Infinite Horizon Linear Optimal Control with Linear Constraints

by

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November 30, 2025.

Last date on which typos were corrected: December 24, 2025.

Abstract

We define infinite horizon linear optimal control problem with linear constraints. We provide a necessary condition for an optimal trajectory in terms of an infinite sequence of linear programming problems. We also provide a similar sufficient condition for optimality in terms of a related infinite sequence of linear programming problems. We define a “bang-bang sequence of decision rules”, and provide sufficient conditions for the existence of a unique optimal trajectory that is generated by such a sequence of decision rules. We also provide a “robust” approximation result in terms of a linear programming problem with a sufficiently long time horizon. We use the strong duality theorem and complementary slackness condition of linear programming to obtain necessary conditions for an optimal trajectory. These necessary conditions lead to a very general “transversality condition”, the satisfaction of which is a characteristic feature of optimal trajectories in infinite horizon optimization. A more compact transversality condition is realized when along an optimal trajectory the control variable is “eventually” strictly positive. Under suitable assumptions we prove that there is a infinite horizon “implied dual linear programming problem” which has a solution and which along with the optimal trajectory satisfies the complementary slackness conditions. Further, the optimal value of the implied dual linear programming problem is equal to the optimal value of the maximization problem that gives rise to it. We obtain sufficient conditions for a trajectory to be an optimal trajectory by using the strong duality theorem and complementary slackness condition of linear programming.

Keywords: infinite horizon linear programming, linear optimal control, linear constraints, duality, infinite horizon dual linear programming problem

AMS Subject Classification: 90C05, 90C46

JEL Codes: C44, C61, C62.

1. Introduction:

The earliest work on infinite horizon linear programming that is known to us is the 1973 Ph.d thesis of Joseph J. M. Evers, hereafter referred to as Evers (1973). We pursue a similar line of work in the present paper.

In Lahiri (2025c) there is a discussion of infinite horizon linear optimal control, with one state variable and one control variable. In that model, the objective function is assumed to be linear, without such restrictions being imposed on the constraints. In our work that follows, we assume that that there is one state variable and one control variable with the dynamics of the state variable determined by a first order linear difference equation and the constraint set for the control variable is bounded above by a finite set of affine functions of the state variable.

Evers (1973) allows for multi-dimensional state and control variables but the parameters defining the constraints are invariant over time. The model we work with here, allows the parameters defining the constraints to vary with the time period. Although, Evers (1973) does not explicitly allow for control variables, viewed from the perspective of our model, the control variables are the "slack variables" that are required to express the dynamics of the state variables as linear equations. The control variables implied by the model in Evers (1973), do not contribute to the objective function. Thus, while our framework of analysis is less general than that of Evers (1973) as far as dimensions of the variables are concerned, it allows for considerably greater flexibility as far as modeling the underlying dynamics of the infinite horizon linear programming problem.

There is one important difference between infinite horizon linear programming and optimization in infinite dimensional spaces that ought to be taken note of. Unlike optimization in infinite dimensional spaces that can accommodate both optimization with infinite number of variables and a finite number of constraints and optimization with a finite number of variables and infinitely many constraints, there is no possibility of meaningfully accommodating such "half-way houses" in infinite horizon linear programming. Whether, that makes infinite horizon linear programming restrictive or more useful than infinite dimensional optimization is irrelevant, since chapter 1 of Evers (1973) discusses four "growth models" which can be modeled as infinite horizon linear programming problems.

The work reported here is similar in nature to chapter 2 of Evers (1973). In section 2 we present the framework of analysis in which both state variable and control variable are restricted to belong to the same closed and bounded interval of real numbers containing zero. We define infinite horizon linear optimal control problem with linear constraints - hereafter referred to as linear optimal control problem with linear constraints- as well as a convenient special case case in which the there is only one affine function of the state variable that is an upper bound for the control variable. In section 3, we introduce the concept of optimality and provide a necessary condition for an optimal trajectory in terms of an infinite sequence of linear programming problems. In the same section, we also provide a similar sufficient condition for optimality in terms of a related infinite sequence of linear programming problems. Also included in this section, is a proposition that provides a set of sufficient conditions under which all feasible trajectories are optimal trajectories.

In section 4, we introduce absolutely convergent linear optimal control problems with linear constraints for which it is well known that the set of optimal solutions is always non-empty. We define a "bang-bang sequence of decision rules", and provide sufficient conditions for the existence of a unique optimal trajectory that is generated by such a sequence of decision rules. We also provide a “robust” approximation result (proposition 4.2) in terms of a linear programming problem with a sufficiently long time horizon. This result is quite plausible and very likely well known in the existing literature on infinite horizon linear programming..

In addition to proposition 4.2, somewhat significant results are discussed in sections sections 5 and 6 of this paper. In section 5, we use the strong duality theorem and complementary slackness condition of linear programming to obtain necessary conditions for an optimal trajectory. These necessary conditions lead to a very general “transversality condition”, the satisfaction of which is a characteristic feature of optimal trajectories in infinite horizon optimization. A more compact transversality condition is realized when along an optimal trajectory the control variable is “eventually” strictly positive for ever. A more striking result is the one in section 6, under the assumptions in section 5 plus the assumption that the control variable affects the evolution of the state variable at each and every time period. We refer to a trajectory in which both state and control variables are positive at every period and the control variable is always strictly less than its upper bound as “interiority condition”. We show that if there exists an optimal trajectory satisfying interiority condition, then there is a infinite horizon “implied dual linear programming problem” which has a solution and which along with the optimal trajectory satisfies the complementary slackness conditions. Further, the optimal value of the implied dual linear programming problem is equal to the optimal value of the maximization problem that gives rise to it. It is important to note, that this result depends on the initial value, not only because the initial value is required to be strictly positive, but also because the solution arising from it needs to satisfy the “interiority condition”.

In section 7, we once again use the strong duality theorem and complementary slackness condition of linear programming to obtain sufficient conditions for a trajectory to be an optimal trajectory.

Work related to, but different from what is presented here- at the very least as far as results are concerned- is available in Romeijn, Smith and Bean (1992). Our approach to infinite horizon linear programming via truncated and finite horizon versions of the original problem is akin to the approach adopted in the paper that we have just cited.

In a concluding section of this paper, we compare our model with other notable infinite horizon linear programming models that already exist in the literature, such as the ones due to Evers (1973), Grinold (1977) and Romeijn, Smith and Bean (1992). While conceding that unlike our model, these other models allow multi-dimensional variables, we point out that our model is conceptually more general, since it explicitly distinguishes between control and state variables, although each of the two are assumed to be one-dimensional in every time period.

2. Framework of Analysis:

Let $X = [0, b] \subset \mathbb{R}$ (the set of real numbers), with $b > 0$ be such that **set of available alternatives** at any time period is a non-empty subset of $X \times X$. Given a current realization $x \in X$ of the state variable that was chosen in the immediately previous time-period, a typical alternative that is "chosen" during the current period is an ordered pair (u, y)$\in$X×X, where u is the value of the control variable chosen for the current period and 'y' is the value of the state variable that will be realized (as an inheritance) in the immediately next period. Based on the realization (x, u) during the current period an instantaneous pay-off is realized by the decision maker.

With $\mathbb{N}$ denoting the set of natural number (i.e., the set of strictly positive integers) let $\mathbb{N}^0$ denote $\mathbb{N}\cup\{0\}$, i.e., the set of non-negative integers. Time is measured in discrete periods $t\in\mathbb{N}^0$. Beginning with an initial state variable, in each period $t\in\mathbb{N}^0$, an alternative (state variable-control variable pair) is realized, and the chosen alternative is denoted by $(x_t, u_t)\in X\times X$. While at all time periods x_t is an "inheritance" in the current period, u_t is chosen during the current period.

At each time-period $t\in\mathbb{N}^0$, $\Omega_t \subset X\times X\times X$ is the **two-period constraint set at time-period** t, such that for all $t\in\mathbb{N}^0$, there exists a matrix $[A^t \mid B^t]$ with a finite number of rows and two-columns and a point $(c^t, d^t, e^t)\in\mathbb{R}^3$ satisfying the following properties:

(i) For all $t\in\mathbb{N}^0$, $A_i^t \geq 0$ and $A_i^t + B_i^t b \geq 0$ for all i where the ordered pair $(A_i^t|B_i^t)$ is the i^{th} row of the matrix $[A^t \mid B^t]$.

(ii) For all $t\in\mathbb{N}^0$ and $(x, u)\in X\times X$, $c^t + d^t x + e^t u \in [0, b]$.

(iii) For all $t\in\mathbb{N}^0$, $\Omega_t = \{(x, u, y)\in X\times X\times X \mid u \leq A_i^t + B_i^t x$ where for every i the ordered pair $(A_i^t|B_i^t)$ is the i^{th} row of the matrix $[A^t \mid B^t]$ and $y = c^t + d^t x + e^t u\}$.

By (i), $A_i^t + B_i^t x \geq 0$ for all $(x, t)\in X\times\mathbb{N}^0$.

Thus, by (i) and (ii) it follows that for all $(x, t)\in X\times\mathbb{N}^0$, $\{(u, y)\in X\times X \mid (x, u, y)\in\Omega_t\} \neq \phi$ and hence for all $t\in\mathbb{N}^0$, $\Omega_t \neq \phi$.

Note 2.1: The number of rows of the matrix $[A^t \mid B^t]$ may vary with the time-period. For $t\in\mathbb{N}^0$, let $I(t) > 0$ denote the number number of rows of the matrix $[A^t \mid B^t]$, so that for $i\in\{1, \ldots, I(t)\}$, the ordered pair $(A_i^t|B_i^t)$ is the i^{th} row of $[A^t \mid B^t]$. However for the sake of simplicity, for $x\in X$, we will sometimes denote the set $\{A_i^t + B_i^t x \mid i = 1, \ldots, I(t)\}$ by $\{A_i^t + B_i^t x\}$. Further, in the context of the matrix $[A^t \mid B^t]$ instead of writing "for all $i\in\{1, \ldots, I(t)\}$ and $t\in\mathbb{H}$" (where $\mathbb{H}$ is any non-empty subset of $\mathbb{N}^0$) we shall sometimes write "for all i and $t\in\mathbb{H}$".

For $t\in\mathbb{N}^0$, $(x, u, y)\in\Omega_t$ can be interpreted in the following manner: given that $x\in X$ is the realization of the state variable at time-period t, it is possible to choose the pair (u, y)$\in$X×X at time-period t.

For all $(x, t)\in X\times\mathbb{N}^0$, let $\Omega_t(x) = \{(u, y)\in X\times X \mid (x, u, y)\in \Omega_t\} = \{(u, c^t + d^t x + e^t u) \mid u \geq 0, u \leq \min\{b, \min\{A_i^t + B_i^t x\}\}\}$.

Clearly for all $(x, t)\in X\times\mathbb{N}^0$, $\Omega_t(x)$ is a non-empty and closed subset of X×X.

For $x\in X$, let $\mathcal{F}(x) = \{<(x_t, u_t)|t\in\mathbb{N}^0> \mid (x_t, u_t, x_{t+1})\in\Omega_t, t\in\mathbb{N}^0, x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$ as a **trajectory starting at (from)** x.

Clearly, $\mathcal{F}$(x) is non-empty for all x∈X.

Let <($p_1^{(t)}$, $p_2^{(t)}$) | t∈ $\mathbb{N}^0$> be a sequence of pairs of real numbers. If x is the realization of the state variable at time-period t and u is the choice of the control variable at time period 't', the instantaneous pay-off received by the decision-maker at time period 't' is $p_1^{(t)}$x + $p_2^{(t)}$u.

