Infinite Horizon Linear Optimal Control with Linear Constraints

by

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Abstract

We define infinite horizon linear optimal control problem with linear constraints. We provide a necessary condition for an optimal trajectory in terms of an infinite sequence of linear programming problems. We also provide a similar sufficient condition for optimality in terms of a related infinite sequence of linear programming problems. We provide a "robust" approximation result in terms of a linear programming problem with a sufficiently long time horizon and another approximation result in terms of a sequence of dual linear programming problems. We prove that the optimal value of the duals of the truncated linear programming problems with "fixed end-point" converge to the optimal value of the linear optimal control problem with linear constraints beginning from period 1 if and only if a weak transversality condition is satisfied. We prove that if there exists a solution satisfying "interiority condition" for an absolutely convergent linear optimal control problem with exactly one inequality constraint for the control variable in each period, then there is an "implied infinite dual linear programming problem" which has a solution and the optimal value of this dual is equal to the optimal value of the optimal control problem if and only if the same weak transversality condition is satisfied. In the general case of optimal control problems that allow more than one inequality constraint for the control variable in each period, the satisfaction of a boundedness condition for the optimal dual variables leads to the existence of an "implied infinite horizon dual linear programming problem" which has a solution and the optimal value of this dual is equal to the optimal value of the optimal control problem if and only if the weak transversality condition is satisfied.

Keywords: infinite horizon linear programming, linear optimal control, linear constraints, duality, interiority condition, boundedness condition, infinite horizon dual linear programming problem, transversality condition

AMS Subject Classification: 90C05, 90C46

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1. Introduction:

The earliest work on infinite horizon linear programming that is known to us is Hopkins (1969). This was followed by the work of Grinold (1971) and the 1973 Ph.d thesis of Joseph J. M. Evers, hereafter referred to as Evers (1973). Notable contributions in this area of research, that have been reported thereafter, are Grinold (1977), Evers (1973), Romeijn, Smith and Bean (1992), Romeijn and Smith (1998). We pursue a similar line of work in the present paper, a fairly good motivation for which is chapter 2 of Evers (1973).

In Lahiri (2025c) there is a discussion of infinite horizon linear optimal control, with one state variable and one control variable. In that model, the objective function is assumed to be linear, without such restrictions being imposed on the constraints. In this paper, we assume that that there is one state variable and one control variable with the dynamics of the state variable determined by a first order linear difference equation and the constraint set for the control variable is bounded above by a finite set of affine functions of the state variable. Thus, the upper bound function for the available values for the control variable in any time period is a continuous, piece-wise affine and concave function on the closed interval from which the value of the state variable during the same period can be conceivably chosen.

In section 2 we present the framework of analysis in which both state variable and control variable are restricted to belong to the same closed and bounded interval of real numbers containing zero. We define infinite horizon linear optimal control problem with linear constraints - hereafter referred to as linear optimal control problem with linear constraints (LOC-LC problem)- as well as a convenient special case case in which the there is only one affine function of the state variable that is an upper bound for the control variable. In section 3, we introduce the concept of optimality and provide a necessary condition for an optimal trajectory in terms of an infinite sequence of linear programming problems. In the same section, we also provide a similar sufficient condition for optimality in terms of a related infinite sequence of linear programming problems. Also included in this section, is a proposition that provides a set of sufficient conditions under which all feasible trajectories are optimal trajectories.

In section 4, we introduce absolutely convergent linear optimal control problems with linear constraints (AC-LOC-LC problems) for which it is well known that the set of optimal solutions is always non-empty. We define a "bang-bang sequence of decision rules", and provide sufficient conditions for the existence of a unique optimal trajectory that is generated by such a sequence of decision rules. We also provide a "robust" approximation result (proposition 4.2) in terms of truncated free end-point linear programming problems with a sufficiently long time horizon. This result is quite plausible and very likely well known in the existing literature on infinite horizon linear programming. The approximation result has a corollary that provides a set of necessary conditions for optimality in terms of the sequence solutions of the dual linear programs corresponding to the truncated free end-point linear programming problems that approximate the infinite horizon linear programming problem.

In addition to proposition 4.2, somewhat significant results are discussed in sections sections 5, 6 and 7 of this paper. In section 5, our first main result is a proposition that states that the "*optimal value of the dual*" of the truncated "fixed end-point" linear programming problem with the initial and terminal values of the state variable in the latter being the same as the initial and terminal values of the state variable in the optimal trajectory , converges to the optimal value of the AC-LOC-LC problem beginning from period 1 if and only if a "weak transversality" condition is satisfied. A more striking result is the one in section 6, about the existence of a solution for the "implied infinite horizon dual linear programming problem" of an AC-LOC-LC problem "*with one inequality constraint for the control variable*". Under the assumptions that the value of the state variable is strictly positive beginning with the first time period, the value of the control variable is strictly positive beginning with the intial time period, the value of the control variable is strictly less than its upper bound in "*at least one*" time period and the difference equation for the state variable has a non-zero co-efficient for the control variable, we prove that a necessary and sufficient condition for the existence of a solution for the "implied infinite horizon dual linear programming problem", is the satisfaction of the "weak transversality condition". We refer to the property of a trajectory requiring the state variable to be positive from the first time period, the control variable to be positive from the initial period, the control variable to be strictly less than its upper bound in at least one time period and the difference equation for the state variable has a non-zero co-efficient for the control variable, as "interiority condition". Thus, if there exists an optimal trajectory satisfying "interiority condition", then the "implied infinite horizon dual linear programming problem" has a solution. Further, the optimal value of the infinite horizon dual linear programming problem, is "equal" to the optimal value of the absolutely continuous optimal control problem with linear constraints. However, we prove this result in the context of AC-LOC-LC problems that have only one inequality condition for the control variable.

In section 7, we prove a similar result in the general case of a AC-LOC- LC problem with one or more inequality constraints for its control variables. We prove that if a AC-LOC- LC problem satisfies a "boundedness condition" for its sequence of optimal dual variables then its "implied infinite horizon dual linear programming problem" has a solution and the optimal value of the infinite horizon dual linear programming problem, is "equal" to the optimal value of the absolutely continuous optimal control problem with linear constraints.

Sections 6 and 7 are concerned with what is known as the "duality gap problem". Duality gap is about the possibility of the optimal value of an infinite horizon linear programming problem being different from the optimal value of its infinite horizon dual linear programming problem. In linear programming, primal and dual linear programming problems are defined in the context of finite number of unknown variables. Hence, we refer to the dual linear programming problem in the infinite horizon context as "implied infinite horizon dual linear programming problem".

There is one important difference between infinite "horizon" linear programming and infinite linear programming that ought to be taken note of. Unlike infinite linear programming that can accommodate both optimization with infinite number of variables and a finite number of constraints and optimization with a finite number of variables and infinitely many constraints, there is no possibility of meaningfully accommodating such "half-way houses" in infinite horizon linear programming.

Whether, that makes infinite horizon linear programming restrictive or more useful than infinite dimensional optimization is irrelevant, since chapter 1 of Evers (1973) discusses four “growth models” which can be modeled as infinite horizon linear programming problems.

Our approach to infinite horizon linear programming via truncated and finite horizon versions of the original problem is akin to the approach adopted in the papers that we cited in the first paragraph of this section. In a concluding section of this paper, we compare our model with these other notable infinite horizon linear programming models. While conceding that unlike our model, these other models allow multi-dimensional variables, we point out that our model is conceptually more general, since it explicitly distinguishes between control variable and state variable with each evolving in its own distinct way, although each of the two are assumed to be one-dimensional in every time period.

Note 1.1: This paper is a generalization of another paper entitled “Infinite Horizon Linear Programming With One-Dimensional Control Variable”. DOI: https://doi.org/10.31224/6208

2. Framework of Analysis:

Let $X = [0, b] \subset \mathbb{R}$ (the set of real numbers), with $b > 0$ be such that **the set of available alternatives** at any time period is a non-empty subset of $X \times X$. Given a current realization $x \in X$ of the state variable that was chosen in the immediately previous time-period, a typical alternative that is “realized” during the current period is an ordered pair $(u, y) \in X \times X$, where ‘u’ is the value of the control variable chosen for the current period and this ‘u’ along with ‘x’, determines ‘y’ as the value of the state variable for the immediately next period. Based on the pair (x, u) during the current period an instantaneous pay-off is realized by the decision maker.

With $\mathbb{N}$ denoting the set of natural number (i.e., the set of strictly positive integers) let $\mathbb{N}^0$ denote $\mathbb{N} \cup \{0\}$, i.e., the set of non-negative integers. Time is measured in discrete periods $t \in \mathbb{N}^0$. Beginning with an initial value of the state variable, in each period $t \in \mathbb{N}^0$, an alternative (state variable-control variable pair) is realized, and the chosen alternative is denoted by $(x_t, u_t) \in X \times X$. While at all time periods x_t is an “inheritance” in the current period, u_t is chosen during the current period.

At each time-period $t\in \mathbb{N}^0$, $\Omega_t \subset X\times X\times X$ is the **two-period constraint set at time-period** t, such that for all $t\in \mathbb{N}^0$, there exists a matrix $[A^{(t)} \mid B^{(t)}]$ of dimension $2\times m(t)$ for some positive integer m(t) with I(t) denoting the set of first m(t) positive integers, and a point $(c^{(t)}, d^{(t)}, e^{(t)})\in \mathbb{R}^3$ satisfying the following properties:

(i) For all $t\in \mathbb{N}^0$ and $i\in I(t)$, $A_i^{(t)} \geq 0$ and $A_i^{(t)} + B_i^{(t)}b \geq 0$, where the ordered pair $(A_i^t|B_i^t)$ is the i^{th} row of the matrix $[A^t \mid B^t]$. Further, $\{m_t|t\in\mathbb{N}^0\}$ is a bounded set of positive integers and $\{M_t|t\in\mathbb{N}^0\}$ is a bounded set of real numbers, where for all $t\in\mathbb{N}^0$, $M_t = \max\{\max\{A_i^t|i\in I(t)\}, \max\{A_i^t + B_i^t b|i\in I(t)\}\}$.

(ii) For all $t\in \mathbb{N}^0$ and $(x, u)\in X\times X$, $c^{(t)} + d^{(t)}x + e^{(t)}u \in [0, b]$.

(iii) For all $t\in \mathbb{N}^0$, $\Omega_t = \{(x, u, y)\in X\times X\times X|\ u \leq A_i^{(t)} + B_i^{(t)}x$ for all $i\in I(t)$, and $y = c^{(t)} + d^{(t)}x + e^{(t)}u\}$.

For $t\in \mathbb{N}^0$, $(x, u, y)\in \Omega_t$ can be interpreted in the following manner: given that $x\in X$ is the realization of the state variable at time-period t, it is possible to choose the pair $(u, y)\in X\times X$ at time-period t.

Further, the boundedness assumption on $\{m_t|t\in\mathbb{N}^0\}$ is included to avoid needless complications related to the dimensions of the variables in the "dual linear programming problems" particularly in sections 6 and 7.

Thus, there exists a natural number $m\in\{m_t|t\in\mathbb{N}^0\}$, such that $m \leq m_t$ for all $t\in\mathbb{N}^0$.

By (i), $A_i^{(t)} + B_i^{(t)}x \geq 0$ for all $(x, t)\in X\times\mathbb{N}^0$ and $i\in I(t)$.

For all $(x, t)\in X\times\mathbb{N}^0$, let $\Omega_t(x) = \{(u, y)\in X\times X|\ (x, u, y)\in \Omega_t\} = \{(u, c^{(t)} + d^{(t)}x + e^{(t)}u)|\ u \geq 0, u \leq \min\{b, \min\{A_i^{(t)} + B_i^{(t)}x|i\in I(t)\}\}\}$.

Thus, by (i) and (ii) it follows that for all $(x, t)\in X\times\mathbb{N}^0$, $\Omega_t(x) \neq \phi$ and hence for all $t\in \mathbb{N}^0$, $\Omega_t \neq \phi$.

Clearly, for all $(x, t)\in X\times\mathbb{N}^0$, $\Omega_t(x)$ is a non-empty and closed subset of $X\times X$.

For $x\in X$, let $\mathcal{F}(x) = \{<(x_t, u_t)|t\in \mathbb{N}^0>|\ (x_t, u_t, x_{t+1})\in \Omega_t, t\in \mathbb{N}^0, x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $<(x_t, u_t)|t\in \mathbb{N}^0 > \in \mathcal{F}(x)$ as a **trajectory starting at (from)** x and to $\mathcal{F}(x)$ as **the set of trajectories starting at (from)** x.

Clearly, $\mathcal{F}(x)$ is non-empty for all $x\in X$.

An *alternative representation* of **the set of trajectories starting at (from)** x could be the following: $\mathcal{F}^*(x) = \{<(x_t, u_{t-1})|t\in \mathbb{N}>| (x_{t-1}, u_{t-1}, x_t)\in \Omega_{t-1}, t\in \mathbb{N}, x_0 = x\}$.

Let $<(p_1^{(t)}, p_2^{(t)}) \mid t\in \mathbb{N}^0>$ be a sequence of pairs of real numbers. If x is the realization of the state variable at time-period t and u is the choice of the control variable at time period ‘t’, the instantaneous pay-off received by the decision-maker at time period ‘t’ is $p_1^{(t)}x + p_2^{(t)}u$.

We shall refer to the array $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)| t\in \mathbb{N}^0>$ or alternatively $<((p_1^{(t)}, p_2^{(t)}), [A^{(t)} | B^{(t)}], (c^{(t)},d^{(t)},e^{(t)}))| t\in \mathbb{N}^0>$ as a **(infinite horizon) linear optimal control problem with linear constraints (LOC-LC problem).**

Since we have assumed in our definition of an LOC-LC problem $<((p_1^{(t)}, p_2^{(t)}), [A^{(t)}| B^{(t)}], (c^{(t)},d^{(t)}, e^{(t)}))| t\in \mathbb{N}^0>$ that $\{M_t|t\in\mathbb{N}^0\}$ is a bounded set of real numbers, where for all $t\in\mathbb{N}^0$, $M_t = \max\{\max\{A_i^{(t)}|i\in I(t)\}, \max\{A_i^{(t)} + B_i^{(t)}b|i\in I(t)\}\}$, if whenever $M_t > b$, we replace $A_i^{(t)}$ by $(\frac{b}{M_t})A_i^{(t)}$, $B_i^{(t)}$ by $(\frac{b}{M_t})B_i^{(t)}$ for all $i\in I(t)$, $e^{(t)}$ by $(\frac{M_t}{b})e^{(t)}$ and $p_2^{(t)}$ by $(\frac{M_t}{b})\, p_2^{(t)}$, and otherwise let the parameters and constraints remain the same as in the original problem, then we get an LOC-LC problem $<((p_1^{'(t)}, p_2^{'(t)}), [A^{'(t)} | B^{'(t)}], (c^{'(t)},d^{'(t)}, e^{'(t)}))| t\in \mathbb{N}^0>$ such that for all $x\in X$, $<(x_t, u_t)|t\in \mathbb{N}^0>$ and $<(x_t, v_t)|t\in \mathbb{N}^0 >$ with $v_t = u_t$ if $M_t \leq b$, $v_t = (\frac{b}{M_t})u_t$ if $M_t > b$:

(1) $v_t \in[0, b]$ for all $t\in \mathbb{N}^0$.

