Resources Research*. 61, e2024WR039368. 2025. https://doi.org/10.1029/2024WR039368

[5] Koris, K., Zsuffa, I. "*A víztározás hidrológiája*". Szakmérnöki jegyzet, M. 320. Mérnök-továbbképző Intézet. Budapest. 1965.

[6] Zsuffa, I. "Az árvízcsökkentő tározók hidrológiai kérdései". *Hidrológiai Közlöny*. 49(5), pp. 200-210. 1969.

[7] Kozák, M., Hamvas, F., Sárosi, L., Szalay, M. "*Vízépítési szerkezetek*". Tankönyvkiadó. Budapest. 1978.

[8] Hamvas, F. "*Vízépítési szerkezetek - Dombvidéki tározók*". Tankönyvkiadó. Budapest. 1981.

[9] Szöllősi-Nagy, A. ''The discretization of the continuous linear cascade by means of state space analysis''. *Journal of Hydrology*, 58, 223–236. 1982.

[10] Szilágyi, J., Szöllősi-Nagy, A. "*Recursive Streamflow Forecasting: A State-Space Approach*". Taylor & Francis, pp. 212. 2010. ISBN-13: 978-0-415-56901-9

[11] Szilágyi, J. “State-space discretization of the KMN-cascade in a sample-data system framework for streamflow forecasting”. *Journal of Hydrologic Engineering*. 8(6), pp. 339-347. 2003.

[12] Szilágyi, J. “Accounting for stream-aquifer interactions in the state-space discretization of the KMN-cascade for streamflow forecasting”. *Journal of Hydrologic Engineering*. 9(2), pp. 135-143. 2004.

[13] Szilágyi, J. “Discrete state-space approximation of the continuous Kalinin-Milyukov-Nash Cascade of noninteger storage elements”. *Journal of Hydrology*. 328, pp. 132-140. 2006.

[14] Szilágyi, J. “On the Clark unit hydrograph model of HEC-HMS”. *Periodica Polytechnica Civil Engineering*. 2017, 11,141. 2017. https://doi.org/10.3311/PPci.11141

[15] Dormand, J. R., Prince, P. J. “A family of embedded Runge-Kutta formulae”. *Journal of Computational and Applied Mathematics*, 6, pp. 19–26. 1980.

[16] Shampine, L. F., Reichelt, M. W. “The MATLAB ODE Suite”. *SIAM Journal on Scientific Computing*, 18, pp. 1–22. 1997.