We shall refer to the array <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> or alternatively <(($p_1^{(t)}$, $p_2^{(t)}$), [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> as a **(infinite horizon) linear optimal control problem with linear constraints (LOC-LC problem).**

Another special case of a LOC-LC problem is one when at each time period there is a single upper-bound on the control variable.

Suppose I(t) = 1 for all t∈$\mathbb{N}^0$ and for all t∈$\mathbb{N}^0$, there exists real numbers a^t, b^t such that $a_t + b^t x \geq 0$ for all x∈[0, b].

For each t∈$\mathbb{N}^0$, let Ω_t = {(u, x, y)∈X×X×X| u ≤ $a^t + b^t x$ and y = $c^t + d^t x + e^t u$}.

In this case the array <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> may be represented as <(($p_1^{(t)}$, $p_2^{(t)}$), [a^t| b^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> and referred to as a **one-plus-one linear optimal control problem with linear constraints (1+1-LOC-LC problem).**

Note 2.2: If for some t∈ $\mathbb{N}^0$, $e^t \neq 0$, then for (x, y, u)∈Ω_t, it must be the case that u = $\frac{y-c_t-d_t x}{e_t}$, so that $p_1^{(t)}$x + $p_2^{(t)}$u = $p_1^{(t)}$x + $p_2^{(t)}$($\frac{y-c_t-d_t x}{e_t}$). Thus, the instantaneous pay-off at time t, can be expressed entirely in terms of the state variables. On the other hand, if for some t∈ $\mathbb{N}^0$, $e^t = 0$, then for a given value x of the state variable at time period t, there may be several values of the control variable that can chosen at time-period t. Thus, the model of linear optimal control discussed here is a generalization of the linear dynamic optimization model discussed in Lahiri (2025a, 2025b), the latter being motivated by the reduced form model in Mitra (2000) and Sorger (2015).

For what follows we assume that <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈$\mathbb{N}^0$> or <(($p_1^{(t)}$, $p_2^{(t)}$), [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> is a given LOC-LC problem. As and when necessary, we will impose additional assumptions on this LOC-LC problem.

u ≤ $A_i^t + B_i^t$x for every i, y = $c^t + d^t x + e^t u$ and $d^t \neq 0$ implies u ≤ $A_i^t + B_i^t(\frac{y-c^t-e^t u}{d^t})$ for every i, i.e., u ≤ $\alpha_i^t + \beta_i^t$y for every i, where for every i, $\alpha_i^t = \frac{A_i^t - B_i^t \frac{c^t}{d^t}}{1-B_i^t \frac{e^t}{d^t}}$ and $\beta_i^t = \frac{\frac{B_i^t}{d^t}}{1-B_i^t \frac{e^t}{d^t}}$.
However, since y = $c^t + d^t x + e^t u$, the chosen pair (u, y) depends on x. On the other hand, if $d_t = 0$, then y = $c^t + e^t$ u, and so x does not play any role in determining y. None the less, x figures in the upper bound min{$A_i^t + B_i^t$x} on the choice of u.

In the case of a LOC-LC problem it is easy to see that, for all $t\in\mathbb{N}^0$, Ω_t is a non-empty, closed and "convex" subset of X×X×X. Thus, in the case of a LOC-LC, for all $x\in X$, $\mathcal{F}(x)$ is a non-empty and convex set.

If $e^t = 0$ for all $t\in\mathbb{N}^0$, then for all $x\in X$: $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$ if and only if $u_t \leq \min\{a^t + b^t x_t, b\}$ and $x_{t+1} = c^t + d^t x_t$ for all $t\in\mathbb{N}^0$.

Note 2.3: The infinite horizon linear programming model in Evers (1973) allows the state and control variables- the latter appearing as slack variables- to be multi-dimensional. If we restrict the state and control variables to be one dimensional, then the framework of analysis in Evers (1973) "suitably adapted to the one we are concerned with" would require: I(t) = 1, $A_1^t = b$, $B_1^t = 0$, $d^t = d^{t+1}$, $e^t = e^{t+1} \neq 0$, $p_2^{(t)} = 0$ for all $t\in\mathbb{N}^0$. Such a formulation may be referred to as the "**Evers model**" in our framework. This model is an example of a 1+1-LOC-LC problem.

3. Optimality and associated linear programming problems:

We will now consider the following optimization problem denoted **OPT**:

Given $x\in X$, Maximize $\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$ subject to the infinite sequence satisfying the constraints: $(x_t, u_t, x_{t+1})\in\Omega_t$, $t\in\mathbb{N}^0$, $x_0 = x$.

Note 3.1: The exact mathematical interpretation of the expression (formula) $\sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$ is $\lim_{T\to\infty}(\sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t])$. Thus, the problem we are concerned with here is in the domain of asymptotic analysis, which is very different from infinite dimensional analysis.

An **alternative and very useful version of OPT** is the following optimization problem:

Given $x\in X$, Maximize $\sum_{t=1}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_{t-1}]$ subject to the infinite sequence satisfying the constraints: $(x_t, u_t, x_{t+1})\in\Omega_t$, $t\in\mathbb{N}^0$, $x_0 = x$.

Let $\mathcal{S}(x) = \{<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)|\ \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] \geq \sum_{t=0}^{\infty}[p_1^{(t)}y_t + p_2^{(t)}v_t]$ for all $<(y_t, v_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)\}$, i.e., $\mathcal{S}(x) = \underset{<(x_t,u_t)|\ t\in\mathbb{N}^0>\in\mathcal{F}(x)}{\operatorname{argmax}} \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t]$..

$\mathcal{S}(x)$ is the **set of solutions starting from** x **for** OPT.

For all $T\in\mathbb{N}^0$, and $x\in X$, let $\mathcal{F}^T(x) = \{<(x_t, u_t)\ |t \geq T>|\ (x_t, u_t, x_{t+1})\in\Omega_t$ for all $t \geq T$ and $x_T = x\}$.

For $T\in\mathbb{N}^0$ and $y\in X$, $<(x_t, u_t)|t \geq T>\in\mathcal{F}^T(x)$ may be referred to as a **trajectory starting at (from)** x **at time-period** T.

It is easy to see that for all $T\in\mathbb{N}^0$ and $x\in X$, $\mathcal{F}^T(x)$ is non-empty.

Given $(x, T)\in X\times\mathbb{N}^0$, we will denote the following optimization problem by **OPT-T**:

Maximize $\sum_{t=T}^{\infty}[p_1^{(t)} y_t + p_2^{(t)} v_t]$ subject to $<(y_t, v_t)|\ t \geq T> \in \mathcal{F}^T(x)$.

OPT-T can also be expressed as follows:

Given (x, T)∈ X×$\mathbb{N}^0$: Maximize $\sum_{t=T+1}^{\infty}[p_1^{(t)} y_t + p_2^{(t)} v_{t-1}]$ subject to $<(y_t, v_t)|\ t \geq T> \in \mathcal{F}^T(x)$.

For (x, T)∈ X×$\mathbb{N}^0$, $\mathcal{S}^T(x)$ is the **set of solutions starting from** x **for** OPT-T, i.e., $\mathcal{S}^T(x)$ $= \underset{<(x_t,u_t)|\ t\geq T>\in\mathcal{F}^T(x)}{\text{argmax}} \sum_{t=T}^{\infty}[p_1^{(t)} x_t + p_2^{(t)} u_t]$.

Clearly, $\mathcal{F}^0(x) = \mathcal{F}(x)$ and $\mathcal{S}^0(x) = \mathcal{S}(x)$ for all x∈ X.

For all T∈ $\mathbb{N}^0$ the correspondence h^T: X →→X×X defined by $h^T(x) = \{(u_T, x_{T+1})|<(x_t,u_t)|\ t\geq T>\in\mathcal{S}^T(x)$ is said to be the **optimal period-T decision rule**.

We now provide one necessary condition and a somewhat stronger sufficient condition for optimality for an LOC-LC in terms of linear programming problems.

Proposition 3.1: Let $<((p_1^{(t)}, p_2^{(t)}), [A^t \mid B^t], (c^t,d^t,e^t))|\ t\in \mathbb{N}^0>$ be an LOC-LC problem and suppose that for some x∈X, $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{F}(x)$.

Part 1: <u>If</u> $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$ <u>then</u> for all T∈ ℕ, $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves the following linear programming problem: Maximize $\sum_{t=0}^{T}[p_1^{(t)} y_t + p_2^{(t)} v_t]$, subject to $v_t \leq A_i^t + B_i^t y_t$ for every i and $v_t \leq b$ for all t = 0, 1, …, T, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T, <u>&</u> both $y_T = x_T$, $v_T = u_T$.

Part 2: <u>If</u> there exists $T^* \in \mathbb{N}$ such that for all T∈ ℕ satisfying $T \geq T^*$, $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves the following linear programming problem: Maximize $\sum_{t=0}^{T}[p_1^{(t)} y_t + p_2^{(t)} v_t]$, subject to $v_t \leq A_i^t + B_i^t y_t$ for every i and $v_t \leq b$ for all t = 0, 1, …, T, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T, <u>then</u> $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$.

Proof: Part 1: Suppose $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$ and towards a contradiction suppose that for some T∈ ℕ,, there exists $<(y_t, v_t)|t = 0, 1, \ldots, T>$ such that

(i) $v_t \leq A_i^t + B_i^t y_t$ for every i and $v_t \leq b$ for all t = 0, 1, …, T,

(ii) $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1,

(iii) $y_0 = x_0 = x$, $y_T = x_T$, $v_T = u_T$,

(iv) $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T, and

(v) $\sum_{t=0}^{T}[p_1^{(t)} y_t + p_2^{(t)} v_t] > \sum_{t=0}^{T}[p_1^{(t)} x_t + p_2^{(t)} u_t]$.

Let $<(z_t, w_t)|t\in \mathbb{N}^0>$ be such that that $(z_t, w_t) = (y_t, v_t)$ for all t = 0, 1, …, T, $(z_t, w_t) = (x_t, u_t)$ for all t > T.

Since $y_0 = x_0 = x$, $y_T = x_T$, $v_T = u_T$, it is easily verified that $<(z_t, w_t)|t\in \mathbb{N}^0> \in \mathcal{F}(x)$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t+p_2^{(t)}w_t] = \sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t] + \sum_{t=T+1}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t] >$ $\sum_{t=0}^{T}[p_1^{(t)}x_t+p_2^{(t)}u_t] + \sum_{t=T+1}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t] = \sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t]$.

This contradicts, $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$ and proves Part 1.

Part 2: Suppose that there exists $T^* \in \mathbb{N}$ such that for all $T\in\mathbb{N}$ satisfying $T \geq T^*$, $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves the linear programming problem: Maximize $\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t]$, subject to $v_t \leq A_i^t + B_i^t y_t$ for every i and $v_t \leq b$ for all t = 0, 1, …, T, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T. Towards a contradiction suppose that $<(x_t, u_t)|t\in\mathbb{N}^0>\notin S(x)$.

Thus, there exists $<(y_t, v_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$ such that $\sum_{t=0}^{\infty}[p_1^{(t)}y_t+p_2^{(t)}v_t] >$ $\sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t]$.

By note 3.1, $\lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t] = \sum_{t=0}^{\infty}[p_1^{(t)}y_t+p_2^{(t)}v_t] > \sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t] =$ $\lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}x_t+p_2^{(t)}u_t]$.