(2) $<(x_t, u_t)|t\in \mathbb{N}^0 >\in \mathcal{F}(x)$ if and only if $<(x_t, v_t)|t\in \mathbb{N}^0 >$ satisfies $x_{t+1} = c^{'(t)} + d^{'(t)}x_t + e^{'(t)}v_t$ for all $t\in \mathbb{N}^0$, $v_t \leq \min\{A_i^{'(t)} + B_i^{'(t)}x_t|i\in I(t)\}$.

(3) $\sum_{t=0}^{\infty} [p_1^{(t)} x_t+ p_2^{(t)} u_t] < +\infty$ <u>if and only if</u> $\sum_{t=0}^{\infty} [p_1^{'(t)} x_t+ p_2^{'(t)} v_t] < +\infty$, in which case $\sum_{t=0}^{\infty} [p_1^{(t)} x_t+ p_2^{(t)} u_t] = \sum_{t=0}^{\infty} [p_1^{'(t)} x_t+ p_2^{'(t)} v_t]$.

Hence, without any loss of generality we may assume that $M_t \leq b$ for all $t \epsilon \mathbb{N}^0$.

Further, we have assumed that there exists there exists a natural number $m \epsilon \{m_t | t \epsilon \mathbb{N}^0\}$, such that $m \leq m_t$ for all $t \epsilon \mathbb{N}^0$.

Hence, if for some $t \epsilon \mathbb{N}^0$, it is the case that m(t) < m, then we can append to the m(t) inequalities of the form $u_t \leq A_i^{'(t)} + B_i^{'(t)} x_t$, $i \epsilon I(t)$ a set of m-m(t) inequalities of the form $u_t \leq b + 0x_t$, without affecting the LOC-LC problem or its outcomes in any manner.

Thus, for what follows in this work, we may with loss of generality assume the following:

Assumption on Control Variable Constraints: (i) $M_t \leq b$ for all $t \epsilon \mathbb{N}^0$. (ii) $m_t = m$ for all $t \epsilon \mathbb{N}^0$, so that $I(t) = I = \{1, \ldots, m\}$ for all $t \epsilon \mathbb{N}^0$.

A special case of a LOC-LC problem is one when at each time period there is a single upper-bound on the control variable, i.e., m = 1.

Suppose I= {1} for all $t \epsilon \mathbb{N}^0$ and for all $t \epsilon \mathbb{N}^0$, there exists real numbers $a^{(t)}$, $b^{(t)}$ such that $a^{(t)} + b^{(t)} x \epsilon [0, b]$ for all $x \epsilon [0, b]$.

For each $t \epsilon \mathbb{N}^0$, let $\Omega_t = \{(u, x, y) \epsilon X \times X \times X | \; u \leq a^{(t)} + b^{(t)} x \text{ and } y = c^{(t)} + d^{(t)} x + e^{(t)} u\}$.

In this case the array $<((p_1^{(t)}, p_2^{(t)}), \Omega_t) | \; t \in \mathbb{N}^0>$ may be represented as $<((p_1^{(t)}, p_2^{(t)}), (a^{(t)}, b^{(t)}), (c^{(t)}, d^{(t)}, e^{(t)})) | \; t \in \mathbb{N}^0>$ and referred to as a **one-plus-one linear optimal control problem with linear constraints ((1+1) -LOC-LC problem).**

Note 2.1: If for some $t \in \mathbb{N}^0$, $e^t \neq 0$, then for $(x, y, u) \epsilon \Omega_t$, it must be the case that $u = \frac{y - c^{(t)} - d^{(t)} x}{e_t}$, so that $p_1^{(t)} x + p_2^{(t)} u = p_1^{(t)} x + p_2^{(t)} (\frac{y - c^{(t)} - d^{(t)} x}{e_t})$. Thus, the instantaneous pay-off at time t, can be expressed entirely in terms of the state variables. On the other hand, if for some $t \in \mathbb{N}^0$, $e^{(t)} = 0$, then for a given value x of the state variable at time period t, there may be several values of the control variable that can chosen at time-period t. Thus, the model of linear optimal control discussed here is a generalization of the linear dynamic optimization model discussed in Lahiri (2025a, 2025b), the latter being motivated by the reduced form model in Mitra (2000) and Sorger (2015).

u ≤ $A_i^{(t)}$ + $B_i^{(t)}$x for every i∈I, y = $c^{(t)}$ + $d^{(t)}$x + $e^{(t)}$u and $d^{(t)} \neq 0$ implies u ≤ $A_i^{(t)}$ + $B_i^{(t)}(\frac{y - c^{(t)} - e^{(t)}u}{d^{(t)}})$ for every i∈I, i.e., u ≤ $\alpha_i^{(t)}$ + $\beta_i^{(t)}$y for every i∈I, satisfying $1 - B_i^{(t)}\frac{e^{(t)}}{d^{(t)}} >$ 0, where for every i∈I, $\alpha_i^{(t)} = \frac{A_i^{(t)} - B_i^{(t)}\frac{c^{(t)}}{d^{(t)}}}{1 - B_i^{(t)}\frac{e^{(t)}}{d^{(t)}}}$ and $\beta_i^{(t)} = \frac{\frac{B_i^{(t)}}{d^{(t)}}}{1 - B_i^t\frac{e^t}{d^t}}$. However, since y = $c^{(t)}$ + $d^{(t)}$x + $e^{(t)}$u, the chosen pair (u, y) depends on x. On the other hand, if $d^{(t)} = 0$, then y = $c^{(t)}$ + $e^{(t)}$u, and so x does not play any role in determining y. None the less, x figures in the upper bound min{$A_i^{(t)}$ +$B_i^{(t)}$x|i∈I} on the choice of u.

In the case of a LOC-LC problem it is easy to see that, for all t∈ $\mathbb{N}^0$, Ω_t is a non-empty, closed and "convex" subset of X×X×X. Thus, in the case of a LOC-LC problem, for all x∈X, $\mathcal{F}$(x) is a non-empty and convex set.

If $e^{(t)} = 0$ for all t∈ $\mathbb{N}^0$, then for all x∈X: <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈$\mathcal{F}$(x) <u>if and only if</u> $u_t \leq a^{(t)} + b^{(t)}x_t$ and $x_{t+1} = c^{(t)} + d^{(t)}x_t$ for all t∈ $\mathbb{N}^0$.

Note 2.2: If we restrict the state and control variables to be one dimensional, then the framework of analysis in Evers (1973)"suitably adapted to the one we are concerned with" would require: I = {1}, $A_1^{(t)}$ = b, $B_1^{(t)}$ = 0, $d^{(t)} = d^{(t+1)}$, $e^{(t)} = e^{(t+1)} \neq 0$, $p_2^{(t)} = 0$ for all t∈$\mathbb{N}^0$. Such a formulation may be referred to as the "**Evers model**" in our framework. This model is an example of a (1+1)-LOC-LC problem.

3. Optimality and associated linear programming problems:

We will now consider the following optimization problem denoted **OPT**:

Given x∈X, Maximize $\sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$ subject to the infinite sequence satisfying the constraints: (x_t, u_t, x_{t+1})∈ Ω_t, t∈ $\mathbb{N}^0$, x_0 = x.

Note 3.1: The exact mathematical interpretation of the expression (formula) $\sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$ is $\lim_{T\to\infty} (\sum_{t=0}^{T} [p_1^{(t)} x_t + p_2^{(t)} u_t])$. Thus, the problem we are concerned with here is in the domain of asymptotic analysis, which is very different from infinite dimensional analysis.

An **alternative and very useful version of OPT** is the following optimization problem:

Given x∈X, Maximize $\sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]$ subject to the infinite sequence satisfying the constraints: $(x_t, u_t, x_{t+1}) \in \Omega_t$, $t \in \mathbb{N}^0$, $x_0 = x$.

We will refer to this alternative version of OPT as **Alt-OPT**.

Let $\mathcal{S}(x) = \{<x_t|t\in\mathbb{N}^0> \in \mathcal{F}(x) | \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] \geq \sum_{t=0}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]$ for all $<(y_t, v_t)|t\in\mathbb{N}^0> \in \mathcal{F}(x)\}$, i.e., $\mathcal{S}(x) = \underset{<(x_t,u_t)|\,t\in\mathbb{N}^0>\in\mathcal{F}(x)}{\operatorname{argmax}} \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$ and $\mathcal{S}^*(x) = \underset{<(x_t,u_{t-1})|t\in\mathbb{N}>\in\mathcal{F}^*(x)}{\operatorname{argmax}} \sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]$.

$\mathcal{S}(x)$ is the **set of solutions starting from** x **for** OPT and $\mathcal{S}^*(x)$ is the **set of solutions starting from** x **for** Alt-OPT.

For all $T\in\mathbb{N}^0$, and x∈X, let $\mathcal{F}^{(T)}(x) = \{<(x_t, u_t)\,|t \geq T> | (x_t, u_t, x_{t+1}) \in \Omega_t$ for all $t \geq T$ and $x_T = x\}$.

For $T\in\mathbb{N}^0$ and x∈X, $<(x_t, u_t)|t \geq T> \in \mathcal{F}^{(T)}(x)$ may be referred to as a **trajectory starting at (from)** x **at time-period** T and to $\mathcal{F}^{(T)}(x)$ as **the set of trajectories starting at (from)** x **at time-period** T.

It is easy to see that for all $T\in\mathbb{N}^0$ and x∈X, $\mathcal{F}^{(T)}(x)$ is non-empty.

Given $(x, T) \in X\times\mathbb{N}^0$, we will denote the following optimization problem by **OPT-T**:

Maximize $\sum_{t=T}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]$ subject to $<(y_t, v_t)|\, t \geq T> \in \mathcal{F}^{(T)}(x)$.

For $(x, T) \in X\times\mathbb{N}^0$, $\mathcal{S}^{(T)}(x)$ is the **set of solutions starting from** x **for** OPT-T, i.e.,

$\mathcal{S}^{(T)}(x) = \underset{<(x_t,u_t)|\,t\geq T>\in\mathcal{F}^{(T)}(x)}{\operatorname{argmax}} \sum_{t=T}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$.

Clearly, $\mathcal{F}^{(0)}(x) = \mathcal{F}(x)$ and $\mathcal{S}^{(0)}(x) = \mathcal{S}(x)$ for all x∈X.

Given $(x, T) \in X\times\mathbb{N}^0$, an *alternative version* of **the set of trajectories starting at (from)** x **at time-period** T denoted by $\mathcal{F}^{(T)*}(x) = \{<(x_t, u_{t-1})|t \geq T+1> | <(x_t, u_t)|t \geq T> \in \mathcal{F}^{(T)}(x)\}$.

Clearly, for all $(x, T) \in X\times\mathbb{N}^0$: $<(x_t, u_t)|t \geq T> \in \mathcal{F}^{(T)}(x)$ <u>if and only if</u> $<(x_t, u_{t-1})|t \geq T + 1> \in \mathcal{F}^{T*}(x)$.

Further, for (x, T)$\in$ X×$\mathbb{N}^0$ and <(x_t, u_t)|t ≥T>$\in$ $\mathcal{F}^{(T)}$(x) satisfying $|\sum_{t=T}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t]|$ $< +\infty$, $\sum_{t=T+1}^{\infty}[p_1^{(t)}x_t+p_2^{(t-1)}u_{t-1}] = \sum_{t=T}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t] - p_1^{(T)}x_T$ and $|\sum_{t=T+1}^{\infty}[p_1^{(t)}x_t+p_2^{(t-1)}u_{t-1}]| < +\infty$.

An alternative version of OPT-T is the following problem denoted by **Alt-OPT-T**:

Given (x, T)$\in$ X×$\mathbb{N}^0$: Maximize $\sum_{t=T+1}^{\infty}[p_1^{(t)}y_t+ p_2^{(t-1)}v_{t-1}]$ subject to $<(y_t, v_{t-1})| \ t \geq T+1> \in \mathcal{F}^{(T)*}(x)$.

For (x, T)$\in$ X×$\mathbb{N}^0$, $\mathcal{S}^{T*}$(x) is the **set of solutions starting from** x **for** Alt-OPT-T, i.e.,

$$\mathcal{S}^{T*}(x) = \underset{<(x_t,u_{t-1})|\ t\geq T+1>\in\mathcal{F}^{T*}(x)}{\operatorname{argmax}} \sum_{t=T+1}^{\infty}[p_1^{(t)}x_t+p_2^{(t-1)}u_{t-1}].$$

Thus, $\mathcal{S}^{T*}(x) = \{<(x_t, u_{t-1})|t \geq T+1> \in \mathcal{F}^{(T)*}(x)|<(x_t, u_t)|t \geq T> \in\mathcal{S}^{(T)}(x)\}$.

Clearly, $\mathcal{F}^0(x) = \mathcal{F}(x)$, $\mathcal{S}^0(x) = \mathcal{S}(x)$, $\mathcal{F}^{0*}(x) = \mathcal{F}^*(x)$ and $\mathcal{S}^{0*}(x) = \mathcal{S}^*(x)$ for all x$\in$ X.

For all T$\in$ $\mathbb{N}^0$ the correspondence $h^{(T)}$: X →→X×X defined by $h^{(T)}(x) = \{(u_T, x_{T+1})|<(x_t,u_{t-1})|\ t\geq T+1>\in\mathcal{S}^{T*}(x)$ is said to be the **optimal period-T decision rule**.

We now provide one necessary condition and a somewhat stronger sufficient condition for optimality for an LOC-LC problem in terms of linear programming problems.

Proposition 3.1: Let $<((p_1^{(t)}, p_2^{(t)}), [A^{(t)} \mid B^{(t)}], (c^{(t)},d^{(t)},e^{(t)}))| \ t\in \mathbb{N}^0>$ be an LOC-LC problem and suppose that for some x$\in$X, <(x_t, u_{t-1})|t$\in$ $\mathbb{N}$>$\in$ $\mathcal{F}^*$(x).

