

**COMPOSITE PLATES IN POSTBUCKLING: DUAL EXTREMAL VARIATIONAL
PRINCIPLES, ENERGY FEATURES, STABILITY, LAY-UP OPTIMALITY
CONDITIONS VIA COMPLEMENTARY ENERGY APPROACH**

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Abstract

In the present paper the composite plates in postbuckling are explored. The dual extremal variational principles are created. The principles generalize the stationary ones obtained earlier. The stability of the plate near the single-modal bifurcation point is proven. Some useful energy relations are established. It is also demonstrated that the total complementary energy may be considered as a measure of the compliance for the post-buckled composite plate. The latter measure leads to the same lay-up optimality conditions as obtained earlier via the maximization of the total potential energy.

Keywords: composite plate, postbuckling, variational principles, extremum, duality, stability

1. INTRODUCTION

Postbuckling behavior of composite structures, in particular, plates, attracts attention of many researches. The basics of postbuckling are described in such books as Hutchinson and Koiter (1970), Budiansky (1974). Some aspects of the phenomenon are discussed in the books of Turvey and Marshall (1995), Falzon and Aliabadi (2008), Kassapoglou (2010), Alfutov (2000), Shen (2016). There are several reviews covering the subject, the non-exhaustive list of them contains the papers of Bruneel (2008) and Ghiasi et al. (2009, 2010).

A brief review of the original studies close to the subject of the present paper is presented below.

In the paper of Stein (1983) the nonlinear large-deflection equations of von Karman are written for specially orthotropic plates. By assuming trigonometric functions in one direction, the plate equations are converted into ordinary nonlinear differential equations that are solved numerically using Newton's method. The postbuckling behavior is obtained for simply supported and clamped, long, rectangular, orthotropic plates covering the complete range of dimensions and material properties.

In the important theoretical paper of Haslach (1986) the potential energy function in a neighborhood of the buckling point for a biaxially loaded, linear elastic, orthotropic, simply supported, rectangular plate is obtained from a perturbation analysis. The postbuckling behavior is proven to be stable. It is also possible to have a bimodal buckling point for certain combinations of loads; this point is also stable.

In the paper of Zhang and Shen (1991) the buckling and postbuckling behavior of orthotropic composite rectangular plates with simply supported edges in biaxial compression are investigated using a deflection-type perturbation technique. Using the technique, the

nonlinear generalized von Karman equations of orthotropic plates are solved. The effects of geometrical imperfections are also considered.

The paper of Bisagni and Lanzi (2002) deals with the definition of a post-buckling optimisation procedure for the design of composite stiffened panels subjected to compression loads. The optimised structures are then characterised by a local skin buckling between the stiffeners and by a high ratio between the collapse load and the buckling load. An optimisation procedure is developed. It is based on a global approximation strategy, where the structure response is given by a system of neural networks and the finite element analysis. The optimisation procedure shows that a local skin buckling between the stiffeners allows a weight reduction equal to 18%.

The aim of the paper of Rahman et al. (2011) is to study the postbuckling behavior of the optimized plates using a perturbation method. The perturbation approach is used to compute postbuckling coefficients, which are used to make a quick estimate of the postbuckling stiffness of the panel and to establish a reduced-order model. In the paper, the postbuckling analysis of variable stiffness plates is carried out using the reduced-order model, and the potential of the approach for incorporation within the optimization process is demonstrated.

In the paper of Mittelstedt et al. (2011), the postbuckling behavior of rectangular orthotropic laminated composite plates with initial imperfections under inplane shear load is investigated. The considered plates are infinitely long. The boundary conditions are taken into account at the longitudinal edges only. Using Timoshenko-type shape functions for both the initial bifurcational buckling analysis and the subsequent Marguerre-type postbuckling studies, the closed-form analytical solutions for the buckling loads and for the postbuckling state variables are derived. A comparison with the geometrically non-linear finite element computations shows that the derived analysis approaches are suitable for postbuckling studies in the load ranges not too far beyond bifurcational buckling.

The paper of Wu et al. (2014) studies the optimal design of the postbuckling behavior of rectangular laminated composite plates under axial compression. In the numerical optimization process the lamination parameters are used. Using perturbation analysis, an asymptotic closed-form solution is derived to model the postbuckling behaviors of the orthotropic laminated composite plates under the axial compression. The postbuckling design criteria of the composite laminated plates are based on the minimization of either the maximum normal displacement or the end-shortening strain for a given compressive load. The optimization process is a numerical one.

In the paper of Raju et al. (2015) a geometrically nonlinear analysis of the symmetric variable angle tow (VAT) composite von Karman plates under the in-plane shear is investigated. The differential quadrature method is used for the numerical analysis. The buckling and postbuckling behaviour of the VAT plates under the positive and the negative shear is studied for different VAT fibre orientations, aspect ratios, combined axial compression; and their performance is compared with that of straight fibre composites. It is shown that there can be enhanced shear buckling and postbuckling performance for both displacement-control and load-control and that the underpinning driving mechanics are different for each.