Thus, there exists $T^0\in\mathbb{N}$, such that for all $T\in\mathbb{N}$ satisfying $T \geq T^0$, it must be the case that $\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t] > \sum_{t=0}^{T}[p_1^{(t)}x_t+p_2^{(t)}u_t]$.

Let $T^{**} = \max\{T^*, T^0\}$.

Since $<(y_t, v_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)$, $v_t \leq A_i^t + B_i^t y_t$ for every $i\in\{1, \ldots, I(t)\}$ and $v_t \leq b$ for all t = 0, 1, …, T^{**}, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T^{**}-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T^{**}.

This contradicts our assumption that $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves the linear programming problems in the statement of Part 2 of this proposition and thus proves Part 2. Q.E.D.

Note 3.2: In one way the sufficient condition is stronger than the necessary condition because the constraints on the terminal values of the state and control variables in the linear programming problem in the necessary condition (i.e., in part 1) are "absent" from the constraints in the linear programming problem in the sufficient condition (i.e., in part 2). Since, the evolution of the state variable is determined by an equation, if the terminal period for linear programming problem in part 1, is T, then the value of the state variable in period T + 1 has to be x_{T+1} and this also determines the constraints for the control variable that needs to be chosen in period T+1. Further, the requirement in part 1 that $x_T = y_T = c_{T-1} + d_{T-1}y_{T-1} + e_{T-1}v_{T-1}$, does impose "some restriction" on the choice of y_{T-1}. None of these restrictions apply for the linear programming problem in part 2. At the same time, the conditions in part 2 are required to hold "eventually", where as the conditions in part 1 hold from period 1 onwards.

We close this section with an interesting observation.

Proposition 3.2: Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> is an LOC-LC and let x∈X. If $\mathcal{S}$(x) ≠ ϕ, d^t = 0 and for some α∈ℝ\{0}. e^t = α, $p_2^{(t)} = -\alpha p_1^{(t+1)}$ for all t∈ $\mathbb{N}^0$, then $\mathcal{S}$(x) = $\mathcal{F}$(x).

Proof: For all <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈$\mathcal{F}$(x), $\sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t] = p_1^{(0)}$x + $\sum_{t=0}^{\infty}[p_1^{(t+1)}x_{t+1}+p_2^{(t)}u_t] = p_1^{(0)}$x + $\sum_{t=0}^{\infty}[p_1^{(t+1)}x_{t+1}-\alpha p_1^{(t+1)}u_t] = p_1^{(0)}$x + $\sum_{t=0}^{\infty}p_1^{(t+1)}[x_{t+1}-\alpha u_t] = p_1^{(0)}$x + $\sum_{t=0}^{\infty}p_1^{(t+1)}c_t$, since $p_2^{(t)}=-\alpha p_1^{(t+1)}$ and x_{t+1} - αu_t = c_t for all t∈ $\mathbb{N}^0$.

We have assumed that $\mathcal{S}$(x) ≠ ϕ. Further, $\mathcal{S}$(x) ⊂ $\mathcal{F}$(x).

Since, $\sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t] = p_1^{(0)}$x + $\sum_{t=0}^{\infty}p_1^{(t+1)}c_t$ for all <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈$\mathcal{F}$(x), it follows that $\mathcal{S}$(x) = $\mathcal{F}$(x). Q.E.D.

4. Existence of optimal solution, bang-bang sequence of decision rules and a "robust" approximation result:

As in Mitra (2000), Sorger (2015) and Lahiri (2025a) (among numerous others) we will, in the rest of this paper be concerned with an optimality criterion that requires the following "Absolute Convergence" condition.

The LOC-LC problem <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> is said to satisfy **Absolute Convergence** if for i∈{1,2}, $\sum_{t=0}^{\infty}|p_i^{(t)}| < +\infty$.

If <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> satisfies Absolute Convergence, then we may refer to <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> as an **Absolutely Convergent LOC-LC (AC-LOC-LC) problem**.

If <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> is an A**bsolutely Convergent 1+1-LOC-LC**, then we will refer to it as an **AC-1+1-LOC-LC**.

Let <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> be an AC-LOC-LC problem. Thus, for all sequence <(x_t, u_t)|t∈ $\mathbb{N}^0$> with (x_t, u_t)∈X×X for all t∈ $\mathbb{N}^0$, it must be the case that $\lim_{t\to\infty}|p_1^{(t)}x_t| = 0$, $\lim_{t\to\infty}|p_2^{(t)}u_t| = 0$, $\sum_{t=0}^{\infty}|p_1^{(t)}x_t| \in [0, \text{b}\sum_{t=0}^{\infty}|p_1^{(t)}|]$ and $\sum_{t=0}^{\infty}|p_1^{(t)}u_t| \in [0, \text{b}\sum_{t=0}^{\infty}|p_2^{(t)}|]$.

Let M = max{$\text{b}\sum_{t=0}^{\infty}|p_1^{(t)}|$, $\text{b}\sum_{t=0}^{\infty}|p_2^{(t)}|$} < +∞.

Thus, for all sequence <(x_t, u_t)|t∈ $\mathbb{N}^0$> with (x_t, u_t)∈X×X for all t∈ $\mathbb{N}^0$, it must be the case that $|\sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t]| \leq \sum_{t=0}^{\infty}|p_1^{(t)}x_t+p_2^{(t)}u_t| \leq \sum_{t=0}^{\infty}[|p_1^{(t)}x_t|+|p_2^{(t)}u_t|] \leq 2\text{M}$.

The following results is immediate consequences of a corresponding result (proposition 5.1) in a more general framework, available in Lahiri (2025c).

Proposition 4.1: Let <(($p_1^{(t)}$, $p_2^{(t)}$), Ω_t)| t∈ $\mathbb{N}^0$> be an AC-LOC-LC problem. $\mathcal{S}^T$(x) ≠ ϕ for all (x, T)∈ X×$\mathbb{N}^0$. Hence, the optimal period-T decision rule h^T is non-empty valued for all T∈ $\mathbb{N}^0$.

The following definition is based on pages 202–208 in Kamien and Schwartz (1991).

For T∈$\mathbb{N}^0$, the array <h^t| t ≥ T> of decision rules is said to be a **bang-bang sequence of decision rules** if for all (x, t)∈X×$\mathbb{N}^0$ satisfying t ≥ T and $p_2^{(t)}$ ≠ 0, h^t(x) = (max{$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{b, min{A_i^t + B_i^tx}}, 0}, c^t + d^tx).

An immediate and interesting consequence of proposition 4.1 is the following corollary.

Corollary 1 of proposition 4.1: Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> is an AC-LOC-LC. If for some T∈ $\mathbb{N}^0$ it is the case that e^t = 0 and $p_2^{(t)}$≠ 0 for all t ≥ T, then, for all x∈ X: <(x_t, u_t)|t ≥ T>∈ $\mathcal{S}^T(x)$ if and only if for all t ≥ T: x_{t+1} = c^t + $d^t x_t$ and u_t = max{$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{b, min{A_i^t + $B_i^t x_t$}},0}. Thus, for all t ≥ T and x∈ X, it must be that h^t(x) = (max {$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{b,{A_i^t + B_i^tx}}, 0}, c^t + d^tx), so that <h^t| t ≥ T> is a bang-bang sequence of decision rules.

Proof: We know from proposition 4.1 that for all x∈ X, $\mathcal{S}^T(x)$ ≠ ϕ. Recall that for all (x, t)∈ X×$\mathbb{N}^0$, A_i^t+$B_i^t x$ ≥ 0 for all i, and e_t = 0 for all t ≥ T implies, x_{t+1} = c^t + d^tx∈ [0, b] for all (x, t)∈ X×$\mathbb{N}^0$ and u_t∈[0, min{b, {A_i^t + B_i^tx}}]. The rest follows from the requirements in the statement of this corollary. Q.E.D.

Note 4.1: For a AC-1+1-LOC-LC problem, if <h^t| t ≥ T> is a bang-bang sequence of decision rules, then for all t ≥ T and x∈ X, it must be the case that h^t(x) = (max {$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{b, a^t + b^tx}, 0}, c^t + d^tx). Further, $\sum_{t=T}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t]$ = $p_1^{(T)}x_T$+ $p_2^{(T)}$(max{$\frac{p_2^{(T)}}{|p_2^{(T)}|}$ min{b,{A_i^T + $B_i^T x_t$}}, 0}) + $\sum_{t=T+1}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}$(max{$\frac{p_2^{(t)}}{|p_2^{(t)}|}$ min{b,{A_i^t + $B_i^t x_t$}}, 0
.

Proposition 4.1 implies the following “robust” approximation result.

Proposition 4.2 (Approximation Result): Suppose <(($p_1^{(t)}$, $p_2^{(t)}$), [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> is an AC-LOC-LC and let x∈ X. Let V(x) = $\sum_{t=0}^{\infty}[p_1^{(t)}y_t+ p_2^{(t)}v_t]$ for all <(y_t, v_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x). Then for all ε > 0, there exists T^*(ε)∈ℕ such that for all T∈ℕ with T ≥ T^*, the linear programming problem [Maximize $\sum_{t=0}^{T}[p_1^{(t)}y_t+ p_2^{(t)}v_t]$ subject to v_t ≤ A_i^t + $B_i^t y_t$ for every i and v_t ≤ b for all t = 0, 1, …, T, y_{t+1} = c^t + $d^t y_t$ + $e^t v_t$ for all t = 0, …, T-1, y_0 = x_0 = x, y_t ≥ 0, v_t ≥ 0, for all t = 0, 1, …, T] has a solution <($x_t^{(T)}$, $u_t^{(T)}$)| t = 0, 1, ..., T> and |$\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)}+ p_2^{(t)}u_t^{(T)}]$- V(x)| < ε.

Proof: Since <(($p_1^{(t)}$, $p_2^{(t)}$), [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> is an AC-LOC-LC, $\mathcal{S}$(x)≠ ϕ and for all T∈ℕ the linear programming problem [Maximize $\sum_{t=0}^{T}[p_1^{(t)}y_t + p_2^{(t)}v_t]$ subject to $v_t \leq A_i^t + B_i^t y_t$ for every i and $v_t \leq b$ for all t = 0, 1, …, T, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T] has a solution <($x_t^{(T)}$, $u_t^{(T)}$)| t = 0, ..., T>.

Towards a contradiction suppose there exists ε > 0 such that $|\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] - V(x)| \geq \varepsilon$ infinitely often, i.e., <u>either</u> $\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] - \varepsilon \geq V(x)$ infinitely often <u>or</u> $V(x) \geq \sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] + \varepsilon$ infinitely often.

Let <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x). Since <(x_t, u_t)|t = 0, 1, …, T> satisfies the constraints of the linear programming problem for all T∈ℕ, it must be the case that $\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] \geq \sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t]$ for all T∈ℕ.

Since, $V(x) = \sum_{t=0}^{\infty}[p_1^{(t)}x_t + p_2^{(t)}u_t] = \lim_{T\to\infty}\sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t]$, there exists T^0∈ℕ such that for all T∈ℕ with $T \geq T^0$ it is the case that $\sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t] + \frac{\varepsilon}{4} > V(x) > \sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t] - \frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] + \frac{\varepsilon}{4} \geq \sum_{t=0}^{T}[p_1^{(t)}x_t + p_2^{(t)}u_t] + \frac{\varepsilon}{4} > V(x)$ for all T∈ℕ with $T \geq T^0$.