Part 1 (The Fixed End-Point Problem): <u>If</u> <(x_t, u_{t-1})|t$\in$ $\mathbb{N}$>$\in$ $\mathcal{S}^*$(x) <u>then</u> for all T$\in$ $\mathbb{N}$ with T ≥ 3, <(x_t, u_{t-1})|t = 1, …, T> solves the following linear programming problem: Maximize $\sum_{t=1}^{T-1}[p_1^{(t)}y_t+p_2^{(t-1)}v_{t-1}] +p_2^{(T-1)}v_{T-1}$, subject to $v_t \leq A_i^{(t)} + B_i^{(t)}y_t$ for every i$\in$I and t = 0, 1, …, T-1, $y_{t+1} = c^{(t)} + d^{(t)}y_t + e^{(t)}v_t$ for all t = 0, …, T-2, $x_T = c^{(T-1)} + d^{(T-1)}y_{T-1} + e^{(T-1)}u_{T-1}$, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T-1.

Part 2 (The Free End-Point Problem): <u>If</u> there exists T^* $\in$ $\mathbb{N}$ such that for all T$\in$ $\mathbb{N}$ satisfying T ≥ T^*, <(x_t, u_t)|t = 0, 1, …, T> solves the following linear programming problem: Maximize $\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t]$, subject to $v_t \leq A_i^{(t)} + B_i^{(t)}y_t$ for every i$\in$I and t

$= 0, 1, \ldots, T$, $y_{t+1} = c^{(t)} + d^{(t)}y_t + e^{(t)}v_t$ for all $t = 0, \ldots, T-1$, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all $t = 0, 1, \ldots, T$, <u>then</u> $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{S}(x)$.

Proof: Part 1: Suppose $<(x_t, u_{t-1})|t \in \mathbb{N}> \in \mathcal{S}^*(x)$ and towards a contradiction suppose that for some $T \in \mathbb{N}$ with $T \geq 3$ there exists $<(y_t, v_{t-1})|t = 1, \ldots, T>$ such that

(i) $v_t \leq A_i^{(t)} + B_i^{(t)}y_t$ for every $i \in I$ and $t = 0, 1, \ldots, T-1$,

(ii) $y_{t+1} = c^{(t)} + d^{(t)}y_t + e^{(t)}v_t$ for all $t = 0, \ldots, T-2$, $x_T = c^{(T-1)} + d^{(T-1)}y_{T-1} + e^{(T-1)}u_{T-1}$,

(iii) $y_0 = x_0 = x$, $y_T = x_T$,

(iv) $y_t \geq 0$ for all $t = 1, \ldots, T-1$, $v_t \geq 0$ for all $t = 0, 1, \ldots, T-1$, and

(v) $\sum_{t=1}^{T-1} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] + p_2^{(T-1)} v_{T-1} > \sum_{t=1}^{T-1} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}] + p_2^{(T-1)} u_{T-1}$.

Let $<(z_t, w_t)|t \in \mathbb{N}^0>$ be such that that $(z_t, w_t) = (y_t, v_t)$ for all $t = 0, 1, \ldots, T-1$, $(z_t, w_t) = (x_t, u_t)$ for all $t \geq T$.

Since $y_0 = x_0 = x$, $y_T = x_T$, $v_T = u_T$ and $x_T = c^{(T-1)} + d^{(T-1)}y_{T-1} + e^{(T-1)}u_{T-1}$, it is easily verified that $<(z_t, w_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$.

Thus, $\sum_{t=1}^{\infty} [p_1^{(t)} z_t + p_2^{(t-1)} w_{t-1}] = \sum_{t=1}^{T-1} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] + p_2^{(T-1)} v_{T-1} + p_1^{(T)} x_T$
$\sum_{t=T+1}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_{t-1}] > \sum_{t=1}^{T-1} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}] + p_2^{(T-1)} u_{T-1} + p_1^{(T)} x_T +$
$\sum_{t=T+1}^{\infty} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}] = \sum_{t=1}^{\infty} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}]$.

This contradicts, $<(x_t, u_{t-1})|t \in \mathbb{N}> \in \mathcal{S}^*(x)$ and proves Part 1.

Part 2: Suppose that there exists $T^* \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ satisfying $T \geq T^*$, $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves the linear programming problem: Maximize $\sum_{t=0}^{T} [p_1^{(t)} y_t + p_2^{(t)} v_t]$, subject to $v_t \leq A_i^{(t)} + B_i^{(t)}y_t$ for every $i \in I$ and $t = 0, 1, \ldots, T$, $y_{t+1} = c^{(t)} + d^{(t)}y_t + e^{(t)}v_t$ for all $t = 0, \ldots, T-1$, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all $t = 0, 1, \ldots, T$. Towards a contradiction suppose that $<(x_t, u_t)|t \in \mathbb{N}^0> \notin S(x)$.

Thus, there exists $<(y_t, v_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$ such that $\sum_{t=0}^{\infty} [p_1^{(t)} y_t + p_2^{(t)} v_t] >$ $\sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t]$.

By note 3.1, $\lim_{T\to\infty} \sum_{t=0}^{T} [p_1^{(t)} y_t + p_2^{(t)} v_t] = \sum_{t=0}^{\infty} [p_1^{(t)} y_t + p_2^{(t)} v_t] > \sum_{t=0}^{\infty} [p_1^{(t)} x_t + p_2^{(t)} u_t] = \lim_{T\to\infty} \sum_{t=0}^{T} [p_1^{(t)} x_t + p_2^{(t)} u_t]$.

Thus, there exists $T^0 \in \mathbb{N}$, such that for all $T \in \mathbb{N}$ satisfying $T \geq T^0$, it must be the case that $\sum_{t=0}^{T} [p_1^{(t)} y_t + p_2^{(t)} v_t] > \sum_{t=0}^{T} [p_1^{(t)} x_t + p_2^{(t)} u_t]$.

Let $T^{**} = \max\{T^*, T^0\}$.

Since $<(y_t, v_t)|t \in \mathbb{N}^0> \in \mathcal{F}(x)$, $v_t \leq A_i^{(t)} + B_i^{(t)} y_t$ for every $i \in I$ and $t = 0, 1, \ldots, T^{**}$, $y_{t+1} = c^{(t)} + d^{(t)} y_t + e^{(t)} v_t$ for all $t = 0, \ldots, T^{**}-1$, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all $t = 0, 1, \ldots, T^{**}$.

This contradicts our assumption that $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves the linear programming problems in the statement of Part 2 of this proposition and thus proves Part 2. Q.E.D.

Note 3.2: In one way the sufficient condition is stronger than the necessary condition because the constraints on the terminal values of the state and control variables in the linear programming problem in the necessary condition (i.e., in part 1) are "absent" from the constraints in the linear programming problem in the sufficient condition (i.e., in part 2). Since, the evolution of the state variable is determined by an equation, if the terminal period for linear programming problem in part 1, is T, then the value of the state variable in period T + 1 has to be x_{T+1} and this also determines the constraints for the control variable that needs to be chosen in period T+1. Further, the requirement in part 1 that $x_T = y_T = c^{(T-1)} + d^{(T-1)} y_{T-1} + e^{(T-1)} v_{T-1}$, does impose "some restriction" on the choice of y_{T-1}. None of these restrictions apply for the linear programming problem in part 2. At the same time, the conditions in part 2 are required to hold "eventually", where as the conditions in part 1 hold from time period 3 onwards.

We close this section with an interesting observation.

Proposition 3.2: Suppose $<((p_1^{(t)}, p_2^{(t)}), [A^{(t)} \mid B^{(t)}], (c^{(t)}, d^{(t)}, e^{(t)}))| t \in \mathbb{N}^0>$ is an LOC-LC problem and let $x \in X$. If $\mathcal{S}(x) \neq \phi$, $d^{(t)} = 0$ and for some $\alpha \in \mathbb{R}\backslash\{0\}$, $e^{(t)} = \alpha$, $p_2^{(t)} = -\alpha p_1^{(t+1)}$ for all $t \in \mathbb{N}^0$, then $\mathcal{S}(x) = \mathcal{F}(x)$.

Proof: For all $<(x_t, u_t)|t\in \mathbb{N}^0> \epsilon \mathcal{F}(x)$, $\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] = p_1^{(0)}x$ + $\sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = p_1^{(0)}x$ + $\sum_{t=1}^{\infty}[p_1^{(t)}x_t - \alpha p_1^{(t)}u_{t-1}] = p_1^{(0)}x + \sum_{t=1}^{\infty}p_1^{(t)}[x_t - \alpha u_{t-1}] = p_1^{(0)}x + \sum_{t=1}^{\infty}p_1^{(t)}c^{(t-1)}$, since $p_2^{(t-1)} = -\alpha p_1^{(t)}$ and $x_t - \alpha u_{t-1} = c^{(t-1)}$ for all $t\in \mathbb{N}$.

We have assumed that $\mathcal{S}(x) \neq \phi$. Further, $\mathcal{S}(x) \subset \mathcal{F}(x)$.

Since, $\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] = p_1^{(0)}x + \sum_{t=1}^{\infty}p_1^{(t)}c^{(t-1)}$ for all $<(x_t, u_t)|t\in \mathbb{N}^0> \epsilon \mathcal{F}(x)$, it follows that $\mathcal{S}(x) = \mathcal{F}(x)$. Q.E.D.

4. Existence of optimal solution, bang-bang sequence of decision rules and a "robust" approximation result:

As in Mitra (2000), Sorger (2015) and Lahiri (2025a) (among numerous others) we will, in the rest of this paper be concerned with an optimality criterion that requires the following "Absolute Convergence" condition.

The LOC-LC problem $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in \mathbb{N}^0>$ is said to satisfy **Absolute Convergence** if for $i\in\{1,2\}$, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$.

If $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in \mathbb{N}^0>$ satisfies Absolute Convergence, then we may refer to $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in \mathbb{N}^0>$ as an **Absolutely Convergent LOC-LC (AC-LOC-LC) problem**.

Thus, for any AC-LOC-LC problem $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in \mathbb{N}^0>$ it must be the case that for $i\epsilon\{1,2\}$: $\lim_{t\to\infty} p_i^{(t)} = 0$.

If $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in \mathbb{N}^0>$ is an A**bsolutely Convergent (1+1)-LOC-LC problem**, then we will refer to it as an **AC-(1+1)-LOC-LC problem**.

Let $<((p_1^{(t)}, p_2^{(t)}), \Omega_t)|\ t\in \mathbb{N}^0>$ be an AC-LOC-LC problem. Thus, for all sequence $<(x_t, u_t)|t\in \mathbb{N}^0>$ with $(x_t, u_t)\in X\times X$ for all $t\in \mathbb{N}^0$, it must be the case that: $\lim_{t\to\infty}|p_1^{(t)}x_t| = 0$, $\lim_{t\to\infty}|p_2^{(t)}u_t| = 0$, $\sum_{t=0}^{\infty}|p_1^{(t)}x_t| \in [0, b\sum_{t=0}^{\infty}|p_1^{(t)}|]$ and $\sum_{t=0}^{\infty}|p_2^{(t)}u_t| \in [0, b\sum_{t=0}^{\infty}|p_2^{(t)}|]$.

Let M = max{b$\sum_{t=0}^{\infty}|p_1^{(t)}|$, b$\sum_{t=0}^{\infty}|p_2^{(t)}|$} < +∞.

Thus, for all sequence <(x_t, u_t)|t∈$\mathbb{N}^0$> with (x_t, u_t)∈X×X for all t∈$\mathbb{N}^0$, it must be the case that $|\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]| \leq \sum_{t=0}^{\infty}|p_1^{(t)}x_t + p_2^{(t)}u_t| \leq \sum_{t=0}^{\infty}[|p_1^{(t)}x_t| + |p_2^{(t)}u_t|] \leq 2M$.

The following result is an immediate consequence of a corresponding result in a more general framework (proposition 5.1 in Lahiri (2025c)).

Proposition 4.1: Let <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈$\mathbb{N}^0$> be an AC-LOC-LC problem. $\mathcal{S}^{(T)}$(x) ≠ ϕ for all (x, T)∈X×$\mathbb{N}^0$. Hence, $\mathcal{S}^{(T)*}$(x) ≠ ϕ for all (x, T)∈X×$\mathbb{N}^0$ and the optimal period-T decision rule $h^{(T)}$ is non-empty valued for all T∈$\mathbb{N}^0$.

Given an AC-LOC-LC problem <(($p_1^{(t)}$, $p_2^{(t)}$), [$A^{(t)}$ | $B^{(t)}$], ($c^{(t)}$,$d^{(t)}$,$e^{(t)}$))| t∈$\mathbb{N}^0$>, the function V:X→ℝ such for x∊X, V(x) = $\sum_{t=0}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]$ for <(y_t, v_t)|t∈$\mathbb{N}^0$>∈$\mathcal{S}$(x) is said to be the **optimal value function**.

The following definition is based on pages 202–208 in Kamien and Schwartz (1991).

For T∊$\mathbb{N}^0$, the array <$h^{(t)}$| t ≥ T> of decision rules is said to be a **bang-bang sequence of decision rules** if for all (x, t)∊X×$\mathbb{N}^0$ satisfying t ≥ T and $p_2^{(t)} \neq 0$, $h^{(t)}$(x) = (max{$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{$A_i^{(t)} + B_i^{(t)}$x|i∊I}, 0}, $c^{(t)} + d^{(t)}$x + $e^{(t)}$max{$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{$A_i^{(t)} + B_i^{(t)}$x|i∊I}, 0}).

An immediate and interesting consequence of proposition 4.1 is the following corollary.

Corollary 1 of proposition 4.1: Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), [$A^{(t)}$ | $B^{(t)}$], ($c^{(t)}$,$d^{(t)}$,$e^{(t)}$))| t∈$\mathbb{N}^0$> is an AC-LOC-LC problem. If for some T∈$\mathbb{N}^0$ it is the case that $e^{(t)} = 0$ and $p_2^{(t)} \neq 0$ for all t ≥ T, then, for all x∈X: <(x_t, u_t)|t ≥ T>∈$\mathcal{S}^{(T)}(x)$ <u>if and only if</u> for all t ≥ T: $x_{t+1} = c^{(t)} + d^{(t)}x_t$ and u_t = max{$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{$A_i^{(t)} + B_i^{(t)}x_t$|i∊I},0}. Thus, for all t ≥ T and x∈X, it must be that $h^{(t)}$(x) = (max {$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{$A_i^{(t)} + B_i^{(t)}$x|i∊I}, 0}, $c^{(t)} + d^{(t)}$x), so that <$h^{(t)}$| t ≥ T> is a bang-bang sequence of decision rules.