The paper of Henrichsen et al. (2016) is a numerical one and presents a novel method for gradient based design optimization of the post-buckling performance of the structures. The post-buckling analysis is based on Koiter's asymptotic method. To perform the gradient based optimization, the design sensitivities of the Koiter factors are derived and new design optimization formulations based on the Koiter factors are presented. The proposed optimization formulations are demonstrated on a composite square plate and a curved panel where the post-buckling stability is optimized. No attempts of the theoretical analysis of the deflection-load curve near the bifurcation point are made.

In the paper of Selyugin (2019a) two variational principles for the von Karman composite plates in postbuckling are derived and presented. The principles are stationary ones. One of them is a kinematic principle, another one is a static principle. The principles lead to the results which may be not unique.

The present paper deals with the generalization of the variational principles (described by Selyugin 2019a) to extremum-type ones. The generalization leads to the useful energy relations and allows proving the postbuckling stability near the bifurcation point. Also, using the complementary energy as a measure of the structural compliance and choosing the laminate lay-up by minimizing the compliance, we come up to the same optimality conditions as in (Selyugin 2019b).

In Section 2 the extremum-type variational principles for the composite plates in postbuckling are developed.

Section 3 is devoted to the proof of postbuckling stability of the symmetrically laminated plates near the single-modal bifurcation point (see also Alfutov 2000).

Section 4 deals with a discussion of some energy relations.

Section 5 describes the use of the total complementary energy for choosing the lay-up of the considered laminates.

Section 6 formulates the conclusions.

2. VARIATIONAL PRINCIPLES IN COMPOSITE PLATE POSTBUCKLING

This Section deals with the generalization of the variational principles for post-buckled plates (Selyugin 2019a) from the stationary-value type to the extremum type.

The assumptions of the present paper are: the symmetric point-wise arbitrary lay-up, the single-modal bifurcation point (Ohsaki and Ikeda 2007), symmetric buckling, the Cartesian

orthogonal coordinates XYZ with respective displacements u, v, w ; the plate with piece-wise-smooth boundary contour is located in the XY plane; the in-plane loading is applied with the value not much higher than the bifurcation load; the boundary conditions are the simple support or clamped.

We explore the plate near the bifurcation point using the Timoshenko approach (described by Alfutov 2000, Section 7.2) and the classic laminated plate theory (Gibson 1994). In the Alfutov's description it is supposed that near the bifurcation point the structure is stressed but not deformed. In our opinion, the assumption is reasonable and is widely used. According to the Timoshenko approach the displacements of the plate points in their new (deviated) state are:

$$\begin{aligned} u &= u_0 + \alpha u_1 + \alpha^2 u_2 \\ v &= v_0 + \alpha v_1 + \alpha^2 v_2 \\ w &= w_0 + \alpha w_1 + \alpha^2 w_2 \end{aligned} \tag{1}$$

where α is the infinitesimal small parameter, (u_0, v_0, w_0) are the non-disturbed functions of the initial point locations, (u_1, v_1, w_1) and (u_2, v_2, w_2) are the finite functions of coordinates.

Applying the Timoshenko approach, we obtain the displacement functions $w_1(x, y), u_2(x, y), v_2(x, y)$ and the Airy function $\Phi(x, y)$ satisfying all boundary conditions of the problem.

We accept approximately that for small but finite plate deflections the plate state is described by the following functions (ξ is the small parameter):

$$\begin{aligned} w(x, y) &= \xi w_1(x, y) \\ u(x, y) &= \xi^2 u_2(x, y) \\ v(x, y) &= \xi^2 v_2(x, y) \\ \Phi(x, y) &= \xi^2 \Phi(x, y) \end{aligned} \tag{2}$$

In this case the change of the total plate potential energy E due to the deviations from the original plane equilibrium state reads:

$$\Delta E = E - E_0 = \xi^2 \frac{1}{2} \int_S dx dy \left\{ D_{11} \left(\frac{\partial^2 w_1}{\partial x^2} \right)^2 + 2D_{12} \frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_1}{\partial y^2} + 4D_{16} \frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_1}{\partial x \partial y} + \right. \\ \left. + D_{22} \left(\frac{\partial^2 w_1}{\partial y^2} \right)^2 + 4D_{26} \frac{\partial^2 w_1}{\partial y^2} \frac{\partial^2 w_1}{\partial x \partial y} + 4D_{66} \left(\frac{\partial^2 w_1}{\partial x \partial y} \right)^2 \right\} - \\ - \xi^2 \oint_{\Gamma} (q_x u_2 + q_y v_2) d\Gamma + \xi^4 W_4 \quad (3)$$

where E_0 is the total plate potential energy independent of ξ ; D is the plate bending matrix;

q_x, q_y are stress flows at the boundary contour Γ around the plate S . The value of W_4 is given