Thus, $\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] + \varepsilon > \sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] + \frac{\varepsilon}{4} > V(x)$ for all T∈ℕ with $T \geq T^0$.

Thus, $|\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] - V(x)| \geq \varepsilon$ infinitely often is incompatible with $V(x) \geq \sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] + \varepsilon$ infinitely often.

Thus, $|\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] - V(x)| \geq \varepsilon$ infinitely often implies $\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] \geq V(x) + \varepsilon$ infinitely often.

Since, <(($p_1^{(t)}$, $p_2^{(t)}$), [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> is an AC-LOC-LC, there exists T^1∈ℕ such that for all T∈ℕ satisfying $T \geq T^1$, $b\sum_{t=T}^{\infty}|p_i^{(t)}| < \frac{\varepsilon}{8}$ for i∈{1,2}.

$\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] \geq V(x) + \varepsilon$ infinitely often implies $\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] \geq V(x) + \varepsilon$ with $T \geq T^1$ infinitely often.

Let $T \geq T^1$ be such that $\sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)} + p_2^{(t)}u_t^{(T)}] \geq V(x) + \varepsilon$

Let $x_{T+1}^{(T)} = c^T + d^T x_T^{(T)} + e^T u_T^{(T)}$ and <($x_t^{(T)}$, $u_t^{(T)}$| t ≥ T+1> ∈$\mathcal{F}^{T+1}(x_{T+1}^{(T)})$.

Thus, <($x_t^{(T)}$, $u_t^{(T)}$| t∈$\mathbb{N}^0$>∈$\mathcal{F}$(x).

Now $|\sum_{t=T+1}^{\infty}[p_1^{(t)}x_t^{(T)}+p_2^{(t)}u_t^{(T)}]| \leq \sum_{t=T+1}^{\infty}[|p_1^{(t)}|x_t^{(T)}+|p_2^{(t)}|u_t^{(T)}] \leq$
$b[\sum_{t=T+1}^{\infty}[|p_1^{(t)}|+|p_2^{(t)}|] = b\sum_{t=T}^{\infty}|p_1^{(t)}| + b\sum_{t=T}^{\infty}|p_2^{(t)}| < \frac{\varepsilon}{8}+\frac{\varepsilon}{8}=\frac{\varepsilon}{4}$.

Thus, $\frac{\varepsilon}{4} > \sum_{t=T+1}^{\infty}[p_1^{(t)}x_t^{(T)}+p_2^{(t)}u_t^{(T)}] > -\frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}x_t^{(T)}+p_2^{(t)}u_t^{(T)}] = \sum_{t=0}^{T}[p_1^{(t)}x_t^{(T)}+p_2^{(t)}u_t^{(T)}] +$
$\sum_{t=T+1}^{\infty}[p_1^{(t)}x_t^{(T)}+p_2^{(t)}u_t^{(T)}] \geq V(x)+\varepsilon+\sum_{t=T+1}^{\infty}[p_1^{(t)}x_t^{(T)}+p_2^{(t)}u_t^{(T)}] > V(x)+\varepsilon-\frac{\varepsilon}{4} =$
$V(x)+\frac{3\varepsilon}{4} > V(x)$.

This, contradicts the definition of V(x) in the statement of this proposition and proves the proposition. Q.E.D.

5. Duality Theory for AC-LOC-LC problems and "necessary" conditions for optimality:

Let $<((p_1^{(t)}, p_2^{(t)}), [A^t \mid B^t], (c^t,d^t,e^t))|\ t\in\mathbb{N}^0>$ be an AC-LOC-LC problem and suppose that for all $t\in\mathbb{N}^0$, A_i^t, $A_i^t + B_i^t b \in (0, b]$ for all $i\in\{1, \ldots, I(t)\}$, so that for all $x\in X$, $A_i^t + B_i^t x \in (0, b]$ for all i and $t\in\mathbb{N}^0$.

For some $x\in X$, let $<(x_t, u_t)|t\in\mathbb{N}^0> \in\mathcal{F}(x)$.

For $T\in\mathbb{N}$ with $T\geq 3$ consider the linear programming problem in part 1 of proposition 3.1.

Maximize $\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t]$, subject to $v_t \leq A_i^t + B_i^t y_t$ for every i and $v_t \leq b$ for all t = 0, 1, …, T, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T, <u>&</u> both $y_T = x_T$, $v_T = u_T$.

Since $\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t] = p_1^{(0)}x + \sum_{t=1}^{T}[p_1^{(t)}y_t+p_2^{(t-1)}v_{t-1}]$ along with $y_0 = x_0 = x$ and $y_T = x_T$, $v_T = u_T$, the linear programming problem in part 1 of proposition 3.1 reduces to the following.

Maximize $p_2^{(T-1)}v_{T-1}+\sum_{t=1}^{T-1}[p_1^{(t)}y_t+p_2^{(t-1)}v_{t-1}]$, subject to $v_0 \leq A_i^0 + B_i^0 x$ for every $i\in\{1, \ldots, I(0)\}$, $v_t - B_i^t y_t \leq A_i^t$ for every i and t = 1, …, T-1, $v_t \leq b$ for all t = 0, 1, …, T-1, $y_1 - e^0 v_0 = c^0 + d^0 x$, $y_{t+1} - d^t y_t - e^t v_t = c^t$ for all t = 1, …, T-2, $-d^{T-1}y_{T-1} - e^{T-1}v_{T-1} = -x_T + c^{T-1}$, $y_t \geq 0$, for all t = 1, …, T-1, $v_t \geq 0$, for all t = 0, 1, …, T-1.

Further, since for all $x\in X$, $A_i^t + B_i^t x \in (0, b]$ for all i and $t\in\mathbb{N}^0$, "$v_0 \leq A_i^0 + B_i^0 x$ for every $i\in\{1, \ldots, I(0)\}$, $v_t - B_i^t y_t \leq A_i^t$ for every i and t = 1, …, T-1" implies "$v_t \leq b$ for all $t\in\mathbb{N}^0$.

Thus, the linear programming maximization problem reduces to the following.

Maximize $p_2^{(T-1)}v_{T-1} + \sum_{t=1}^{T-1}[p_1^{(t)}y_t+p_2^{(t-1)}v_{t-1}]$, subject to $v_0 \leq A_i^0 + B_i^0 x$ for every $i\in\{1, \ldots, I(0)\}$, $v_t - B_i^t y_t \leq A_i^t$ for every i and t = 1, …, T-1, $y_1 - e^0 v_0 = c^0 + d^0 x$, $y_{t+1} - d^t y_t - e^t v_t = c^t$ for all t = 1, …, T-2, $-d^{T-1}y_{T-1} - e^{T-1}v_{T-1} = -x_T + c^{T-1}$, $y_t \geq 0$, for all t = 1, …, T-1, $v_t \geq 0$, for all t = 0, 1, …, T-1.

The dual of this linear programming problem is the following linear programming problem.

Minimize $\alpha_0^{(T)}(c^0 + d^0x) + \sum_{t=1}^{T-2}\alpha_t^{(T)}c^t + \alpha_{T-1}^{(T)}(c^{T-1} - x_T) + \sum_{i=1}^{I(0)}\beta^{(0,i|T)}(A_i^0 + B_i^0x) + \sum_{t=1}^{T-1}\sum_{i=1}^{I(t)}\beta^{(t,i|T)}A_i^t$, subject to $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)}\beta^{(t,i|T)}B_i^t \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)}\beta^{(t,i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta^{(t,i|T)} \geq 0$ for all i and t = 0, …, T-1, $\alpha_t^{(T)} \in \mathbb{R}$ for all t = 0, …, T-1.

In the context of this section we have the following lemma.

Lemma 5.1: If <$(x_t, u_t)|t\in \mathbb{N}^0$>$\in \mathcal{S}(x)$ then for all $T\in \mathbb{N}$, with $T \geq 3$, there exist arrays <$\alpha_t^{(T)}|\alpha_t^{(T)}\in\mathbb{R}$ for all t = 0, …, T-1> and <$\beta^{(t,i|T)}|\beta^{(t,i|T)}\in\mathbb{R}_+$ for all i and t = 0, …, T-1> such that:

(i) $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)}\beta^{(t,i|T)}B_i^t \geq p_1^{(t)}$ and $(\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)}\beta^{(t,i|T)}B_i^t - p_1^{(t)})x_t = 0$, for all t = 1, …, T-1.

(ii) $-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)}\beta^{(t,i|T)} \geq p_2^{(t)}$ and $(-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)}\beta^{(t,i|T)} - p_2^{(t)})u_t = 0$ for all t = 1, …, T-1.

(iii) $u_0 \leq A_i^0 + B_i^0x$ and $(u_0 - A_i^0 - B_i^0x)\beta^{(0,i|T)} = 0$ for every $i\in\{1, …, I(0)\}$.

(iv) $u_t - B_i^tx_t \leq A_i^t$ and $(u_t - A_i^t - B_i^tx_t)\beta^{(t,i|T)} = 0$ for every i and t = 1, …, T-1.

Proof: By part 1 of proposition 3.1, if <$(x_t, u_t)|t\in \mathbb{N}^0$>$\in \mathcal{S}(x)$ then for all $T\in \mathbb{N}$, <$(x_t, u_t)|t = 0, 1, …, T$> solves the following linear programming problem: Maximize $\sum_{t=0}^{T}[p_1^{(t)}y_t + p_2^{(t)}v_t]$, subject to $v_t \leq A_i^t + B_i^ty_t$ for every i and $v_t \leq b$ for all t = 0, 1, …, T, $y_{t+1} = c^t + d^ty_t + e^tv_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$, for all t = 0, 1, …, T, & both $y_T = x_T$, $v_T = u_T$.

Thus, it follows from the discussion preceding the statement of this proposition, that for all $T\in \mathbb{N}$ with $T \geq 3$, <$(x_t, u_t)|t = 0, 1, …, T$> with $x_0 = x$ solves: Maximize $p_2^{(T-1)}v_{T-1} + \sum_{t=1}^{T-1}[p_1^{(t)}y_t + p_2^{(t-1)}v_{t-1}]$, subject to $v_0 \leq A_i^0 + B_i^0x$ for every $i\in\{1, …, I(0)\}$, $v_t - B_i^ty_t \leq A_i^t$ for every i and t = 1, …, T-1, $y_1 - e^0v_0 = c^0 + d^0x$, $y_{t+1} - d^ty_t - e^tv_t = c^t$ for all t = 1, …, T-2, $-d^{T-1}y_{T-1} - e^{T-1}v_{T-1} = -x_T + c^{T-1}$, $y_t \geq 0$, for all t = 1, …, T-1, $v_t \geq 0$, for all t = 0, 1, …, T-1.

By the weak duality theorem and complementary slackness conditions of linear programming (see topic 2 of Lahiri (2020)), it follows that for all $T\in \mathbb{N}$ with $T \geq 3$, there exist arrays <$\alpha_t^{(T)}|\alpha_t^{(T)}\in\mathbb{R}$ for all t = 0, …, T-1> and <$\beta^{(t,i|T)}|\beta^{(t,i|T)}\in\mathbb{R}_+$ for all i and t = 0, …, T-1> such that:

(i) $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(1)}\beta^{(t,i|T)}B_i^t \geq p_1^{(t)}$ and $(\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)}\beta^{(t,i|T)}B_i^t - p_1^{(t)})x_t = 0$, for all t = 1, …, T-1.