Proof: We know from proposition 4.1 that for all x∈X, $\mathcal{S}^{(T)}(x) \neq \phi$. Recall that for all (x, t)∈X×$\mathbb{N}^0$, $A_i^{(t)}+B_i^{(t)}x \geq 0$ for all i∊I, and $e^{(t)} = 0$ for all t ≥ T implies, $x_{t+1} = c^{(t)} + d^{(t)}x \in [0, b]$ for all (x, t)∈X×$\mathbb{N}^0$ and $u_t \in [0, \min\{A_i^{(t)} + B_i^{(t)}x | i \in I\}]$. The rest follows from the requirements in the statement of this corollary. Q.E.D.

Note 4.1: For a AC-(1+1)-LOC-LC problem, if $<h^{(t)}| t \geq T>$ is a bang-bang sequence of decision rules, then for all t ≥ T and x∈X, it must be the case that $h^{(t)}(x) = (\max\{\frac{p_2^{(t)}}{|p_2^{(t)}|}(a^{(t)} + b^{(t)}x), 0\}, c^{(t)} + d^{(t)}x + e^{(t)}\max\{\frac{p_2^{(t)}}{|p_2^{(t)}|}(a^{(t)} + b^{(t)}x), 0\})$. Further,

$\sum_{t=T}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t] = p_1^{(T)}x_T+p_2^{(T)}(\max\{\frac{p_2^{(T)}}{|p_2^{(T)}|}(a^{(t)}+b^{(t)}x), 0\}) +$

$\sum_{t=T+1}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}(\max\{\frac{p_2^{(t)}}{|p_2^{(t)}|}(a^{(t)}+b^{(t)}x_t), 0\})]$.

Recall the definition of the optimal value function V:X→ℝ immediately after the statement of proposition 4.1.

Proposition 4.1 implies the following "robust" approximation result.

Proposition 4.2 (Approximation Result): Suppose $<((p_1^{(t)}, p_2^{(t)}), [A^{(t)} | B^{(t)}], (c^{(t)},d^{(t)},e^{(t)}))| t\in \mathbb{N}^0>$ is an AC-LOC-LC problem and let x∈X. Then for all ε > 0, there exists $T^*(\varepsilon)\in\mathbb{N}$ such that for all T∊ℕ satisfying $T \geq T^*(\varepsilon)$, the linear programming problem [Maximize $\sum_{t=1}^{T}[p_1^{(t)}y_t+ p_2^{(t-1)}v_{t-1}]$ subject to $v_t \leq A_i^t + B_i^t y_t$ for every i∊I and t = 0, 1, …, T-1, $y_{t+1} = c^{(t)} + d^{(t)}y_t + e^{(t)}v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$ for all t = 1, …, T, $v_t \geq 0$, for all t = 0, 1, …, T-1] has a solution $<(x_t^{(T)}, u_{t-1}^{(T)})| t = 1, ..., T>$ and $|p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)}+ p_2^{(t-1)}u_{t-1}^{(T)}]- V(x)| < \varepsilon$. Hence, $p_1^{(0)}x + \lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)}+ p_2^{(t)}u_t^{(T)}] = V(x)$.

Proof: Since $<((p_1^{(t)}, p_2^{(t)}), [A^{(t)} | B^{(t)}], (c^{(t)},d^{(t)},e^{(t)}))| t\in \mathbb{N}^0>$ is an AC-LOC-LC problem, $\mathcal{S}^*(x)\neq \phi$ and for all T∊ℕ the linear programming problem [Maximize $\sum_{t=1}^{T}[p_1^{(t)}y_t+ p_2^{(t-1)}v_{t-1}]$ subject to $v_t \leq A_i^t + B_i^t y_t$ for every i∊I and t = 0, 1, …, T-1, $y_{t+1} = c^{(t)} + d^{(t)}y_t + e^{(t)}v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$ for all t = 1, …, T, $v_t \geq 0$, for all t = 0, 1, …, T-1] has a solution $<(x_t^{(T)}, u_{t-1}^{(T)})| t = 1, ..., T>$.

Towards a contradiction suppose there exists $\varepsilon > 0$ such that $|p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] - V(x)| \geq \varepsilon$ infinitely often, i.e., <u>either</u> $p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] - \varepsilon \geq V(x)$ infinitely often <u>or</u> $V(x) \geq p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] + \varepsilon$ infinitely often.

Let $<(x_t, u_t)|t\in \mathbb{N}^0> \in \mathcal{S}(x)$ so that $<(x_t, u_{t-1})|t\in \mathbb{N}> \in \mathcal{S}^*(x)$. Since $<(x_t, u_{t-1})|t = 1, \ldots, T>$ satisfies the constraints of the linear programming problem for all $T\in\mathbb{N}$, it must be the case that $\sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] \geq \sum_{t=1}^{T}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]$ for all $T\in\mathbb{N}$.

Since, $V(x) = \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] = \lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t] = p_1^{(0)}x + \lim_{T\to\infty}\sum_{t=1}^{T}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1} + p_2^{(T)}u_T]$ and since $u_T \in [0, b]$ for all $T\in\mathbb{N}$ along with $\lim_{T\to\infty} p_2^{(T)} = 0$ implies $\lim_{T\to\infty}\sum_{t=1}^{T}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1} + p_2^{(T)}u_T] = \lim_{T\to\infty}\sum_{t=1}^{T}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]$, there exists $T^0\in\mathbb{N}$ such that for all $T\in\mathbb{N}$ with $T \geq T^0$ it is the case that $\sum_{t=1}^{T}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] + \frac{\varepsilon}{4} > V(x) - p_1^{(0)}x > \sum_{t=1}^{T}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] - \frac{\varepsilon}{4}$.

Thus, $\sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] + \frac{\varepsilon}{4} \geq \sum_{t=1}^{T}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] + \frac{\varepsilon}{4} > V(x) - p_1^{(0)}x$ for all $T\in\mathbb{N}$ with $T \geq T^0$.

Thus, $\sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] + \varepsilon > \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] + \frac{\varepsilon}{4} > V(x) - p_1^{(0)}x$ for all $T\in\mathbb{N}$ with $T \geq T^0$.

Thus, $|p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] - V(x)| \geq \varepsilon$ infinitely often is incompatible with $V(x) \geq p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] + \varepsilon$ infinitely often.

Thus, $|p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] - V(x)| \geq \varepsilon$ infinitely often implies that there exists $T^1\in\mathbb{N}$ such that for all $T\in\mathbb{N}$ with $T \geq T^1$: $p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] \geq V(x) + \varepsilon$.

Since, $<((p_1^{(t)}, p_2^{(t)}), [A^t \mid B^t], (c^{(t)}, d^{(t)}, e^{(t)}))| t\in \mathbb{N}^0>$ is an AC-LOC-LC problem, there exists $T^2\in\mathbb{N}$ such that for all $T\in\mathbb{N}$ satisfying $T \geq T^2$, $b\sum_{t=T}^{\infty}|p_i^{(t)}| < \frac{\varepsilon}{8}$ for $i\in \{1,2\}$.

Let $T = 1 + \max\{T^1, T^2\}$, $x_{T+1}^{(T)} = c^{(T)} + d^{(T)}x_T^{(T)} + e^{(T)}u_T^{(T)}$ and $<(x_t^{(T)}, u_{t-1}^{(T)}| t \geq T+2> \in\mathcal{F}^{(T+1)*}(x_{T+1}^{(T)})$.

Thus, $<(x_t^{(T)}, u_{t-1}^{(T)} | t \epsilon \mathbb{N}> \epsilon \mathcal{F}^*(x)$.

Further, $p_1^{(0)}x + \sum_{t=1}^{T} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}] \geq V(x) + \varepsilon$ and $b\sum_{t=T}^{\infty} |p_i^{(t)}| < \frac{\varepsilon}{8}$ for

$i \in \{1,2\}$. Since $b > 0$, it must be the case that $b\sum_{t=T+1}^{\infty} |p_i^{(t)}| < \frac{\varepsilon}{8}$.

Now $|\sum_{t=T+1}^{\infty} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}]| \leq \sum_{t=T+1}^{\infty} [|p_1^{(t)}| x_t^{(T)} + |p_2^{(t-1)}| u_{t-1}^{(T)}] \leq$

$b[\sum_{t=T+1}^{\infty} [|p_1^{(t)}| + |p_2^{(t-1)}|] = b\sum_{t=T+1}^{\infty} |p_1^{(t)}| + b\sum_{t=T+1}^{\infty} |p_2^{(t-1)}| < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}$.

Thus, $\frac{\varepsilon}{4} > \sum_{t=T+1}^{\infty} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}] > -\frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{\infty} [p_1^{(t)} x_t^{(T)} + p_2^{(t)} u_t^{(T)}] = p_1^{(0)}x + \sum_{t=1}^{\infty} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}] = p_1^{(0)}x +$
$\sum_{t=1}^{T} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}] + \sum_{t=T+1}^{\infty} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}] \geq V(x) + \varepsilon +$
$\sum_{t=T+1}^{\infty} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}] > V(x) + \varepsilon - \frac{\varepsilon}{4} = V(x) + \frac{3\varepsilon}{4} > V(x)$.

This, contradicts the definition of V(x) in the statement of this proposition and proves the proposition. Q.E.D.

Since for the "free end-point problem", $y_T = c^{(T-1)} + d^{(T-1)}y_{T-1} + e^{(T-1)}v_{T-1}$, for $x \epsilon X$ and $T \in \mathbb{N}$ with $T \geq 3$, the linear programming problem in the statement of proposition 4.2, is equivalent to the following linear programming maximization problem:

Maximize $\sum_{t=1}^{T-1} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] + (p_2^{(T-1)} + e^{(T-1)} p_1^{(T)}) v_{T-1} + p_1^{(T)} d^{(T-1)} y_{T-1}$

subject to $v_0 \leq A_i^{(0)} + B_i^{(0)}x$ for every $i \epsilon I$, $v_t - B_i^t y_t \leq A_i^t$ for every $i \epsilon I$ and $t = 0, 1, \ldots, T-1$, $y_1 - e^{(0)}v_0 = c^{(0)} + d^{(0)}x$, $y_{t+1} = c^{(t)} + d^{(t)}y_t + e^{(t)}v_t$ for all $t = 0, 1, \ldots, T-1$, $y_0 = x_0 = x$, $y_t \geq 0$ for all $t = 1, \ldots, T$, $v_t \geq 0$, for all $t = 0, 1, \ldots, T-1$.

In this maximization problem, the co-efficient of y_{T-1} in the objective function is $(p_1^{(T-1)} + p_1^{(T)} d^{(T-1)})$ and the co-efficient of v_{T-1} is $(p_2^{(T-1)} + e^{(T-1)} p_1^{(T)})$.

Further, if for $T \in \mathbb{N}$ with $T \geq 3$, $<(x_t^{(T)}, u_{t-1}^{(T)})| t = 1, \ldots, T>$ solves the linear programming problem in the statement of proposition 4.2, then it continues to solve the equivalent problem mentioned above.

The dual of this equivalent linear programming problem is the linear programming minimization problem [Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^{(T)} c^{(t)} +$

$\sum_{i=1}^{m} \beta_0^{(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m} \beta_t^{(i|T)} A_i^{(t)}$, subject to $\alpha_{t-1}^{(T)}$ - $d^{(t)}\alpha_t^{(T)}$ - $\sum_{i=1}^{m} \beta_t^{(i|T)} B_i^{(t)}$ $\geq p_1^{(t)}$ for all t = 1, …, T-2, $\alpha_{T-2}^{(T)}$ - $d^{(T-1)}\alpha_{T-1}^{(T)}$ - $\sum_{i=1}^{m} \beta_t^{(i|T)} B_i^{(t)} \geq p_1^{(T)}$ - $+p_1^{(T)}d^{(T-1)}$, $\alpha_t^{(T)}e^{(t)}$ + $\sum_{i=1}^{m} \beta_t^{(i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-2, $\alpha_{T-1}^{(T)}e^{(t)} + \sum_{i=1}^{m} \beta_{T-1}^{(i|T)} \geq (p_2^{(T-1)} + e^{(T-1)}p_1^{(T)})$, $\beta_t^{(i|T)} \geq 0$ for all iϵI and t = 0, …, T-1, $\alpha_t^{(T)} \epsilon \mathbb{R}$ for all t = 0, …, T-1].

Important Notation: For T$\epsilon\mathbb{N}$ and each t = 0, …, T-1, let the array <$\beta_t^{(i|T)}$|iϵI> be denoted by $\beta_t^{(T)}$. Thus for T$\epsilon\mathbb{N}$, T ≥ 3 and t = 0, …, T-1, $\beta_t^{(T)} \epsilon \mathbb{R}_+^m$.

An immediate consequence of proposition 4.2, the weak duality theorem and complementary slackness conditions of linear programming (as in topic 2 of Lahiri (2020)) is the following corollary.

Corollary of Proposition 4.2: Let <(($p_1^{(t)}$, $p_2^{(t)}$), [$A^{(t)}$ | $B^{(t)}$], ($c^{(t)}$, $d^{(t)}$, $e^{(t)}$))| t$\in \mathbb{N}^0$> be an AC-LOC-LC problem and suppose that for some xϵX, <x_t|t$\in \mathbb{N}^0$>$\in \mathcal{S}$(x). Then for all T$\epsilon\mathbb{N}$ with T ≥ 3, there exists a finite array <($\alpha_t^{*(T)}$, $\beta_t^{*(T)}$)$\epsilon\mathbb{R}\times \mathbb{R}_+^m$|t = 0,1, 2, …, T-1> satisfying the following conditions:

(G-1) The array <($\alpha_t^{*(T)}$, $\beta_t^{*(T)}$)$\epsilon\mathbb{R}\times \mathbb{R}_+^m$|t = 0,1, 2, …, T-1> solve:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^{(T)} c^{(t)} + \sum_{i=1}^{m} \beta_0^{(i|T)}(A_i^{(0)}+B_i^{(0)}x)$ + $\sum_{t=1}^{T-1}\sum_{i=1}^{m} \beta_t^{(i|T)} A_i^{(t)}$, subject to $\alpha_{t-1}^{(T)}$ - $d^{(t)}\alpha_t^{(T)}$ - $\sum_{i=1}^{m} \beta_t^{(i|T)} B_i^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-2, $\alpha_{T-2}^{(T)}$ - $d^{(T-1)}\alpha_{T-1}^{(T)}$ - $\sum_{i=1}^{m} \beta_t^{(i|T)} B_i^{(t)} \geq p_1^{(T)}$ - $+p_1^{(T)}d^{(T-1)}$, $\alpha_t^{(T)}e^{(t)} + \sum_{i=1}^{m} \beta_t^{(i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-2, $\alpha_{T-1}^{(T)}e^{(t)} + \sum_{i=1}^{m} \beta_{T-1}^{(i|T)} \geq (p_2^{(T-1)} + e^{(T-1)}p_1^{(T)}$, $\beta_t^{(i|T)} \geq 0$ for all iϵI and t = 0, …, T-1, $\alpha_t^{(T)} \epsilon \mathbb{R}$ for all t = 0, …, T-1

(G-2) $\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^{*(T)} c^{(t)} + \sum_{i=1}^{m} \beta_0^{*(i|T)}(A_i^{(0)}+B_i^{(0)}x)$ + $\sum_{t=1}^{T-1}\sum_{i=1}^{m} \beta_t^{*(i|T)} A_i^{(t)} = \sum_{t=1}^{T-1} [p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] + (p_2^{(T-1)} + e^{(T-1)}p_1^{(T)})u_{T-1}$ + $p_1^{(T)}d^{(T-1)}x_{T-1}$.