(ii) $-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)}\beta^{(t,i|T)} \geq p_2^{(t)}$ and $(-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)}\beta^{(t,i|T)} - p_2^{(t)})u_t = 0$ for all t = 1, …, T-1.

(iii) $u_0 \leq A_i^0 + B_i^0 x$ and $(u_0 - A_i^0 - B_i^0 x)\beta^{(0,i|T)} = 0$ for every $i \in \{1, \ldots, I(0)\}$.

(iv) $u_t - B_i^t x_t \leq A_i^t$ and $(u_t - A_i^t - B_i^t x_t)\beta^{(t,i|T)} = 0$ for every i and t = 1, …, T-1.

This proves the lemma. Q.E.D.

In the context of this section, the following proposition follows from lemma 5.1.

Proposition 5.1: If $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{S}(x)$ then there exists $\alpha_0^{(3)} \in \mathbb{R}$ and arrays $<(\alpha_{T-2}^{(T)}, \alpha_{T-1}^{(T)})|(\alpha_{T-2}^{(T)}, \alpha_{T-1}^{(T)}) \in \mathbb{R}^2$ for all $T \in \mathbb{N}$, $T \geq 3>$, $<\beta(T, i)|\ \beta(T, i) \in \mathbb{R}_+$ for all i and $T \in \mathbb{N}^0>$ such that:

(i*) $\alpha_0^{(3)} - d^1\alpha_1^{(3)} - \sum_{i=1}^{I(1)} \beta(1, i)B_i^1 \geq p_1^{(1)}$ and $(\alpha_0^{(3)} - d^1\alpha_1^{(3)} - \sum_{i=1}^{I(1)} \beta(1, i)B_i^1 - p_1^{(1)})x_1 = 0$.

(ii*) $u_0 \leq A_i^0 + B_i^0 x$ and $(u_0 - A_i^0 - B_i^0 x)\beta(0, i) = 0$ for every $i \in \{1, \ldots, I(0)\}$.

(iii*) $\alpha_{T-2}^{(T)} - d^{T-1}\alpha_{T-1}^{(T)} - \sum_{i=1}^{I(T-1)} \beta(T-1, i)B_i^{T-1} \geq p_1^{(T-1)}$ and $(\alpha_{T-2}^{(T)} - d^{T-1}\alpha_{T-1}^{(T)} - \sum_{i=1}^{I(T-1)} \beta(T-1, i)B_i^{T-1} - p_1^{(T-1)})x_{T-1} = 0$ for all $T \in \mathbb{N}$ with $T \geq 3$.

(iv*) $-\alpha_{T-1}^{(T)}e^t + \sum_{i=1}^{I(T-1)} \beta(T-1, i) \geq p_2^{(T-1)}$ and $(-\alpha_{T-1}^{(T)}e^t + \sum_{i=1}^{I(T-1)} \beta(T-1, i) - p_2^{(T-1)})u_{T-1} = 0$ for all $T \in \mathbb{N}$ with $T \geq 3$.

(v*) $u_{T-1} - B_i^{T-1}x_{T-1} \leq A_i^{T-1}$ and $(u_{T-1} - A_i^{T-1} - B_i^{T-1}x_{T-1})\beta(T-1, i) = 0$ for every $i \in \{1, \ldots, I(T-1)\}$ and $T \in \mathbb{N}$ with $T \geq 2$.

Proof: Suppose $<(x_t, u_t)|t \in \mathbb{N}^0> \in \mathcal{S}(x)$.

From lemma 5.1 it follows that for all $T \in \mathbb{N}$, with $T \geq 3$, there exist arrays $<\alpha_t^{(T)}|\alpha_t^{(T)} \in \mathbb{R}$ for all t = 0, …, T-1> and $<\beta^{(t,i|T)}|\beta^{(t,i|T)} \in \mathbb{R}_+$ for all i and t = 0, …, T-1> such that:

(i) $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)} \beta^{(t,i|T)}B_i^t \geq p_1^{(t)}$ and $(\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)} \beta^{(t,i|T)}B_i^t - p_1^{(t)})x_t = 0$, for all t = 1, …, T-1.

(ii) $-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)} \beta^{(t,i|T)} \geq p_2^{(t)}$ and $(-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)} \beta^{(t,i|T)} - p_2^{(t)})u_t = 0$ for all t = 1, …, T-1.

(iii) $u_0 \leq A_i^0 + B_i^0 x$ and $(u_0 - A_i^0 - B_i^0 x)\beta^{(0,i|T)} = 0$ for every $i \in \{1, \ldots, I(0)\}$.

(iv) $u_t - B_i^t x_t \leq A_i^t$ and $(u_t - A_i^t - B_i^t x_t)\beta^{(t,i|T)} = 0$ for every i and t = 1, …, T-1.

Let $\beta(0, i) = \beta^{(0,i|3)} = 0$ for every $i \in \{1, \ldots, I(0)\}$, $\beta(1, i) = \beta^{(1,i|3)} = 0$ for every $i \in \{1, \ldots, I(1)\}$ and for $T \in \mathbb{N}$, with $T \geq 3$, $\beta(T-1, i)$ denote $\beta^{(T-1,i|T)}$ for all $i \in \{1, \ldots, I(T-1)\}$.

For T = 3 and t = 0, (i) is equivalent to (i*).

(iii) is equivalent to (ii*) when T = 3 and t = 0.

(iii*), (iv*), (v*) are equivalent to (i), (ii) and (iv) for t = T-1 and $T \geq 3$. Q.E.D.

Note 5.1: If for some T∈ ℕ, with T ≥ 3, it is the case that $x_{T-1} > 0$, then $p_1^{(T-1)} = \alpha_{T-2}^{(T)}$ - $d^t\alpha_{T-1}^{(T)}$ - $\sum_{i=1}^{I(T-1)} \beta(T-1, i)B_i^t$. If for some T∈ ℕ, with T ≥ 3 it is the case that $u_{T-1} > 0$, then $p_2^{(T-1)}$ = $-\alpha_{T-1}^{(T)}e^t + \sum_{i=1}^{I(T-1)} \beta(T-1, i)$. If for some T∈ ℕ, with T ≥ 3, and i∊{1, …, I(T-1)}, $u_{T-1} < A_i^{T-1} + B_i^{T-1}x_{T-1}$, then β(T−1, i) = 0.

Note 5.2: For a AC-1+1-LOC-LC problem, the four conditions in the statement of proposition 5.1, reduce to the following: there exists $\alpha_0^{(3)}$ ∊ ℝ and arrays <($\alpha_{T-2}^{(T)}$, $\alpha_{T-1}^{(T)}$)|($\alpha_{T-2}^{(T)}$, $\alpha_{T-1}^{(T)}$)∊$\mathbb{R}^2$ for all T∈ ℕ, T ≥ 3>, <β(T)| β(T)∊$\mathbb{R}_+$for all T∊$\mathbb{N}^0$> such that:

(i*) $\alpha_0^{(3)}$ - $d^1\alpha_1^{(3)}$ - $\beta(1)b^1 \geq p_1^{(1)}$and ($\alpha_0^{(3)}$ - $d^1\alpha_1^{(3)}$ - $\beta(1)b^1$ - $p_1^{(1)}$)x_1 = 0.

(ii*) $u_0 \leq a^0 + b^0x$ and $(u_0 - a^0 - b^0x)\beta(0) = 0$.

(iii*) $\alpha_{T-2}^{(T)}$ - $d^t\alpha_{T-1}^{(T)}$ - $\beta(T-1)b^{T-1} \geq p_1^{(T-1)}$and ($\alpha_{T-2}^{(T)}$ - $d^t\alpha_{T-1}^{(T)}$ - $\beta(T-1)b^{T-1}$ - $p_1^{(T-1)}$)x_{T-1} = 0 for all T∈ ℕ with T ≥ 3.

(iv*) $-\alpha_{T-1}^{(T)}e^t + \beta(T-1) \geq p_2^{(T-1)}$ and ($-\alpha_{T-1}^{(T)}e^t + \beta(T-1)$ - $p_2^{(T-1)}$)u_{T-1} = 0 for all T∈ ℕ with T ≥ 3.

(v*) $u_{T-1} - b^{T-1}x_{T-1} \leq a^{T-1}$ and $(u_{T-1} - a^{T-1} - b^{T-1}x_{T-1})\beta(T-1) = 0$, for every T∈ ℕ with T ≥ 2.

An interesting corollary of proposition 5.1 is the following.

Corollary 1 of proposition 5.1: <u>If</u> <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x) <u>then</u> for all T∈ ℕ, with T ≥ 3, there exist sequences <($\alpha_{T-2}^{(T)}$, $\alpha_{T-1}^{(T)}$)|($\alpha_{T-2}^{(T)}$, $\alpha_{T-1}^{(T)}$)∊$\mathbb{R}^2$ for all T∈ ℕ, T ≥ 3> and <β(T, i)| β(T, i)∊$\mathbb{R}_+$ for all i and T∊$\mathbb{N}^0$> such that
$\lim_{T\to\infty} [(c_{T-1}-x_T)\alpha_{T-1}^{(T)}+\alpha_{T-1}^{(T)}x_{T-2}+ \sum_{i=1}^{I(T-1)} \beta(T-1, i)A_i^{T-1}] = 0$.

Further, if there exists T^*∊ $\mathbb{N}^0$, such that $u_T < A_i^T + B_i^T x_T$ for all i∊{1, …, I(T)} and T ≥ T^*, then $\lim_{T\to\infty} [(c_{T-1} - x_T)\alpha_{T-1}^{(T)}+ \alpha_{T-2}^{(T)}x_{T-1}] = 0$.

Proof: <u>If</u> <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x), then from proposition 5.1, we know that for all T∈ ℕ, with T ≥ 3, there exists arrays <($\alpha_{T-2}^{(T)}$, $\alpha_{T-1}^{(T)}$)|($\alpha_{T-2}^{(T)}$, $\alpha_{T-1}^{(T)}$)∊$\mathbb{R}^2$ for all T∈ ℕ, T ≥ 3>, <β(T, i)| β(T, i)∊$\mathbb{R}_+$ for all i and T∊$\mathbb{N}^0$> such that:

(iii*) $\alpha_{T-2}^{(T)}$ - $d^{T-1}\alpha_{T-1}^{(T)}$ - $\sum_{i=1}^{I(T-1)} \beta(T-1, i)B_i^{T-1} \geq p_1^{(T-1)}$and ($\alpha_{T-2}^{(T)}$ - $d^{T-1}\alpha_{T-1}^{(T)}$ - $\sum_{i=1}^{I(T-1)} \beta(T-1, i)B_i^{T-1}$ - $p_1^{(T-1)}$)x_{T-1} = 0 for all T∈ ℕ with T ≥ 3.

(iv*) $-\alpha_{T-1}^{(T)}e^t + \sum_{i=1}^{I(T-1)} \beta(T-1, i) \geq p_2^{(T-1)}$ and ($-\alpha_{T-1}^{(T)}e^t + \sum_{i=1}^{I(T-1)} \beta(T-1, i)$ - $p_2^{(T-1)}$)u_{T-1} = 0 for all T∈ ℕ with T ≥ 3.

(v*) $u_{T-1} - B_i^{T-1}x_{T-1} \leq A_i^{T-1}$ and ($u_{T-1} - A_i^{T-1} - B_i^{T-1}x_{T-1}$)β(T-1, i) = 0 for every i∊{1, …, I(T-1)} and T∈ ℕ with T ≥ 2.

Adding the two equalities ($\alpha_{T-2}^{(T)}$ - $d^{T-1}\alpha_{T-1}^{(T)}$ - $\sum_{i=1}^{I(T-1)} \beta(T-1, i)B_i^{T-1}$ - $p_1^{(T-1)}$)x_{T-1} = 0 and ($-\alpha_{T-1}^{(T)}e^t + \sum_{i=1}^{I(T-1)} \beta(T-1, i)$ - $p_2^{(T-1)}$)u_{T-1} = 0 for all T∈ ℕ with T ≥ 3 we get

$\alpha_{T-2}^{(T)}$xT-1 - $d^{T-1}\alpha_{T-1}^{(T)}$xT-1 - $\sum_{i=1}^{I(T-1)}\beta(T-1,i)B_i^{T-1}x_{T-1}$ - $p_1^{(T-1)}$xT-1 - $\alpha_{T-1}^{(T)}e^{T-1}u_{T-1}$ + $\sum_{i=1}^{I(T-1)}\beta(T-1,i)u_{T-1}$ - $p_2^{(T-1)}u_{T-1} = 0$

Since, $<(x_t, u_t)|t\in\mathbb{N}^0>\in\mathcal{S}(x)$, it must be the case that $x_{t+1} = c_t + d_tx_t + e_tu_t$ for all $t\in\mathbb{N}^0$, by substituting $c_{T-1} - x_T$ for $-d^{T-1}x_{T-1} - e^{T-1}u_{T-1}$, we get, $(c_{T-1} - x_T)\alpha_{T-1}^{(T)} + \alpha_{T-2}^{(T)}x_{T-1}$ - $\sum_{i=1}^{I(T-1)}\beta(T-1,i)B_i^{T-1}x_{T-1} + \sum_{i=1}^{I(T-1)}\beta(T-1,i)u_{T-1}$ - $p_1^{(T-1)}x_{T-1} - p_2^{(T-1)}u_{T-1} = 0$.

Adding the equalities $(u_{T-1} - A_i^{T-1} - B_i^{T-1}x_{T-1})\beta(T-1, i) = 0$ over $i\in\{1, \ldots, I(T-1)\}$ we get

$\sum_{i=1}^{I(T-1)}\beta(T-1,i)u_{T-1} - \sum_{i=1}^{I(T-1)}\beta(T-1,i)B_i^{T-1}x_{T-1} = \sum_{i=1}^{I(T-1)}\beta(T-1,i)A_i^{T-1} = 0$, i.e.,

$\sum_{i=1}^{I(T-1)}\beta(T-1,i)u_{T-1} - \sum_{i=1}^{I(T-1)}\beta(T-1,i)B_i^{T-1}x_{T-1} = \sum_{i=1}^{I(T-1)}\beta(T-1,i)A_i^{T-1}$.

Substituting for $\sum_{i=1}^{I(T-1)}\beta(T-1,i)u_{T-1} - \sum_{i=1}^{I(T-1)}\beta(T-1,i)B_i^{T-1}x_{T-1}$ in the equation $(c_{T-1} - x_T)\alpha_{T-1}^{(T)} + \alpha_{T-2}^{(T)}x_{T-1} - \sum_{i=1}^{I(T-1)}\beta(T-1,i)B_i^{T-1}x_{T-1} + \sum_{i=1}^{I(T-1)}\beta(T-1,i)u_{T-1}$ - $p_1^{(T-1)}x_{T-1} - p_2^{(T-1)}u_{T-1} = 0$, we get $(c_{T-1} - x_T)\alpha_{T-1}^{(T)} + \alpha_{T-2}^{(T)}x_{T-1} + \sum_{i=1}^{I(T-1)}\beta(T-1,i)A_i^{T-1}$ - $p_1^{(T-1)}x_{T-1} - p_2^{(T-1)}u_{T-1} = 0$

Since $<((p_1^{(t)}, p_2^{(t)}), [A^t \mid B^t], (c^t,d^t,e^t))|\ t\in\mathbb{N}^0>$ is an AC-LOC-LC problem and $x_t, u_t \in X = [0, b]$ for all $t\in\mathbb{N}^0$, $\lim_{T\to\infty}[p_1^{(T-1)}x_{T-1} - p_2^{(T-1)}u_{T-1}] = 0$.

Thus, $\lim_{T\to\infty}[(c_{T-1} - x_T)\alpha_{T-1}^{(T)} + \alpha_{T-2}^{(T)}x_{T-1} + \sum_{i=1}^{I(T-1)}\beta(T-1,i)A_i^{T-1}] = 0$.

Further, if if there exists $T^*\in\mathbb{N}^0$, such that $u_T < A_i^T + B_i^Tx_T$ for all $i\in\{1, \ldots, I(T)\}$ and $T \geq T^*$, then $\beta(T, i) = 0$, then for all $i\in\{1, \ldots, I(T)\}$ and $T \geq T^*$.

Thus, $\sum_{i=1}^{I(T-1)}\beta(T-1,i)A_i^{T-1} = 0$ for all $T \geq T^* + 1$.

Hence, $\lim_{T\to\infty}[(c_{T-1} - x_T)\alpha_{T-1}^{(T)} + \alpha_{T-2}^{(T)}x_{T-1}] = 0$. Q.E.D.

Note 5.3: We may refer to the condition "$\lim_{T\to\infty}[(c_{T-1} - x_T)\alpha_{T-1}^{(T)} + \alpha_{T-2}^{(T)}x_{T-1} + \sum_{i=1}^{I(T-1)}\beta(T-1,i)A_i^{T-1}] = 0$" as a ***transversality condition***. For the 1+1-LOC-LC problem, the transversality condition reduces to $\lim_{T\to\infty}[(c_{T-1} - x_T)\alpha_{T-1}^{(T)} + \alpha_{T-2}^{(T)}x_{T-1} + \beta(T-1)a^{T-1}] = 0$.

6. Interiority condition and infinite horizon dual linear programming problem:

In this section we consider AC-LOC-LC problems for which, in addition to the conditions assumed in section 5, we assume that the evolution of the state variable is always dependent on the chosen value of the control control variable. Further, the consequences of optimality are only applicable only to those optimal trajectories for which both the state and control variable are always positive and the control variable always satisfy its inequality constraints with strict inequality. We refer to this condition as "interiority condition". In such a situation lemma 5.1 leads to a much stronger necessary condition than in proposition 5.1.

Proposition 6.1: Let <$((p_1^{(t)}, p_2^{(t)})$, [A^t | B^t], (c^t,d^t,e^t))| t∈ $\mathbb{N}^0$> be an AC-LOC-LC problem and suppose that for all t∈$\mathbb{N}^0$, $e^t \neq 0$, A_i^t, $A_i^t + B_i^t b \in (0, b]$ for all i∈{1, …, I(t)}.

Suppose that for some x∈(0, b] and <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x), <(x_t, u_t)|t∈ $\mathbb{N}^0$> satisfies the following "**interiority condition**": For all t∈$\mathbb{N}^0$, $u_t > 0$, $x_t > 0$ and $u_t < A_i^t - B_i^t x_t$ for all i∈{1, …, I(t)}. Then, there exists a sequence <α_t^*| $\alpha_t^* \in \mathbb{R}$, t∈$\mathbb{N}^0$> such that:

(i**) $\alpha_0^* = p_1^{(1)} - d^1\frac{p_2^{(1)}}{e^1}$, $\alpha_t^* = -\frac{p_2^{(t)}}{e^t}$ for all t∈ $\mathbb{N}$.

(ii**) $\alpha_{t-1}^* - d^t\alpha_t^* = p_1^{(t)}$ for all t ∈$\mathbb{N}$.

(iii**) $-\alpha_t^* e^t = p_2^{(t)}$ for all t ∈$\mathbb{N}$.

Further:

(iv**) <α_t^*| t∈$\mathbb{N}^0$> and <β*(t, i)| β*(t, i) = 0 for all i∈{1, …, t} and t ∈$\mathbb{N}^0$> solve the following linear programming problem for all T∈ $\mathbb{N}$, T ≥ 3:

Minimize $\alpha_0^{(T)}(c^0 + d^0x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^t + \alpha_{T-1}^{(T)}(c^{T-1} - x_T) + \sum_{i=1}^{I(0)} \beta^{(0,i|T)}(A_i^0 + B_i^0 x) + \sum_{t=1}^{T-1}\sum_{i=1}^{I(t)} \beta^{(t,i|T)} A_i^t$, subject to $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)} \beta^{(t,i|T)} B_i^t \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)} \beta^{(t,i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta^{(t,i|T)} \geq 0$ for all i and t = 0, …, T-1, $\alpha_t^{(T)} \in \mathbb{R}$ for all t = 0, …, T-1.

(v**) For all T∈ $\mathbb{N}$, T ≥ 3, $p_2^{(T-1)}u_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \alpha_0^*(c^0 + d^0x) + \sum_{t=1}^{T-2} \alpha_t^* c^t + \alpha_{T-1}^*(c^{T-1} - x_T)$.

Proof: From the discussion preceding lemma 5.1, we know if <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x), then that for all T∈ $\mathbb{N}$ with T ≥ 3, <(x_t, u_t)|t = 0, …, T> solves:

Maximize $p_2^{(T-1)} v_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}]$, subject to $v_0 \leq A_i^0 + B_i^0 x$ for every i∈{1, …, I(0)}, $v_t - B_i^t y_t \leq A_i^t$ for every i and t = 1, …, T-1, $y_1 - e^0v_0 = c^0 + d^0x$, $y_{t+1} - d^ty_t - e^tv_t = c^t$ for all t = 1, …, T-2, $-d^{T-1}y_{T-1} - e^{T-1}v_{T-1} = -x_T + c^{T-1}$, $y_t \geq 0$, for all t = 1, …, T-1, $v_t \geq 0$, for all t = 0, 1, …, T-1.

By the strong duality theorem of linear programming we know that <(x_t, u_t)|t = 0, …, T> solves the above problem if and only if its dual has a solution, in which case the optimal value of the maximization problem and the optimal value of its dual are equal. The dual of the linear programming maximization problem is the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^0 + d^0x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^t + \alpha_{T-1}^{(T)}(c^{T-1} - x_T) + \sum_{i=1}^{I(0)} \beta^{(0,i|T)}(A_i^0 + B_i^0 x) + \sum_{t=1}^{T-1}\sum_{i=1}^{I(t)} \beta^{(t,i|T)} A_i^t$, subject to $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)} \beta^{(t,i|T)} B_i^t \geq p_1^{(t)}$ for all t = 1, …, T-1, $-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)} \beta^{(t,i|T)} \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta^{(t,i|T)} \geq 0$ for all i and t = 0, …, T-1, $\alpha_t^{(T)} \in \mathbb{R}$ for all t = 0, …, T-1.

From the strong duality theorem and the complementary slackness condition we know that since <(x_t, u_t)|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x) ⊂ $\mathcal{F}$(x), <(x_t, u_t)|t = 0, …, T> solves the maximization problem <u>if and only if</u> there exist $<\alpha_t^{(T)}|\alpha_t^{(T)}\epsilon\mathbb{R}$ for all t = 0, …, T-1> and $<\beta^{(t,i|T)}|\beta^{(t,i|T)}\epsilon\mathbb{R}_+$ for all i and t = 0, …, T-1> such that:

(i) $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(1)} \beta^{(t,i|T)} B_i^t \geq p_1^{(t)}$ and $(\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \sum_{i=1}^{I(t)} \beta^{(t,i|T)} B_i^t - p_1^{(t)})x_t = 0$, for all t = 1, …, T-1.