(G-3) $V(x) - p_1^{(0)}x = \lim_{T\to\infty} [\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^{*(T)} c^{(t)}$ + $\sum_{i=1}^{m} \beta_0^{*(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m} \beta_t^{*(i|T)} A_i^{(t)}]$.

Note 4.1: G-3, follows from the fact that for all T$\epsilon\mathbb{N}$ with T ≥ 3,
$\sum_{t=1}^{T} [p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] = \sum_{t=1}^{T-1} [p_1^{(t)}x_t^{(T)} + p_2^{(t-1)}u_{t-1}^{(T)}] + (p_2^{(T-1)} + e^{(T-1)}p_1^{(T)})u_{T-1}$

$+ p_1^{(T)}$d$^{(T-1)}$x$_{T-1}$ + $p_1^{(T)}$c$^{(T-1)}$, so that since c$^{(T-1)}$ ϵ[0, b] and $\lim_{T\to\infty} p_1^{(T)} = 0$ implies $\lim_{T\to\infty} p_1^{(T)} c^{(T-1)} = 0$, it follows that V(x) - $p_1^{(0)}$x = $\lim_{T\to\infty} \sum_{t=1}^{T} [p_1^{(t)} x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}]$ = $\lim_{T\to\infty} [\sum_{t=1}^{T-1} (x_t^{(T)} + p_2^{(t-1)} u_{t-1}^{(T)}) + (p_2^{(T-1)} + e^{(T-1)} p_1^{(T)}) u_{T-1} + p_1^{(T)}d^{(T-1)}x_{T-1}$)].

Note 4.2: What the corollary of proposition 4.2 says is that, the optimal value of the dual of the truncated "free end-point" linear programming problem with the initial value of the state variable in the latter being the same as the initial value of the state variable in the optimal trajectory (i.e., dual of the the linear programming problems in the "approximation result" proposition 4.2), converges to the optimal value of the AC-LOC-LC problem.

5. Duality Theory for AC-LOC-LC problems and "necessary" conditions for optimality:

Let <(($p_1^{(t)}$, $p_2^{(t)}$), [A$^{(t)}$ | B$^{(t)}$], (c$^{(t)}$,d$^{(t)}$,e$^{(t)}$))| t∈ $\mathbb{N}^0$> be an AC-LOC-LC problem. For some x∈ X, let <(x$_t$, u$_t$)|t∈ $\mathbb{N}^0$> ϵ$\mathcal{F}$(x).

For T∈ ℕ with T ≥ 3 consider the linear programming problem in part 1 of proposition 3.1.

Maximize $p_2^{(T-1)} v_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}]$, subject to $v_0 \leq A_i^{(0)} + B_i^{(0)}$x for every iϵI, $v_t - B_i^{(t)} y_t \leq A_i^{(t)}$ for every iϵI and t = 1, …, T-1, $y_1 - e^{(0)}v_0 = c^{(0)} + d^{(0)}x$, $y_{t+1} - d^{(t)}y_t - e^{(t)}v_t = c^{(t)}$ for all t = 1, …, T-2, $-d^{(T-1)}y_{T-1} - e^{(T-1)}v_{T-1} = -x_T + c^{(T-1)}$, $y_t \geq 0$, for all t = 1, …, T-1, $v_t \geq 0$, for all t = 0, 1, …, T-1.

The dual of this linear programming maximization problem is the following linear programming minimization problem.

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m} \beta_0^{(i|T)}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m} \beta_t^{(i|T)} A_i^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \sum_{i=1}^{m} \beta_t^{(i|T)} B_i^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)} e^{(t)} + \sum_{i=1}^{m} \beta_t^{(i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta_t^{(i|T)} \geq 0$ for all iϵI and t = 0, …, T-1, $\alpha_t^{(T)} \in \mathbb{R}$ for all t = 0, …, T-1.

A first result which is very general is the following.

Proposition 5.1: Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), [$A^{(t)}$ | $B^{(t)}$], ($c^{(t)}$,$d^{(t)}$,$e^{(t)}$))| t∈ $\mathbb{N}^0$> be an AC-LOC-LC problem. For x∊X, let <(x_t, u_t)|t∈ $\mathbb{N}^0$> ∊$\mathcal{S}$(x).

<u>Then</u>, for all T∊ℕ with T ≥ 3, for all t = 0, 1, …, T- 1 there exists an array <($\alpha_t^{*(T)}$, $\beta_t^{*(T)}$)∊$\mathbb{R}\times \mathbb{R}_+^m$|t = 0,1, 2, …, T> that along with <(x_t, u_{t-1})|t = 1, …, T> satisfy the following properties:

(1) The array <($\alpha_t^{*(T)}$, $\beta_t^{*(T)}$)∊$\mathbb{R}\times \mathbb{R}_+^m$|t = 0,1, 2, …, T> solves the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{(i|T)}A_i^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \sum_{i=1}^{m}\beta_t^{(i|T)}B_i^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)}e^{(t)} + \sum_{i=1}^{m}\beta_t^{(i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta_t^{(i|T)} \geq 0$ for all i∊I and t = 0, …, T-1, $\alpha_t^{(T)}\in\mathbb{R}$ for all t = 0, …, T-1.

(2) $\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{*(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{*(i|T)}A_i^{(t)} = p_2^{(T-1)}u_{T-1} + \sum_{t=1}^{T-1}[p_1^{(t)}x_t+p_2^{(t-1)}u_{t-1}] = \sum_{t=1}^{T-1}[p_1^{(t)}x_t+p_2^{(t)}u_t] - p_1^{(0)}$x.

(3) $\lim_{T\to\infty}[\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{*(i|T)}(A_i^0+B_i^0x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{*(i|T)}A_i^{(t)}]$ exists and is equal to $\lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)}+ p_2^{(t)}u_t^{(T)}] - p_1^{(0)}x$ = V(x) - $p_1^{(0)}x$ <u>if and only if</u> $\lim_{T\to\infty}\alpha_{T-1}^{*(T)}x_T = 0$.

Proof: (1) and (2) follow from the weak duality theorem and complementary slackness conditions of linear programming (see topic 2 in Lahiri (2020)).

Since, <(($p_1^{(t)}$, $p_2^{(t)}$), [$A^{(t)}$ | $B^{(t)}$], ($c^{(t)}$, $d^{(t)}$, $e^{(t)}$))| t∈ $\mathbb{N}^0$> is an AC-LOC-LC problem, by proposition 4.2, $\lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)}+ p_2^{(t)}u_t^{(T)}]$ exists and is equal to V(x).

Thus from (2) it follows that $\lim_{T\to\infty}[\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{*(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{*(i|T)}A_i^{(t)}]$ exists <u>and</u> $\lim_{T\to\infty}[\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{*(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{*(i|T)}A_i^{(t)}] = \lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)}+ p_2^{(t)}u_t^{(T)}] - p_1^{(0)}x$ = V(x) - $p_1^{(0)}x$.

$\lim_{T\to\infty}[\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{*(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{*(i|T)}A_i^{(t)}] = \lim_{T\to\infty}[\alpha_0^{*(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-1}\alpha_t^{*(T)}c^{(t)} + \sum_{i=1}^{m}\beta_0^{*(i|T)}(A_i^{(0)}+B_i^{(0)}x) + \sum_{t=1}^{T}\sum_{i=1}^{m}\beta_t^{*(i|T)}A_i^{(t)}]$ <u>if and only if</u> $\lim_{T\to\infty}\alpha_{T-1}^{*(T)}x_T = 0$.

This proves (3). Q.E.D.

Note 5.1: If for $<(x_t, u_t)|t\in\mathbb{N}^0> \epsilon\mathcal{S}(x)$ and for all $T\epsilon\mathbb{N}$ with $T \geq 3$ there exists an array $<(\alpha_t^{*(T)}, \beta_t^{*(T)})\epsilon\mathbb{R}\times\mathbb{R}_+^m|t = 0,1, 2, \ldots, T>$ satisfying conditions (1) and (2) of proposition 5.1 then we refer to the condition $\lim_{T\to\infty}\alpha_{T-1}^{*(T)}x_T = 0$ as "**weak transversality condition**".

Note 5.2: What proposition 5.1 claims is that the optimal value of the dual of the truncated "free end-point" linear programming problem with the initial value of the state variable in the latter being the same as the initial value of the state variable in the optimal trajectory (i.e., the linear programming problems in the "approximation result" proposition 4.2), converges to the optimal value of the AC-LOC-LC problem beginning from period 1 <u>if and only if</u> weak transversality condition is satisfied.

6. Interiority condition and infinite horizon dual linear programming problem for AC-(1+1)-LOC-LC problems:

In this section we assume for the AC-(1+1)-LOC-LC problem $<((p_1^{(t)}, p_2^{(t)}), (a^{(t)}, b^{(t)}), (c^{(t)},d^{(t)},e^{(t)}))|\ t\in\mathbb{N}^0>$ that the consequences of optimality are only applicable to those optimal trajectories for which both the state and control variable are always positive and the control variable satisfies its inequality constraints with strict inequality in at least one time period. We refer to this condition as "interiority condition". In such a situation we get a very strong necessary condition for optimality.

Proposition 6.1: Let $<((p_1^{(t)}, p_2^{(t)}), (a^{(t)}, b^{(t)}), (c^{(t)},d^{(t)},e^{(t)}))|\ t\in\mathbb{N}^0>$ be an AC-(1+1)-LOC-LC problem. Suppose that for some $x\epsilon X$ and $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$, $<(x_t, u_t)|t\in\mathbb{N}^0>$ satisfies the following "**interiority condition**": For all $t\epsilon\mathbb{N}^0$, $u_t > 0$, for all $t\epsilon\mathbb{N}$, $x_t > 0$ and there exists $t^*\epsilon\mathbb{N}^0$ such that $u_{t^*} < a^{(t^*)} + b^{(t^*)}x_{t^*}$ and $e^{(t^*)}\neq 0$. Then, there exists a sequences $<(\alpha_t^*, \beta_t^*)\epsilon\mathbb{R}\times\mathbb{R}_+|t\epsilon\mathbb{N}^0>$ such that:

(i**) α^*_{t-1} - $(d^{(t)} + e^{(t)}b^{(t)})\alpha^*_t + (p_2^{(t)}b^{(t)} - p_1^{(t)}) = 0$ for all $t \in \mathbb{N}$ and $\alpha^*_{t^*} = \frac{-p_2^{(t^*)}}{e^{(t^*)}}$.

(ii**) $\beta^*_t = \alpha^*_t e^{(t)} + p_2^{(t)}$.

Further:

(iii**) For all $T \in \mathbb{N}$, $T \geq \max\{3, t^*+1\}$, $<(\alpha^*_t, \beta^*_t) \in \mathbb{R}\times\mathbb{R}_+ | t = 0, 1, \ldots, T-1>$ solves the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \beta_0^{(T)}(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1} \beta_t^{(T)} a^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \beta_t^{(T)}b^{(t)} \geq p_1^{(t)}$ for all $t = 1, \ldots, T-1$, $-\alpha_t^{(T)}e^{(t)} + \beta_t^{(T)} \geq p_2^{(t)}$ for all $t = 0, \ldots, T-1$, $\beta_t^{(T)} \geq 0$ for all i and $t = 0, \ldots, T-1$, $\alpha_t^{(T)} \in \mathbb{R}$ for all $t = 0, \ldots, T-1$.

(iv**) For all $T \in \mathbb{N}$, $T \geq 3$, $p_2^{(T-1)}u_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \alpha^*_0(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha^*_t c^{(t)} + \alpha^*_{T-1}(c^{(T-1)} - x_T) + \beta^*_0(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1} \beta^*_t a^{(t)}$.

Proof: From the discussion following note 5.1 (of this paper), we know that if $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{S}(x)$, then that for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*+1\}$, $<(x_t, u_t)|t = 0, \ldots, T>$ solves:

Maximize $p_2^{(T-1)} v_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}]$, subject to $v_0 \leq a^{(0)} + b^{(0)}x$, $v_t - b^{(t)}y_t \leq a^{(t)}$ for $t = 1, \ldots, T-1$, $y_1 - e^{(0)}v_0 = c^{(0)} + d^{(0)}x$, $y_{t+1} - d^{(t)}y_t - e^{(t)}v_t = c^{(t)}$ for all $t = 1, \ldots, T-2$, $-d^{(T-1)}y_{T-1} - e^{(T-1)}v_{T-1} = -x_T + c^{(T-1)}$, $y_t \geq 0$, for all $t = 1, \ldots, T-1$, $v_t \geq 0$, for all $t = 0, 1, \ldots, T-1$.

By the strong duality theorem of linear programming we know that $<(x_t, u_t)|t = 0, \ldots, T>$ solves the above problem <u>if and only if</u> its dual has a solution, in which case the optimal value of the maximization problem and the optimal value of its dual are equal. The dual of the linear programming maximization problem is the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \beta_0^{(T)}(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1} \beta_t^{(T)} a^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \beta_t^{(T)}b^{(t)} \geq p_1^{(t)}$ for all $t = 1, \ldots, T-1$, $-\alpha_t^{(T)}e^{(t)} + \beta_t^{(T)} \geq p_2^{(t)}$ for all $t = 0, \ldots, T-1$, $\beta_t^{(T)} \geq 0$ for all i and $t = 0, \ldots, T-1$, $\alpha_t^{(T)} \in \mathbb{R}$ for all $t = 0, \ldots, T-1$.