(ii) $-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)} \beta^{(t,i|T)} \geq p_2^{(t)}$ and $(-\alpha_t^{(T)}e^t + \sum_{i=1}^{I(t)} \beta^{(t,i|T)} - p_2^{(t)})u_t = 0$ for all t = 1, …, T-1.

(iii) $u_0 \leq A_i^0 + B_i^0x$ and $(u_0 - A_i^0 - B_i^0x)\beta^{(0,i|T)} = 0$ for every i∈{1, …, I(0)}.

(iv) $u_t - B_i^tx_t \leq A_i^t$ and $(u_t - A_i^t - B_i^tx_t)\beta^{(t,i|T)} = 0$ for every i and t = 1, …, T-1.

By the "interiority condition", for all t∈$\mathbb{N}^0$, $u_t > 0$. $u_t < A_i^t - B_i^tx_t$ for all i∈{1, …, I(t)}.

From (iii) and (iv), $\beta^{(t,i|T)} = 0$ for all i∈{1, …, I(0)} and t = 1, …, T-1.

Along with this information, from (i) and (ii) we get: $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} = p_1^{(t)}$ and $-\alpha_t^{(T)}e^t = p_2^{(t)}$ for all t = 1, …, T-1.

From $-\alpha_t^{(T)}e^t = p_2^{(t)}$ for all t = 1, …, T-1 and $e^t \neq 0$ for all t ∈ $\mathbb{N}^0$, we get $\alpha_t^{(T)} = -\frac{p_2^{(t)}}{e^t}$ for all t = 1, …, T-1 and from $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} = p_1^{(t)}$ for all t = 1, …, T-1, we get $\alpha_0^{(T)} = p_1^{(1)} + d^1\alpha_1^{(T)} = p_1^{(1)} - d^1\frac{p_2^{(1)}}{e^1}$.

Thus, values of $\alpha_t^{(T)}$ for t = 0, ..., T-1 and T∈$\mathbb{N}$ with T ≥ 3, is independent of T.

Thus, for all t∈$\mathbb{N}^0$, there exists $\alpha_t^* \in \mathbb{R}$ such that $\alpha_t^{(T)} = \alpha_t^*$ for all T∈$\mathbb{N}$ with T ≥ 3.

Hence, $\alpha_{t-1}^* - d^t\alpha_t^* = p_1^{(t)}$ for all t∈$\mathbb{N}$.

Thus, $<\alpha_t^*| \alpha_t^* \in \mathbb{R}$. t∈$\mathbb{N}^0$> satisfies (i**), (ii**) and (iii**) in the statement of this proposition and along with <β*(t, i)| β*(t, i) = 0 for all i∈{1, …, t} and t ∈$\mathbb{N}^0$> solve the linear programming minimization problem in the statement of this proposition for all T∈ $\mathbb{N}$, T ≥ 3. Hence (iv**) is satisfied.

Thus, for all T∈ $\mathbb{N}$, T ≥ 3, $p_2^{(T-1)}u_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \alpha_0^*(c^0 + d^0x) + \sum_{t=1}^{T-2} \alpha_t^* c^t + \alpha_{T-1}^*(c^{T-1} - x_T)$.

Thus (v**) is satisfied. Q.E.D.

Note 6.1: $\lim_{T\to\infty} p_2^{(T-1)}u_{T-1} = 0$, since $\lim_{T\to\infty} p_2^{(T-1)} = 0$ and $u_{T-1} \in [0, b]$ for all T∈$\mathbb{N}$.

Thus, $\lim_{T\to\infty} (p_2^{(T-1)}u_{T-1} + \sum_{t=1}^{T-1} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]) = \lim_{T\to\infty} \sum_{t=1}^{T-1} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}] = \sum_{t=1}^{\infty} [p_1^{(t)}x_t + p_2^{(t-1)}u_{t-1}]$.

For x∈X, consider the following infinite horizon linear programming problem "implied" by OPT (as defined in section 3).

Maximize $\sum_{t=1}^{\infty}[p_1^{(t)}y_t+p_2^{(t-1)}v_{t-1}]$, subject to $v_0 \leq A_i^0 + B_i^0 x$ for every i∈{1, …, I(0)}, $v_t - B_i^t y_t \leq A_i^t$ for every i∈{1, …, I(t)} and t ∈ℕ, $y_1 - e^0 v_0 = c^0 + d^0 x$, $y_{t+1} - d^t y_t - e^t v_t = c^t$ for all t ∈ℕ, $y_t \geq 0$ for all t ∈ℕ, $v_t \geq 0$ for all t ∈ℕ.

This is the maximization problem that we are really concerned with.

Its "**implied dual linear programming (IDLP) problem**" is the following:

Minimize $\alpha_0(c^0 + d^0x) + \sum_{t=1}^{\infty}\alpha_t c^t + \sum_{i=1}^{I(0)}\beta(0,i)(A_i^0+B_i^0 x) + \sum_{t=1}^{\infty}\sum_{i=1}^{I(t)}\beta(t,i)A_i^t$, subject to $\alpha_{t-1} - d^t\alpha_t - \sum_{i=1}^{I(t)}\beta(t,i)B_i^t \geq p_1^{(t)}$ for all t∈ℕ, $-\alpha_t e^t + \sum_{i=1}^{I(t)}\beta(t,i) \geq p_2^{(t)}$ for all t ∈ ℕ, $\beta(t,i) \geq 0$ for all i∈{1, …, I(t)} and t∈$\mathbb{N}^0$, $\alpha_t \in \mathbb{R}$ for all t∈$\mathbb{N}^0$.

If $<\frac{1}{e^t}|\ e^t \in \mathbb{R}\backslash\{0\}, t\in\mathbb{N}>$ is a bounded sequence, then $\lim_{t\to\infty}\alpha_t^* = \lim_{t\to\infty} -\frac{p_2^{(t)}}{e^t} = 0$, since $\lim_{t\to\infty}p_2^{(t)} = 0$.

Thus, $\alpha_0^*(c^0 + d^0x) + \lim_{T\to\infty}\sum_{t=1}^{T-2}\alpha_t^* c^t + \alpha_{T-1}^*(c^{T-1} - x_T) = \alpha_0^*(c^0 + d^0x) + \sum_{t=1}^{\infty}\alpha_t^* c^t$

Since we have assumed that for all t∈$\mathbb{N}^0$, and (x,u)∈[0,b]×[0, b], $c_t + d_t x + e_t u \in [0, b]$, it must be the case that $<c_t|\ c_t \in \mathbb{R}, t\in\mathbb{N}^0>$ is a bounded sequence.

The implication of proposition 6.1, is that $<\alpha_t^*|\ \alpha_t^* \in \mathbb{R}.\ t\in\mathbb{N}^0>$, $<\beta^*(t, i)|\ \beta^*(t, i) = 0$ for all i∈{1, …, t} and t ∈$\mathbb{N}^0$> satisfy all the constraints of IDLP and $\alpha_0^*(c^0 + d^0x) + \sum_{t=1}^{\infty}\alpha_t^* c^t + \sum_{i=1}^{I(0)}\beta^*(0,i)(A_i^0+B_i^0 x) + \sum_{t=1}^{\infty}\sum_{i=1}^{I(t)}\beta^*(t,i)A_i^t = \alpha_0^*(c^0 + d^0x) + \sum_{t=1}^{\infty}\alpha_t^* c^t = \sum_{t=1}^{\infty}[p_1^{(t)}x_t+p_2^{(t-1)}u_{t-1}]$.

7. Duality Theory for AC-1+1-LOC-LC problems and "sufficient" conditions for optimality:

To facilitate and simplify exposition we will in this section consider sufficient conditions from the perspective of duality for AC-1+1-LOC-LC problems. The results extend in a straightforward manner to AC-LC-LOC roblems.

Let $<((p_1^{(t)}, p_2^{(t)}), [a^t \mid b^t], (c^t,d^t,e^t))|\ t\in \mathbb{N}^0>$ be a AC-1+1-LOC-LC problem and suppose that for all t∈$\mathbb{N}^0$, a^t, $a^t + b^t b \in (0, b]$, so that for all x∈X, $a^t + b^t x \in (0, b]$ for all t∈ $\mathbb{N}^0$.

Once again, for some x∈ X, let $<(x_t, u_t)|t\in \mathbb{N}^0> \in \mathcal{F}(x)$.

For T∈ ℕ with T ≥ 2 consider the linear programming problem in part 2 of proposition 3.1.

Maximize $\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t]$, subject to $v_t \leq a^t + b^t y_t$, $v_t \leq b$ for all t = 0, 1, …, T, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$ for all t = 0, 1, …, T.

Note that $y_T = c_{T-1} + d_{T-1}y_{T-1} + e_{T-1}v_{T-1}$, and so in this linear programming problem y_T is endogenously determined and not chosen. However, v_T is chosen.

$\sum_{t=0}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_t] = p_1^{(0)}x+\sum_{t=1}^{T}[p_1^{(t)}y_t+p_2^{(t)}v_{t-1}] + p_2^{(T)}v_T$ and by hypothesis for all $x\epsilon X$, $a^t + b^t x \epsilon (0, b)$ for all $t\in \mathbb{N}^0$, so that $v_t \leq a^t + b^t y_t$ for t = 0, 1, …, T, implies $v_t < b$ for all t = 0, 1, …, T.

Thus, the linear programming problem reduces to the following.

Maximize $\sum_{t=1}^{T}[p_1^{(t)}y_t+p_2^{(t-1)}v_{t-1}] + p_2^{(T)}v_T$, subject to $v_t \leq a^t + b^t y_t$ for t = 0, 1, …, T, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$ for all t = 0, 1, …, T.

$y_{t+1} = c^t + d^t y_t + e^t v_t$ for all t = 0, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$, for all t = 0, 1, …, T, can be written as, $y_1 - e^0 v_0 = c^0 + d^0 x$, $y_{t+1} - d^t y_t - e^t v_t = c^t$ for all t = 1, …, T-1, $y_t \geq 0$ for all t = 1, …, T.

Further, $v_T \leq a^T + b^T y_T$ and $y_T = c^{T-1} + d^{T-1}y_{T-1} + e^{T-1}v_{T-1}$ implies $v_T - b^T d^{T-1}y_{T-1} - b^T e^{T-1}v_{T-1} \leq a^T + b^T c^{T-1}$.

Hence, the linear programming problem that we are concerned with in this section may be expressed as follows.

Maximize $\sum_{t=1}^{T}[p_1^{(t)}y_t+p_2^{(t-1)}v_{t-1}] + p_2^{(T)}v_T$, subject to $v_0 \leq a^0 + b^0 x$, $v_t - b^t y_t \leq a^t$ for t = 1, …, T, $y_1 - e^0 v_0 = c^0 + d^0 x$, $y_{t+1} - d^t y_t - e^t v_t = c^t$ for all t = 1, …, T-1, $y_t \geq 0$ for all t = 1, …, T and $v_t \geq 0$ for all t = 0, 1, …, T.

The dual of this linear programming problem is the following.

Minimize $\alpha_0^{(T)}(c^0 + d^0 x) + \sum_{t=1}^{T-1}\alpha_t^{(T)}c^t + \beta(0|T)(a^0+b^0 x) + \sum_{t=1}^{T}\beta(t|T)a^t$ subject to $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \beta(t|T)b^t \geq p_1^{(t)}$ for all t = 1, …, T-1, $\alpha_{T-1}^{(T)} - \beta(T|T)b^T \geq p_1^{(T)}$, $-\alpha_t^{(T)}e^t + \beta(t|T) \geq p_2^{(t)}$ for all t = 0, …, T-1, $\beta(T|T) \geq p_2^{(T)}$, $\beta(t|T) \geq 0$ for t = 0, …, T, $\alpha_t^{(T)} \epsilon \mathbb{R}$ for all t = 0, …, T-1.