From the strong duality theorem and the complementary slackness condition we know that since $<(x_t, u_t)|t\in \mathbb{N}^0> \in \mathcal{S}(x) \subset \mathcal{F}(x)$, $<(x_t, u_t)|t = 0, \ldots, T>$ solves the maximization problem <u>if and only if</u> there exists $<(\alpha_t^{(T)}, \beta_t^{(T)}) \in \mathbb{R}\times\mathbb{R}_+|t = 0, \ldots, T-1>$ and which along with $<(x_t, u_t)| t = 0, 1, \ldots, T-1>$ satisfy the following:

(i) $x_t \geq 0$, $u_t \geq 0$ for all $t = 0, 1, \ldots, T-1$, $x_0 = x$.

(ii) $x_{t+1} = c^{(t)} + d^{(t)}x_t + e^{(t)}u_t$ for all $t = 0, 1, \ldots, T-1$.

(iii) $u_0 \leq a^{(0)} + b^{(0)}x$ and $(u_0 - a^{(0)} - b^{(0)}x)\beta_0^{(T)} = 0$.

(iv) $u_t - b^{(t)}x_t \leq a^{(t)}$ and $(u_t - a^{(t)} - b^{(t)}x_t)\beta_t^{(T)} = 0$ for all $t = 1, \ldots, T-1$.

(v) $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \beta_t^{(T)}b^{(t)} \geq p_1^{(t)}$ and $(\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \beta_t^{(T)}b^{(t)} - p_1^{(t)})x_t = 0$, for all $t = 1, \ldots, T-1$.

(vi) $-\alpha_t^{(T)}e^{(t)} + \beta_t^{(T)} \geq p_2^{(t)}$ and $(-\alpha_t^{(T)}e^{(t)} + \beta_t^{(T)} - p_2^{(t)})u_t = 0$ for all $t = 0, 1, \ldots, T-1$.

By the "interiority condition", for all $t\in\mathbb{N}^0$, $u_t > 0$ and for all $t\in\mathbb{N}$, $x_t > 0$.

Thus, from (v) and (vi) we get $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \beta_t^{(T)}b^{(t)} - p_1^{(t)} = 0$, for all $t = 1, \ldots, T-1$ and $-\alpha_t^{(T)}e^{(t)} + \beta_t^{(T)} - p_2^{(t)} = 0$, for all $t = 0, 1, \ldots, T-1$.

From the second set of equations we get $\beta_t^{(T)} = \alpha_t^{(T)}e^{(t)} + p_2^{(t)}$ for all $t = 0, 1, \ldots, T-1$.

Substituting for $\beta_t^{(T)}$, $t = 1, \ldots, T-1$ in the first set of equations we get $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - (\alpha_t^{(T)}e^{(t)} + p_2^{(t)})b^{(t)} - p_1^{(t)} = 0$, for all $t = 1, \ldots, T-1$.

Further, $u_{t^*} < a^{(t^*)} + b^{(t^*)}x_{t^*}$ (iii) and (iv) we get $\beta_{t^*}^{(T)} = 0$, so that $\alpha_{t^*}^{(T)}e^{(t^*)} + p_2^{(t^*)} = 0$.

Since $e^{(t^*)} \neq 0$, we get $\alpha_{t^*}^{(T)} = \frac{-p_2^{(t^*)}}{e^{(t^*)}}$.

Thus, for all $T \geq \max\{3, t^* + 1\}$, $<\alpha_t^{(T)}| t = 0, 1, \ldots, T-1>$ satisfies the system of first order difference equations $\alpha_{t-1} - (d^{(t)} + e^{(t)}b^{(t)})\alpha_t + (p_2^{(t)}b^{(t)} - p_1^{(t)}) = 0$ along with the additional condition $\alpha_{t^*} = \frac{-p_2^{(t^*)}}{e^{(t^*)}}$.

Let $<\alpha_t^*| \alpha_t^* \in \mathbb{R}, t\in\mathbb{N}^0>$ be the unique solution of this system of first order difference equations satisfying the additional constraint.

For each t$\epsilon\mathbb{N}^0$, let $\beta_t^* = \alpha_t^* e^{(t)} + p_2^{(t)}$.

Thus, <α_t^*| $\alpha_t^* \epsilon \mathbb{R}$, t$\epsilon\mathbb{N}^0$> satisfies (i**) and <$\beta_t^*$| $\alpha_t^* \epsilon \mathbb{R}$, t$\epsilon\mathbb{N}^0$> satisfies (ii**)

Thus, for all T ≥ max {3, t* + 1}, <α_t^*| t = 1, …, T-1> and <β_t^*| t = 1, ..., T-1> along with <(x_t, u_t)| t = 0, 1, …, T-1> satisfy (i), (ii), (iii), (iv), (v) and (vi).

By the strong duality theorem and complementary slackness conditions, for all T ≥ max {3, t* + 1}, <α_t^*| t = 1, …, T-1> and <β_t^*| t = 1, ..., T-1> solve:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \beta_0^{(T)}(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1} \beta_t^{(T)} a^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \beta_t^{(T)} b^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)} e^{(t)} + \beta_t^{(T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta_t^{(T)} \geq 0$ for all i and t = 0, …, T-1, $\alpha_t^{(T)} \epsilon\mathbb{R}$ for all t = 0, …, T-1.

Further, For all T$\in\mathbb{N}$, T ≥ 3, $p_2^{(T-1)} u_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}] = \alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \beta_0^*(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1} \beta_t^* a^{(t)}$.

Thus, (iii**) and (iv**) are satisfied by <α_t^*| t = 1, …, T-1> and <β_t^*| t = 1, ..., T-1>. Q.E.D.

Note 6.1: $\lim_{T\to\infty} p_2^{(T-1)} u_{T-1} = 0$, since $\lim_{T\to\infty} p_2^{(T-1)} = 0$ and $u_{T-1} \epsilon [0, b]$ for all T$\epsilon\mathbb{N}$.

Thus, $\lim_{T\to\infty} (p_2^{(T-1)} u_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}]) = \lim_{T\to\infty} \sum_{t=1}^{T-1} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}] = \sum_{t=1}^{\infty} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}]$.

In the context of the AC-(1+1)-LOC-LC problem <(($p_1^{(t)}$, $p_2^{(t)}$), ($a^{(t)}$, $b^{(t)}$), ($c^{(t)}$,$d^{(t)}$,$e^{(t)}$))| t$\in\mathbb{N}^0$>, for xϵX, consider the following infinite horizon linear programming problem "implied" by Alt- OPT (as defined in section 3).

Maximize $\sum_{t=1}^{\infty} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}]$, subject to $v_0 \leq a^{(0)} + b^{(0)}x$, $v_t - b^{(t)}y_t \leq a^{(t)}$ for t$\epsilon\mathbb{N}$, $y_1 - e^{(0)}v_0 = c^{(0)} + d^{(0)}x$, $y_{t+1} - d^{(t)}y_t - e^{(t)}v_t = c^t$ for all t$\epsilon\mathbb{N}$, $y_t \geq 0$, for all t$\epsilon\mathbb{N}$, $v_t \geq 0$, for all t$\epsilon\mathbb{N}^0$.

Its "**implied dual linear programming (IDLP) problem**" is the following:

Minimize $\alpha_0(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t c^{(t)} + \beta_0(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{\infty} \beta_t a^{(t)}$, subject to $\alpha_{t-1} - d^{(t)}\alpha_t - \beta_t b^{(t)} \geq p_1^{(t)}$ for all t$\epsilon\mathbb{N}$, $-\alpha_t e^{(t)} + \beta_t \geq p_2^{(t)}$ for all t ϵ $\mathbb{N}^0$, $\beta_t \geq 0$ and $\alpha_t \epsilon\mathbb{R}$ for all t$\epsilon\mathbb{N}^0$.

An immediate consequence of proposition 6.1 is the following theorem.

Theorem 6.1: Let <(($p_1^{(t)}$, $p_2^{(t)}$), ($a^{(t)}$, $b^{(t)}$), ($c^{(t)}$,$d^{(t)}$,$e^{(t)}$))| t∈ $\mathbb{N}^0$> be an AC-(1+1)-LOC-LC problem. Suppose that for some x∈X and <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x), <(x_t, u_t)|t∈ $\mathbb{N}^0$> satisfies the following "**interiority condition**": For all t∈$\mathbb{N}^0$, $u_t > 0$, for all t∈ℕ, $x_t > 0$ and there exists t^*∈$\mathbb{N}^0$ such that $u_{t^*} < a^{(t^*)} + b^{(t^*)}x_{t^*}$ and $e^{(t^*)} \neq 0$. Then, there exists a sequence <(α_t^*, β_t^*)∈ℝ × $\mathbb{R}_+$| α_t^* ∈ ℝ, t∈$\mathbb{N}^0$> such that:

(i**) α_{t-1}^* - ($d^{(t)} + e^{(t)}b^{(t)}$)α_t^* + ($p_2^{(t)}b^{(t)}$ - $p_1^{(t)}$) = 0 for all t ∈ℕ and $\alpha_{t^*}^* = \frac{-p_2^{(t^*)}}{e^{(t^*)}}$.

(ii**) $\beta_t^* = \alpha_t^* e^{(t)} + p_2^{(t)}$.

Further:

(iii**) For all T∈ ℕ, T ≥ max{3, t^*+1}, <(α_t^*, β_t^*)∈ℝ × $\mathbb{R}_+$| t = 0, 1, …, T-1> solves the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \beta_0^{(T)}(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1} \beta_t^{(T)} a^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \beta_t^{(T)}b^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)}e^{(t)} + \beta_t^{(T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta_t^{(T)} \geq 0$ for all i and t = 0, …, T-1, $\alpha_t^{(T)}$∈ℝ for all t = 0, …, T-1.

(iv**) For all T∈ ℕ, T ≥ 3, $p_2^{(T-1)}u_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \beta_0^*(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1} \beta_t^* a^{(t)}$.

(v**) <(α_t^*, β_t^*)∈ℝ × $\mathbb{R}_+$|t∈$\mathbb{N}^0$> solves the "**implied infinite horizon dual linear programming problem**": Minimize $\alpha_0(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t c^{(t)} + \beta_0(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{\infty} \beta_t a^{(t)}$, subject to $\alpha_{t-1} - d^{(t)}\alpha_t - \beta_t b^{(t)} \geq p_1^{(t)}$ for all t∈ℕ, $-\alpha_t e^{(t)} + \beta_t \geq p_2^{(t)}$ for all t ∈ $\mathbb{N}^0$, $\beta_t \geq 0$ and α_t∈ℝ for all t∈$\mathbb{N}^0$

<u>&</u> hence $V(x) - p_1^{(0)}x = \sum_{t=1}^{\infty} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)} + \beta_0^*(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{\infty} \beta_t^* a^{(t)}$ <u>if and only if</u>

<(α_t^*, β_t^*)∈ℝ × $\mathbb{R}_+$|t∈$\mathbb{N}^0$> along with <(x_t, u_t)|t∈ $\mathbb{N}^0$> satisfy the "**weak transversality condition**" $\lim_{T\to\infty} \alpha_T^* x_T = 0$.

Proof: The first four claims comprise proposition 6.1. Hence, let us prove (v**).

We know from (iii**) and (iv**) that for all T$\in\mathbb{N}$, T ≥ 3, <$(\alpha_t^*, \beta_t^*)\in\mathbb{R}\times\mathbb{R}_+$| t = 0, ..., T-1> solves the linear programming minimization problem in (iii**) and $p_2^{(T-1)}u_{T-1}$ + $\sum_{t=1}^{T-1}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \beta_0^*(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1}\beta_t^* a^{(t)}$.

Since, V(x) - $p_1^{(0)}x = \sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \lim_{T\to\infty} p_2^{(T-1)}u_{T-1}$ + $\sum_{t=1}^{T-1}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]$, it must be the case that $\lim_{T\to\infty}[\alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \beta_0^*(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1}\beta_t^* a^{(t)}]$ exists and $\lim_{T\to\infty}[\alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \beta_0^*(a^{(0)} + b^{(0)}x) + \sum_{t=1}^{T-1}\beta_t^* a^{(t)}]$ =V(x) - $p_1^{(0)}$x.

Further, since for all T$\in\mathbb{N}$, T ≥ 3, <$(\alpha_t^*, \beta_t^*)\in\mathbb{R}\times\mathbb{R}_+$| t = 0, ..., T-1> solves the linear programming minimization problem in (iii**), it follows that <(α_t^*, β_t^*)|t$\in\mathbb{N}^0$> solves the implied infinite horizon dual linear programming problem. Q.E.D.

7. Bounded dual variables and implied infinite horizon dual linear programming problem for AC-LOC-LC problem:

Consider the AC-LOC-LC problem <(($p_1^{(t)}$, $p_2^{(t)}$), [$A^{(t)}$ | $B^{(t)}$], ($c^{(t)}$, $d^{(t)}$, $e^{(t)}$))| t$\in\mathbb{N}^0$>.

For x$\in$X, consider the following infinite horizon linear programming problem "implied" by Alt- OPT (as defined in section 3).

Maximize $\sum_{t=1}^{\infty}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}]$, subject to $v_0 \le A_i^{(0)} + B_i^{(0)}x$ for every i$\in$I, $v_t - B_i^{(t)}y_t \le A_i^{(t)}$ for every i$\in$I and t $\in\mathbb{N}$, $y_1 - e^{(0)}v_0 = c^{(0)} + d^{(0)}x$, $y_{t+1} - d^{(t)}y_t - e^{(t)}v_t = c^{(t)}$ for all t $\in\mathbb{N}$, $y_t \ge 0$ for all t $\in\mathbb{N}$, $v_t \ge 0$ for all t $\in\mathbb{N}^0$.

This is the maximization problem that we are really concerned with.

Its "**implied infinite horizon dual linear programming (IDLP) problem**" is the following:

Minimize $\alpha_0(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{\infty}\alpha_t c^{(t)} + \sum_{i=1}^{m}\beta_0^{(i)}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{\infty}\sum_{i=1}^{m}\beta_t^{(i)}A_i^t$,

subject to $\alpha_{t-1} - d^{(t)}\alpha_t - \sum_{i=1}^{m}\beta_t^{(i)}B_i^t \ge p_1^{(t)}$ for all t$\in\mathbb{N}$, $-\alpha_t e^{(t)} + \sum_{i=1}^{m}\beta_t^{(i)} \ge p_2^{(t)}$ for all t $\in\mathbb{N}^0$, $\beta_t^{(i)} \ge 0$ for all i$\in$I and t$\in\mathbb{N}^0$, $\alpha_t\in\mathbb{R}$ for all t$\in\mathbb{N}^0$.