In the context of this section we have the following proposition.

Proposition 7.1: Suppose that for $<(x_t, u_t)|t\in \mathbb{N}^0> \epsilon \mathcal{F}(x)$ and there exists $T^* \epsilon \mathbb{N}$ such that for all $T \epsilon \mathbb{N}$ with $T \geq T^*$ the following conditions are satisfied:

There exists arrays $<\alpha_t^{(T)}|\alpha_t^{(T)} \epsilon \mathbb{R}$ for all t = 0, …, T-1> and $<\beta(t|T)|\beta(t|T) \epsilon \mathbb{R}_+$ for all t = 0, …, T> such that:

(i***) $\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \beta(t|T)b^t \geq p_1^{(t)}$ and $(\alpha_{t-1}^{(T)} - d^t\alpha_t^{(T)} - \beta(t|T)b^t - p_1^{(t)})x_t = 0$ for all t = 1, …, T-1.

(ii***) $\alpha_{T-1}^{(T)} - \beta(T|T)b^T \geq p_1^{(T)}$ and $(\alpha_{T-1}^{(T)} - \beta(T|T)b^T - p_1^{(T)})x_T = 0$.

(iii***) $-\alpha_t^{(T)}e^t + \beta(t|T) \geq p_2^{(t)}$ and $(-\alpha_t^{(T)}e^t + \beta(t|T) - p_2^{(t)})u_t = 0$ for all t = 0, …, T-1

(iv***) $\beta(T|T) \geq p_2^{(T)}$ and $(\beta(T|T) - p_2^{(T)})u_T = 0$.

(v***) $(u_0 - a^0 - b^0 x)\beta(0|T) = 0$.

(vi***) $(u_t - b^t x_t - a^t)\beta(t|T) = 0$, for t = 1, …, T.

Then, $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$.

Proof: Let $<(x_t, u_t)|t\in \mathbb{N}^0> \epsilon\mathcal{F}(x)$.

By part 2 of proposition 3.1, if there exists $T^*\epsilon\mathbb{N}$ such that for all $T\epsilon\mathbb{N}$ with $T \geq T^*$ such that $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves [Maximize $\sum_{t=0}^{T} [p_1^{(t)} y_t + p_2^{(t)} v_t]$, subject to $v_t \leq a^t + b^t y_t$, $v_t \leq b$ for all $t = 0, 1, \ldots, T$, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all $t = 0, \ldots, T-1$, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$ for all $t = 0, 1, \ldots, T$] then $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$.

In the discussion preceding the statement of the proposition we have shown that $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves [Maximize $\sum_{t=0}^{T} [p_1^{(t)} y_t + p_2^{(t)} v_t]$, subject to $v_t \leq a^t + b^t y_t$, $v_t \leq b$ for all $t = 0, 1, \ldots, T$, $y_{t+1} = c^t + d^t y_t + e^t v_t$ for all $t = 0, \ldots, T-1$, $y_0 = x_0 = x$, $y_t \geq 0$, $v_t \geq 0$ for all $t = 0, 1, \ldots, T$] if and only if $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves [Maximize $\sum_{t=1}^{T} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] + p_2^{(T)} v_T$, subject to $v_0 \leq a^0 + b^0 x$, $v_t - b^t y_t \leq a^t$ for $t = 1, \ldots, T$, $y_1 - e^0 v_0 = c^0 + d^0 x$, $y_{t+1} - d^t y_t - e^t v_t = c^t$ for all $t = 1, \ldots, T-1$, $y_t \geq 0$ for all $t = 1, \ldots, T$ and $v_t \geq 0$ for all $t = 0, 1, \ldots, T$].

However, by the strong duality theorem and the complementary slackness condition of linear programming, $<(x_t, u_t)|t = 0, 1, \ldots, T>$ solves [Maximize $\sum_{t=1}^{T} [p_1^{(t)} y_t + p_2^{(t-1)} v_{t-1}] + p_2^{(T)} v_T$, subject to $v_0 \leq a^0 + b^0 x$, $v_t - b^t y_t \leq a^t$ for $t = 1, \ldots, T$, $y_1 - e^0 v_0 = c^0 + d^0 x$, $y_{t+1} - d^t y_t - e^t v_t = c^t$ for all $t = 1, \ldots, T-1$, $y_t \geq 0$ for all $t = 1, \ldots, T$ and $v_t \geq 0$ for all $t = 0, 1, \ldots, T$] if and only if

(1) $u_0 \leq a^0 + b^0 x$, $u_t - b^t x_t \leq a^t$ for $t = 1, \ldots, T$, $x_1 - e^0 v_0 = c^0 + d^0 x$, $x_{t+1} - d^t x_t - e^t u_t = c^t$ for all $t = 1, \ldots, T-1$, $x_t \geq 0$ for all $t = 1, \ldots, T$ and $u_t \geq 0$ for all $t = 0, 1, \ldots, T$.

(2) There exists arrays $<\alpha_t^{(T)}|\alpha_t^{(T)}\epsilon\mathbb{R}$ for all $t = 0, \ldots, T-1>$ and $<\beta(t|T)|\beta(t|T)\epsilon\mathbb{R}_+$ for all $t = 0, \ldots, T>$ such that (i^{***}), (ii^{***}), (iii^{***}), (iv^{***}), (v^{***}) and (vi^{***}) in the statement of this proposition are satisfied.

Since $<(x_t, u_t)|t\in \mathbb{N}^0> \epsilon\mathcal{F}(x)$, for all $T\epsilon\mathbb{N}$ with $T \geq T^*$, (1) holds and we have assumed in the statement of this proposition that for all $T\epsilon\mathbb{N}$ with $T \geq T^*$, (2) is satisfied.

Thus, $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$. Q.E.D.

8. Conclusion:

While concluding this paper, the first point that needs to be noted is that it is possible to make the analysis appear slightly simpler if we began with the assumption that for all $t\in \mathbb{N}^0$ and $i\in \{1, \ldots, I(t)\}$, $A_i^t \in [0, b]$ and $A_i^t + B_i^t b \in [0, b]$. In fact that can always be implemented by including an additional inequality constraint for the control variable (if necessary) whose parameters are defined by $A_{I(t)+1}^t \in [0, b]$ and $A_{I(t)+1}^t + B_{I(t)+1}^t b \in [0, b]$. Under such circumstances, the bang-bang sequence of decision rules would be defined thus:

For $T\epsilon\mathbb{N}^0$ and for all $(x, t)\epsilon X\times\mathbb{N}^0$ satisfying $t \geq T$ and $p_2^{(t)} \neq 0$, $h^t(x) = (\max\{\frac{p_2^{(t)}}{|p_2^{(t)}|} \min\{A_i^t + B_i^t x\}, 0\}, c^t + d^t x)$.

Not much by way of generality would be lost in the context of this paper, if we restrict our framework of analysis in this manner.

The next point that needs to be noted is that as in Evers (1973), control variables are implicit in the model discussed in Romeijn, Smith and Bean (1992). However, as in the case of Evers (1973), the framework of analysis in Romeijn, Smith and Bean (1992) requires $p_2^{(t)} = 0$ for all $t \in \mathbb{N}^0$, and hence fails to generalize the framework of analysis of this paper. A notable consequence of their assumption that the control variable does not influence the value of the objective function, is that our definition of a bang-bang sequence of decision rules is difficult (if not impossible) to accommodate in their framework. Needless to say, that there are other results in our conceptually more general framework that cannot be framed meaningfully in the Evers model or the one of Romeijn, Smith and Bean (1992).

The final point that we wish to note is about finite horizon approximations of the infinite horizon optimization problem that we are concerned with. A notable precedent in this respect is the infinite horizon linear programming model in Grinold (1977), which in our context would resemble something like the following:

Given a non-zero real number p, a real number $\delta \in (0, 1)$ and $x \in X$: Maximize $\sum_{t=0}^{\infty} \delta^t p u_t$ subject to $u_t \leq x_t$, $x_{t+1} = c^t + e^t u_t$ for all $t \in \mathbb{N}^0$, $x_0 = x$, $x_t \geq 0$ for all $t \in \mathbb{N}$, $u_t \geq 0$ for all $t \in \mathbb{N}^0$.

Note that $u_t \leq x_t$, for all $t \in \mathbb{N}^0$ and our assumption that for all $(y, u) \in X \times X$, $c^t + d^t y + e^t u \in X = [0, b]$ for all $t \in \mathbb{N}^0$ implies $u_t \leq b$ for all $t \in \mathbb{N}^0$. Note that the infinite horizon model in Grinold (1977) when appropriately calibrated to fit into our framework implies $d^t = 0$ for all $t \in \mathbb{N}^0$.

The infinite horizon maximization problem described above can be more simply represented as the following: Maximize $\sum_{t=0}^{\infty} \delta^t p u_t$ subject to $u_0 \leq x_0$, $u_{t+1} \leq c^t + e^t u_t$ for all $t \in \mathbb{N}^0$, $x_0 = x$, $u_t \geq 0$ for all $t \in \mathbb{N}^0$.

The finite horizon approximation suggested in section 5 of Grinold (1977) does not seem to be applicable in our conceptually more general framework of analysis.

If our framework of analysis a suitable finite horizon approximation for $T \in \mathbb{N}$ with T sufficiently large may be formulated as follows:

Given $x \in X$, and $q_i^{(T)} = \sum_{t=T}^{\infty} p_i^{(t)}$, $i \in \{1, 2\}$, Maximize $\sum_{t=0}^{T-1} [p_1^{(t)} x_t + p_2^{(t)} u_t] + q_1^{(T)} x_T + q_2^{(t)} u_T$ subject to $u_t \leq A_i^t + B_i^t x_t$, $i \in \{1, \ldots, I(t)\}$, $t = 0, 1, \ldots, T$, $x_{t+1} = c^t + d^t x_t + e^t u_t$, $t = 0, \ldots, T - 1$, $x_0 = x$, $u_t \leq b$, $u_t \geq 0$ for all $t \in \{0, 1, \ldots, T\}$, $x_t \geq 0$ for all $t \in \{1, \ldots, T\}$.

The implicit assumption in such an approximation is that after a sufficiently long period of time, the state and control variables may be assumed to remain constant, without causing much disparity between the infinite horizon model and its finite horizon approximation. For such an interpretation to be "perfectly consistent" with our AC-LOC-LC problem, we need to impose the additional constraints $u_T \leq A_i^t + B_i^t x_T$, $i \in \{1, \ldots, I(t)\}$, $t \in \mathbb{N}$, $t \geq T+1$, $x_T = c^t + d^t x_T + e^t u_T$, $t \in \mathbb{N}$, $t \geq T$.

Apart from issues concerning the existence of a trajectory $<(x_t, u_t)|t\in \mathbb{N}^0> \epsilon\mathcal{F}(x)$ satisfying $(x_t, u_t) = (x_T, u_T)$ for all $t\in \mathbb{N}$, $t \geq T+1$, the solution to the truncated optimization problem satisfying such additional constraints may underestimate the optimal value of the AC-LOC-LC problem. Thus for T sufficiently large, the decision maker might choose to ignore the feasibility constraints beyond the terminal time period.

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