Part 1 of proposition 3.1 states: If $<(x_t, u_t)|t\in \mathbb{N}^0> \in \mathcal{S}(x)$ then for all $T\in \mathbb{N}$ with $T \geq 3$,$<(x_t, u_{t-1})|t = 1, \ldots, T>$ solves the following linear programming problem: Maximize $\sum_{t=1}^{T-1} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] + p_2^{(T-1)} v_{T-1}$, subject to $v_0 \leq A_i^{(0)} + B_i^{(0)}x$ for every $i\epsilon I$, $v_t - B_i^{(t)} y_t \leq A_i^{(t)}$ for every $i\epsilon I$ and $t = 1, \ldots, T-1$, $y_1 - e^{(0)}v_0 = c^{(0)} + d^{(0)}x$, $y_{t+1} - d^{(t)}y_t - e^{(t)}v_t = c^{(t)}$ for all $t = 1, \ldots, T-2$, $- d^{(T-1)}y_{T-1} - e^{(T-1)}v_{T-1} = c^{(T-1)} - x_T$, $y_t \geq 0$ for all $t = 1, \ldots, T-1$, $v_t \geq 0$, for all $t = 0, 1, \ldots, T-1$.

By the weak duality theorem and complementary slackness conditions of linear programming (see topic 2 of Lahiri (2020)), we know that for $T\in \mathbb{N}$ with $T \geq 3$, $<(x_t, u_{t-1})|t = 1, \ldots, T>$ solves the linear programming problem above if and only if there exists an array $<(\alpha_t^{*(T)}, \beta_t^{*(T)})\epsilon\mathbb{R}\times \mathbb{R}_+^m|t = 0,1, 2, \ldots, T-1>$ that along with $<(x_t, u_{t-1})|t = 1, \ldots, T>$ satisfy the following conditions:

(1) $x_1 - e^{(0)}u_0 = c^{(0)} + d^{(0)}x$, $x_1 \geq 0$.

(2) $x_{t+1} - d^{(t)}x_t - e^{(t)}u_t = c^{(t)}$ for all $t = 1, \ldots, T-2$, $x_t \geq 0$ for all $t = 1, \ldots, T-2$.

(3) $- d^{(T-1)}x_{T-1} - e^{(T-1)}u_{T-1} = c^{(T-1)} - x_T$, $x_{T-1} \geq 0$.

(4) $- d^{(T-1)}x_{T-1} - e^{(T-1)}u_{T-1} = c^{(T-1)} - x_T$

(5) $u_0 \leq A_i^{(0)} + B_i^{(0)}x$, $(u_0 - A_i^{(0)} - B_i^{(0)}x)\beta_0^{*(i|T)} = 0$, for every $i\epsilon I$, $u_0 \geq 0$.

(6) $u_t - B_i^{(t)}x_t \leq A_i^{(t)}$, $(u_t - B_i^{(t)}x_t - A_i^{(t)})\beta_t^{*(i|T)} = 0$, for every $i\epsilon I$, $u_t \geq 0$ and $t = 1, \ldots, T-1$.

(7) $\alpha_{t-1}^{*(T)} - d^{(t)}\alpha_t^{*(T)} - \sum_{i=1}^m \beta_t^{*(i|T)} B_i^{(t)} \geq p_1^{(t)}$, $(\alpha_{t-1}^{*(T)} - d^{(t)}\alpha_t^{*(T)} - \sum_{i=1}^m \beta_t^{*(i|T)} B_i^{(t)} - p_1^{(t)})x_t = 0$ for all $t = 1, \ldots, T-1$, $\alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \ldots, T-1$, $\beta_t^{*(i|T)} \geq 0$ for all $i\epsilon I$, $t = 0, 1, \ldots, T-1$.

(8) $\sum_{i=1}^m \beta_t^{*(i|T)} - \alpha_t^{*(T)}e^{(t)} \geq p_2^{(t)}$, $(\sum_{i=1}^m \beta_t^{*(i|T)} - \alpha_t^{*(T)}e^{(t)} - p_2^{(t)})u_t = 0$ for all $t = 0, 1, \ldots, T-1$.

These “eight” conditions are equivalent to the statement: $<\alpha_t^{*(T)}\epsilon\, \mathbb{R}|t = 0, \ldots, T-1>$ and $<\beta_t^{*(i|T)}\epsilon\mathbb{R}_+|i\epsilon I, t = 0, \ldots, T-1>$ solve the dual of the linear programming maximization problem mentioned above, i.e., $<\alpha_t^{*(T)}\epsilon\, \mathbb{R}|t = 0, \ldots, T-1>$ and $<\beta_t^{*(i|T)}\epsilon\mathbb{R}_+|i\epsilon I, t = 0, \ldots, T-1>$ solve the following linear programming minimization problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m} \beta_0^{(i|T)}(A_i^{(0)} + B_i^{(0)}x)$ $+ \sum_{t=1}^{T-1}\sum_{i=1}^{m} \beta_t^{(i|T)} A_i^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \sum_{i=1}^{m} \beta_t^{(i|T)} B_i^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)} e^{(t)} + \sum_{i=1}^{m} \beta_t^{(i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta_t^{(i|T)} \geq 0$ for all i∈I and t = 0, …, T-1, $\alpha_t^{(T)} \in \mathbb{R}$ for all t = 0, …, T-1.

However, the eight conditions above are also satisfied by the arrays $<\alpha_t^{*(T+s)} \in \mathbb{R} | t =$ 0, …, T-1>, $<\beta_t^{*(i|T+s)} | i \in I$, t = 0, …, T-1> for all s∈ℕ.

Thus, for all $s \in \mathbb{N}^0$, $<\alpha_t^{*(T+s)} \in \mathbb{R} | t = 0, \ldots, T-1>$, $<\beta_t^{*(i|T+s)} | i \in I, t = 0, \ldots, T-1>$ solve the minimization problem.

Hence, for all $s \in \mathbb{N}^0$, $\sum_{t=1}^{T-1} [p_1^{(t)} x_t + p_2^{(t-1)} u_{t-1}] + p_2^{(T-1)} u_{T-1} = \alpha_0^{*(T+s)}(c^{(0)} + d^{(0)}x) +$ $\sum_{t=1}^{T-2} \alpha_t^{*(T+s)} c^{(t)} + \alpha_{T-1}^{*(T+s)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m} \beta_0^{*(i|T+s)}(A_i^{(0)} + B_i^{(0)}x) +$ $\sum_{t=1}^{T-1}\sum_{i=1}^{m} \beta_t^{*(i|T+s)} A_i^{(t)}$.

Thus, we arrive at the following theorem.

Theorem 7.1: Let $<((p_1^{(t)}, p_2^{(t)}), [A^t | B^t], (c^t, d^t, e^t)) | t \in \mathbb{N}^0>$ be a AC-LOC-LC problem and suppose that for some x∈X, $<(x_t, u_t) | t \in \mathbb{N}^0> \in \mathcal{S}(x)$. Then for all T∈ ℕ with T ≥ 3, there exists an array $<(\alpha_t^{*(T)}, \beta_t^{*(T)}) \in \mathbb{R} \times \mathbb{R}_+^m | t = 0, \ldots, T-1>$ such that:

(A) For all T∈ ℕ with T ≥ 3 and $s \in \mathbb{N}^0$, $(\alpha_t^{*(T+s)}, \beta_t^{*(T+s)}) \in \mathbb{R} \times \mathbb{R}_+^m$ along with $<x_t | t =$ 1, …, T-1> satisfy the following conditions:

(1) $x_1 - e^{(0)}u_0 = c^{(0)} + d^{(0)}x$, $x_1 \geq 0$.

(2) $x_{t+1} - d^{(t)}x_t - e^{(t)}u_t = c^{(t)}$ for all t = 1, …, T-2, $x_t \geq 0$ for all t = 1, …, T-2.

(3) $- d^{(T-1)}x_{T-1} - e^{(T-1)}u_{T-1} = c^{(T-1)} - x_T$, $x_{T-1} \geq 0$.

(4) $- d^{(T-1)}x_{T-1} - e^{(T-1)}u_{T-1} = c^{(T-1)} - x_T$

(5) $u_0 \leq A_i^{(0)} + B_i^{(0)}x$, $(u_0 - A_i^{(0)} - B_i^{(0)}x)\beta_0^{*(i|T)} = 0$, for every i∈I, $u_0 \geq 0$.

(6) $u_t - B_i^{(t)}x_t \leq A_i^{(t)}$, $(u_t - B_i^{(t)}x_t - A_i^{(t)})\beta_t^{*(i|T)} = 0$, for every i∈I, $u_t \geq 0$ and t = 1, …, T-1.

(7) $\alpha_{t-1}^{*(T)}$ - $d^{(t)}\alpha_t^{*(T)}$ - $\sum_{i=1}^{m(t)}\beta_t^{*(i|T)}B_i^{(t)} \geq p_1^{(t)}$, ($\alpha_{t-1}^{*(T)}$ - $d^{(t)}\alpha_t^{*(T)}$ - $\sum_{i=1}^{m(t)}\beta_t^{*(i|T)}B_i^{(t)}$ - $p_1^{(t)}$)x_t = 0 for all t = 1, …, T-1, $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, …, T-1, $\beta_t^{*(i|T)} \geq 0$ for all i∊I and t = 0, 1, …, T-1.

(8) $\sum_{i=1}^{m(t)}\beta_i^{*(T,t)}$ - $\alpha_t^{*(T)}e^{(t)} \geq p_2^{(t)}$, ($\sum_{i=1}^{m(t)}\beta_i^{*(T,t)}$ - $\alpha_t^{*(T)}e^{(t)}$ - $p_2^{(t)}$)u_t = 0 for all t = 0, 1, …, T-1.

(B) (A) is equivalent to the following statement: For all T∈ℕ with T ≥ 3 and s∊$\mathbb{N}^0$, <($\alpha_t^{*(T)}$, $\beta_t^{*(T)}$)∊$\mathbb{R} \times \mathbb{R}_+^m$|t = 0, …, T-1> solves the linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{(i|T)}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{(i|T)}A_i^{(t)}$, subject to $\alpha_{t-1}^{(T)}$ - $d^{(t)}\alpha_t^{(T)}$ - $\sum_{i=1}^{m}\beta_t^{t(i|T)}B_i^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)}e^{(t)} + \sum_{i=1}^{m}\beta_t^{(i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta_t^{(i|T)} \geq 0$ for all i∊I and t = 0, …, T-1, $\alpha_t^{(T)}$∊ℝ for all t = 0, …, T-1.

Further, $\sum_{t=1}^{T-1}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] + p_2^{(T-1)}u_{T-1} = \alpha_0^{*(T+s)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{*(T+s)}c^{(t)} + \alpha_{T-1}^{*(T+s)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{*(i|T+s)}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{*(i|T+s)}A_i^{(t)}$.

(C) If the set of real numbers {($\alpha_t^{*(T)}$, $\beta_t^{*(T)}$)∊$\mathbb{R} \times \mathbb{R}_+^m$|t = 0, …, T-1 for T∊ℕ, T ≥ 3} is bounded then there exist an array <(α_t^*, β_t^*)∊$\mathbb{R} \times \mathbb{R}_+^m$ |t∊$\mathbb{N}^0$> such that for all T∈ℕ with T ≥ 3, <(α_t^*, β_t^*)∊$\mathbb{R} \times \mathbb{R}_+^m$ |t = 0, 1, …, T-1> solves the linear programming problem in (B).

<u>Further</u>:

<α_t^*∊ℝ|t∊$\mathbb{N}^0$> along with <(x_t, u_t)|t∈$\mathbb{N}^0$> satisfy the "**weak transversality condition**" $\lim_{T\to\infty}\alpha_T^* x_T = 0$, <u>if and only if</u>

(i) <(α_t^*, β_t^*)∊$\mathbb{R} \times \mathbb{R}_+^{m(t)}$ |t∊$\mathbb{N}^0$> solve the "**implied infinite horizon dual linear programming problem**": Minimize $\alpha_0(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{\infty}\alpha_t c^{(t)} + \sum_{i=1}^{m}\beta_0^{(i)}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{\infty}\sum_{i=1}^{m}\beta_t^{(i)}A_i^{(t)}$, subject to α_{t-1} - $d^{(t)}\alpha_t$ - $\sum_{i=1}^{m}\beta_t^{(i)}B_i^{(t)} \geq p_1^{(t)}$ for all t∊ℕ, $-\alpha_t e^{(t)} + \sum_{i=1}^{m}\beta_t^{(i)} \geq p_2^{(t)}$ for all t∊$\mathbb{N}^0$, $\beta_t^{(i)} \geq 0$ for all i∊I and t∊$\mathbb{N}^0$ 0, α_t∊ℝ for all t∊$\mathbb{N}^0$

&

(ii) $V(x) - p_1^{(0)}x = \sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{\infty}\alpha_t^* c^{(t)} + \sum_{i=1}^{m}\beta_0^{*(i)}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{\infty}\sum_{i=1}^{m}\beta_t^{*(i)}A_i^{(t)}$.

Proof: (A) and (B) follow from the discussion preceding the statement of this theorem. Hence, let us prove (C).

Suppose the set $\{(\alpha_t^{*(T)}, \beta_t^{*(T)}) \epsilon \mathbb{R} \times \mathbb{R}_+^m \mid t = 0, \ldots, T-1 \text{ for } T \epsilon \mathbb{N}, T \geq 3\}$ is bounded.

Thus <u>for each $t\epsilon\mathbb{N}^0$</u>, $<(\alpha_t^{*(T)}, \beta_t^{*(T)}) \mid T\epsilon\mathbb{N}, T \geq \max\{3, t+1\}>$ is a bounded sequence in $\mathbb{R} \times \mathbb{R}_+^m$.

Let t = 0 and consider the sequence $<(\alpha_0^{*(T)}, \beta_0^{*(T)}) \mid T\epsilon\mathbb{N}, T \geq 3>$. Since it is a bounded infinite sequence in $\mathbb{R} \times \mathbb{R}_+^m$, by the Bolzano-Weirstass's theorem, there exists a convergent sub-sequence $<(\alpha_0^{*(R^{(0)}(n))}, \beta_0^{*(R^{(0)}(n))}) \mid n\epsilon\mathbb{N}, R^{(0)}(n) \geq 3>$ converging $(\alpha_0^*, \beta_0^*)\epsilon\mathbb{R} \times \mathbb{R}_+^m$.

Consider $< (\alpha_1^{*(R^{(0)}(n))}, \beta_1^{*(R^{(0)}(n))}) \mid n\epsilon\mathbb{N}, R^{(0)}(n) \geq 3 >$ which is a bounded infinite sequence in $\mathbb{R} \times \mathbb{R}_+^m$.

By the Bolzano-Weirstass's theorem, this sequence has a convergent sub-sequence $<(\alpha_1^{*(R^{(1)}(n))}, \beta_1^{*(R^{(1)}(n))}) \mid n\epsilon\mathbb{N}, R^{(1)}(n) \geq 3>$ converging $(\alpha_1^*, \beta_1^*)\epsilon\mathbb{R} \times \mathbb{R}_+^m$.

Further, $<(\alpha_0^{*(R^{(1)}(n))}, \beta_0^{*(R^{(1)}(n))}) \mid n\epsilon\mathbb{N}, R^{(1)}(n) \geq 3>$ is a sub-sequence of $<(\alpha_0^{*(R^{(0)}(n))}, \beta_0^{*(R^{(0)}(n))}) \mid n\epsilon\mathbb{N}, R^{(0)}(n) \geq 3>$ that converges to $(\alpha_0^*, \beta_0^*)\epsilon\mathbb{R} \times \mathbb{R}_+^m$. Hence, $<(\alpha_0^{*(R^{(1)}(n))}, \beta_0^{*(R^{(1)}(n))}) \mid n\epsilon\mathbb{N}, R^{(1)}(n) \geq 3>$ converges to (α_0^*, β_0^*).

Consider $<(\alpha_2^{*(R^{(1)}(n))}, \beta_2^{*(R^{(1)}(n))}) \mid n\epsilon\mathbb{N}, R^{(1)}(n) \geq 3>$ which is a bounded infinite sequence in $\mathbb{R} \times \mathbb{R}_+^m$.

By the Bolzano-Weirstass's theorem, this sequence has convergent sub-sequence $<(\alpha_2^{*(R^{(2)}(n))}, \beta_2^{*(R^{(2)}(n))}) \mid n\epsilon\mathbb{N}, R^{(2)}(n) \geq 3>$ converging to $(\alpha_2^*, \beta_2^*)\epsilon\mathbb{R} \times \mathbb{R}_+^m$.

Further, $<(\alpha_0^{*(R^{(2)}(n))}, \beta_0^{*(R^{(2)}(n))}) \mid n\epsilon\mathbb{N}, R^{(2)}(n) \geq 3>$ is a sub-sequence of $<(\alpha_0^{*(R^{(0)}(n))}, \beta_0^{*(R^{(0)}(n))}) \mid n\epsilon\mathbb{N}, R^{(0)}(n) \geq 3>$ and $<(\alpha_1^{*(R^{(2)}(n))}, \beta_1^{*(R^{(2)}(n))}) \mid n\epsilon\mathbb{N}, R^{(2)}(n) \geq$

3> is a sub-sequences of $<(\alpha_1^{*(R^{(2)}(n))}, \beta_1^{*(R^{(2)}(n))}) | n \epsilon \mathbb{N}$, $R^{(1)}(n) \geq 3>$. Hence, the first sub-sequence converges to (α_0^*, β_0^*) and the second sub-sequence converges to (α_1^*, β_1^*).

Suppose that for $t \epsilon \mathbb{N}$, with $t \geq 2$ and s = 0, 1, 2, …, t, there are bounded sub-sequences $<(\alpha_s^{*(R^{(t)}(n))}, \beta_s^{*(R^{(t)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t)}(n) \geq t+1>$ in $\mathbb{R} \times \mathbb{R}_+^{m(s)}$ such that $<(\alpha_s^{*(R^{(t)}(n))}, \beta_s^{*(R^{(t)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t)}(n) \geq t+1>$ converges to $(\alpha_s^*, \beta_s^*) \epsilon \mathbb{R} \times \mathbb{R}_+^m$.

Consider the sub-sequence $<(\alpha_{t+1}^{*(R^{(t)}(n))}, \beta_{t+1}^{*(R^{(t)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t)}(n) \geq t+1>$ in $\mathbb{R} \times \mathbb{R}_+^m$. Since it is a bounded sequence, by the Bolzano-Weirstass's theorem it has a convergent sub-sequence $<(\alpha_{t+1}^{*(R^{(t+1)}(n))}, \beta_{t+1}^{*(R^{(t+1)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t+1)}(n) \geq t+2>$ in $\mathbb{R} \times \mathbb{R}_+^m$, converging to $(\alpha_{t+1}^*, \beta_{t+1}^*) \epsilon \mathbb{R} \times \mathbb{R}_+^m$.

Since for s = 0, 1, 2, ..., t, $<(\alpha_s^{*(R^{(t+1)}(n))}, \beta_s^{*(R^{(t+1)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t+1)}(n) \geq t+2>$ is a sub-sequence of $<(\alpha_s^{*(R^{(t)}(n))}, \beta_s^{*(R^{(t)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t)}(n) \geq t+1>$, it follows that $<(\alpha_s^{*(R^{(t+1)}(n))}, \beta_s^{*(R^{(t+1)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t+1)}(n) \geq t+2>$ converges to (α_s^*, β_s^*).

Hence, by a standard induction argument it follows that for all $t \epsilon \mathbb{N}^0$ and s = 0, 1, 2, …, t, there exists a sub-sequence $<(\alpha_s^{*(R^{(t)}(n))}, \beta_s^{*(R^{(t)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t)}(n) \geq t+1>$ in $\mathbb{R} \times \mathbb{R}_+^m$ such that $<(\alpha_s^{*(R^{(t)}(n))}, \beta_s^{*(R^{(t)}(n))}) | n \epsilon \mathbb{N}$, $R^{(t)}(n) \geq t+1>$ converges to $(\alpha_s^*, \beta_s^*) \epsilon \mathbb{R} \times \mathbb{R}_+^m$.

Note that for all $T \epsilon \mathbb{N}$ with $T \geq 3$, $(\alpha_{T-1}^*, \beta_{T-1}^*)$ gets defined by the limit of the sub-sequence $<(\alpha_{T-1}^{*(R^{(T-1)}(n))}, \beta_{T-1}^{*(R^{(T-1)}(n))}) | n \epsilon \mathbb{N}$, $R^{(T-1)}(n) \geq T>$ of the sub-sequence $<(\alpha_{T-1}^{*(R^{(T-2)}(n))}, \beta_{T-1}^{*(R^{(T-2)}(n))}) | n \epsilon \mathbb{N}$, $R^{(T-2)}(n) \geq T-1>$.

We know that, for all $T \epsilon \mathbb{N}$ with $T \geq 3$, $<(\alpha_t^{*(R^{(T-1)}(n))}, \beta_t^{*(R^{(T-1)}(n))}) | t = 0, ..., T-1>$ satisfying $R^{(T-1)}(n) \geq T$ for all $n \epsilon \mathbb{N}$ solves the linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T) + \sum_{i=1}^{m} \beta_0^{(i|T)}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1} \sum_{i=1}^{m} \beta_t^{(i|T)} A_i^{(t)}$, subject to $\alpha_{t-1}^{(T)} - d^{(t)}\alpha_t^{(T)} - \sum_{i=1}^{m} \beta_t^{t(i|T)} B_i^{(t)} \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)} e^{(t)} + \sum_{i=1}^{m} \beta_t^{(i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta_t^{(i|T)} \geq 0$ for all $i \epsilon I$ and t = 0, …, T-1, $\alpha_t^{(T)} \epsilon \mathbb{R}$ for all t = 0, …, T-1.

Further, for all T$\epsilon\mathbb{N}$ with T ≥ 3 and n$\epsilon\mathbb{N}$, $\sum_{t=1}^{T-1}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] + p_2^{(T-1)}u_{T-1} = \alpha_0^{*(R^{(T-1)}(n))}(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{*(R^{(T-1)}(n))}c^{(t)} + \alpha_{T-1}^{*(R^{(T-1)}(n))}(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^{*(i|R^{(T-1)}(n))}(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^{*(i|R^{(T-1)}(n))}A_i^{(t)}$.

Since for all t = 0, 1, 2, …, T-1, $\lim_{n\to\infty}(\alpha_t^{*(R^{(T-1)}(n))}, \beta_t^{*(R^{(T-1)}(n))}) = (\alpha_t^*, \beta_t^*)$, it follows that $<(\alpha_t^*, \beta_t^*)|$ t = 0, ..., T-1$>$ satisfies all the constraints of the linear programming minimization problem in (B) and $\sum_{t=1}^{T-1}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] + p_2^{(T-1)}u_{T-1} = \alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^*(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^* A_i^{(t)}$.

Thus, for all T$\epsilon\mathbb{N}$ with T ≥ 3, $<(\alpha_t^*, \beta_t^*)|$ t = 0, ..., T-1$>$ solves the linear programming problem in (B).

To complete the proof of (C), first note that since for all T$\epsilon\mathbb{N}$, T ≥ 3, $<(\alpha_t^*, \beta_t^*)|$ t = 0, ..., T-1$>$ satisfies all the constraints of the linear programming minimization problem in (B), it must be the case that for all T$\epsilon\mathbb{N}$ with T ≥ 3,
$\sum_{t=1}^{T-1}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] + p_2^{(T-1)}u_{T-1} = \alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \sum_{i=1}^{m(0)}\beta_0^*(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m(t)}\beta_t^* A_i^{(t)}$,

<u>and</u> $<(\alpha_t^*, \beta_t^*)\epsilon\mathbb{R}\times\mathbb{R}_+^{m(t)}\ |t\epsilon\mathbb{N}^0>$ *satisfies all the constraints of the implied infinite horizon dual linear programming problem.*

We know that $\lim_{T\to\infty}\sum_{t=1}^{T-1}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] + p_2^{(T-1)}u_{T-1} = \sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]$ $= V(x) - p_1^{(0)}x$.

Thus, $\lim_{T\to\infty}[\alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^*(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^* A_i^{(t)}]$ exists and is equal to $\sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = V(x) - p_1^{(0)}x$.

However, $\lim_{T\to\infty}[\alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) + \sum_{i=1}^{m}\beta_0^*(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^* A_i^{(t)}] = \lim_{T\to\infty}[\alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-1}\alpha_t^* c^{(t)} ++ \sum_{i=1}^{m}\beta_0^*(A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1}\sum_{i=1}^{m}\beta_t^* A_i^{(t)} - \alpha_{T-1}^* x_T]$.

Thus, V(x) - $p_1^{(0)}x = \lim_{T\to\infty} [\alpha_0^*(c^{(0)} + d^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} + \sum_{i=1}^{m} \beta_0^* (A_i^{(0)} + B_i^{(0)}x) + \sum_{t=1}^{T-1} \sum_{i=1}^{m} \beta_t^* A_i^{(t)}]$ <u>if and only if</u> $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$.

Further, since for all $T \in \mathbb{N}$ with $T \geq 3$, $<(\alpha_t^*, \beta_t^*)|$ t = 0, ..., T-1$>$ solves the linear programming problem in (B), $<(\alpha_t^*, \beta_t^*)| t \in \mathbb{N}^0>$ solves the infinite horizon linear programming minimization problem in (D) <u>if and only if</u> $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$. Q.E.D.

8. Conclusion:

While concluding this paper, the first point that needs to be noted is that in the models discussed in Hopkins (1969), Grinold (1971), Evers (1973), Grinold (1977), Romeijn, Smith and Bean (1992), Romeijn and Smith (1998), the state and control variables may be multi-dimensional, while we work with one-dimensional control variables and one-dimensional state variables. Unlike these models, we explicitly distinguish between state variable and control variable, with both contributing to the objective function. Further, the dynamics of our state variable is governed by a first order linear difference equation which depends on the values of both the state variable and the control variable in the previous period, while our control variable at each time period is chosen from an interval whose left-hand end point is zero and the right-hand end point is determined by a continuous, piece-wise affine and concave function of the value of the state variable in that period. Thus, while our framework of analysis is less general than those mentioned above as far as dimensions of the variables are concerned, it allows for considerably greater clarity as far as modeling the underlying dynamics of the infinite horizon linear programming problem.

The model in Grinold (1971), allows the choice set for the variables at any point in time, to depend on chosen values of the variables in all previous periods. In the other models the choice set for the variables at any point in time depends on chosen values of the variables in the immediately previous period and no further back. Further the parameters defining the constraints in the model due to Evers (1973), are invariant over time. The model we work with here, allows the parameters defining the constraints to vary with the time period.

Another point that we wish to note is about finite horizon approximations of the infinite horizon optimization problem that we are concerned with. A notable precedent in this respect is the infinite horizon linear programming model in Grinold (1977), which in our context would resemble something like the following:

Given a non-zero real numbers p, q a real number $\delta \in (0, 1)$ and $x \in X$: Maximize $\sum_{t=0}^{\infty} \delta^t (px_t + qu_t)$ subject to $u_t - x_t \leq 0$, $x_{t+1} = c^{(t)} + d^{(t)}x_t + e^{(t)}u_t$ for all $t \in \mathbb{N}^0$, $x_0 = x$, $x_t \geq 0$ for all $t \in \mathbb{N}$, $u_t \geq 0$ for all $t \in \mathbb{N}^0$.

In our framework of analysis a suitable finite horizon approximation for $T \in \mathbb{N}$ with T sufficiently large may be formulated as follows:

Given $x \in X$, and $q_i^{(T)} = \sum_{t=T}^{\infty} p_i^{(t)}$, $i \in \{1, 2\}$, Maximize $\sum_{t=0}^{T-1} [p_1^{(t)} x_t + p_2^{(t)} u_t] + q_1^{(T)} x_T + q_2^{(t)} u_T$ subject to $u_t \leq A_i^{(t)} + B_i^{(t)} x_t$, $i \in I$, t = 0, 1, …, T, $x_{t+1} = c^t + d^t x_t + e^t u_t$, t = 0, …, T - 1, $x_0 = x$, $u_t \leq b$, $u_t \geq 0$ for all $t \in \{0, 1, …, T\}$, $x_t \geq 0$ for all $t \in \{1, …, T\}$.

The implicit assumption in such an approximation is that after a sufficiently long period of time, the state and control variables may be assumed to remain constant, without causing much disparity between the infinite horizon model and its finite horizon approximation. For such an interpretation to be "perfectly consistent" with our AC-LOC-LC problem, we need to impose the additional constraints $u_T \leq A_i^t + B_i^t x_T$, $i \in I$, $t \in \mathbb{N}$, $t \geq T+1$, $x_T = c^t + d^t x_T + e^t u_T$, $t \in \mathbb{N}$, $t \geq T$.

Gunter and Bender (1980) discuss at length other approaches to the "finite approximation" problem.

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