by the expression:

$$W_4 = \frac{1}{2} \iint_S dx dy \{ A_{11} \varepsilon_x''^2 + 2A_{12} \varepsilon_x'' \varepsilon_y'' + A_{22} \varepsilon_y''^2 + 4A_{16} \varepsilon_x'' \varepsilon_{xy}'' + 4A_{26} \varepsilon_y'' \varepsilon_{xy}'' + 4A_{66} \varepsilon_{xy}''^2 \} \quad (4)$$

where A is the plate 2D-strain stiffness matrix,

$$\varepsilon_x'' = \frac{\partial u_2}{\partial x} + \frac{1}{2} \left(\frac{\partial w_1}{\partial x} \right)^2 \\ \varepsilon_y'' = \frac{\partial v_2}{\partial x} + \frac{1}{2} \left(\frac{\partial w_1}{\partial y} \right)^2 \\ \varepsilon_{xy}'' = \frac{1}{2} \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} + \frac{\partial w_1}{\partial x} \frac{\partial w_1}{\partial y} \right) \quad (5)$$

Since the functions w_1, u_2, v_2, Φ of (x, y) are known as the results of the linear stability plate problem, the postbuckling behavior near the bifurcation point is determined by the parameter ξ only.

It is obvious that the quantity W_4 (as an energy quantity) is always positively definite; this leads to a conclusion that after passing through the bifurcation point (under the assumptions made) the total plate potential energy is a convex function of the displacements.

Basing on this conclusion, we may state that the kinematic variational principle for plate postbuckling (see Selyugin 2019a) is a minimal variational principle. Moreover, the result (the displacements) coming from the principle is a unique one.

The relation between ξ and the load above the bifurcation point may be found from the condition of the stationary total potential energy. For the example of uniaxially compressed by the force P plate, calculating the derivative of (3) w.r.t. ξ , we have

$$\xi[P_{cr} - P] + 4\xi^3 W_4 \frac{1}{\beta} = 0 \quad (6)$$

where the quantity β and the critical compression force P_{cr} read:

$$\beta = 2 \oint_{\Gamma} (\bar{q}_x u_2 + \bar{q}_y v_2) d\Gamma \quad (7)$$

$$P_{cr} = \frac{\int_S dx dy \left\{ D_{11} \left(\frac{\partial^2 w_1}{\partial x^2} \right)^2 + 2D_{12} \frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_1}{\partial y^2} + 4D_{16} \frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_1}{\partial x \partial y} + \right.}{\beta} \left. + D_{22} \left(\frac{\partial^2 w_1}{\partial y^2} \right)^2 + 4D_{26} \frac{\partial^2 w_1}{\partial y^2} \frac{\partial^2 w_1}{\partial x \partial y} + 4D_{66} \left(\frac{\partial^2 w_1}{\partial x \partial y} \right)^2 \right\} \quad (8)$$

Here $\bar{q}_x, \bar{q}_y$ is the external load distribution for $P=1$.

Making $\xi \rightarrow 0$ we obtain the buckling load. For ξ not equal to zero we have

$$\xi = \frac{K}{2} \sqrt{\frac{\Delta P}{P_{cr}}} \quad (9)$$

where $\Delta P = P - P_{cr}$ and $K = \sqrt{\frac{2V_1}{W_4}}$.

V_1 is given by the formula:

$$V_1 = \int_S dx dy \left\{ D_{11} \left(\frac{\partial^2 w_1}{\partial x^2} \right)^2 + 2D_{12} \frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_1}{\partial y^2} + 4D_{16} \frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_1}{\partial x \partial y} + \right. \quad (10)$$

$$\left. + D_{22} \left(\frac{\partial^2 w_1}{\partial y^2} \right)^2 + 4D_{26} \frac{\partial^2 w_1}{\partial y^2} \frac{\partial^2 w_1}{\partial x \partial y} + 4D_{66} \left(\frac{\partial^2 w_1}{\partial x \partial y} \right)^2 \right\}$$

It follows from the latter relation that in the case of $\frac{\Delta P}{P_{cr}} \ll 1$ the applied load P is a quadratic function of the maximal deflection $w_{\max}$.

The functional of the kinematic variational principle is written as follows:

$$I = \int_S \left(\frac{1}{2} \vec{e}_0^T A \vec{e}_0 + \frac{1}{2} \vec{\omega}^T D \vec{\omega} \right) dS - \int_{\Gamma_1} (\bar{N}_{x\nu} \delta u + \bar{N}_{y\nu} \delta v) d\Gamma \quad (11)$$

where the mid-plane ($z=0$) vector-columns are

$$\vec{e}_0 = \left(\begin{array}{c} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{array} \right) \Big|_{z=0}, \quad \vec{\omega} = \left(\begin{array}{c} -\partial^2 w / \partial x^2 \\ -\partial^2 w / \partial y^2 \\ -2\partial^2 w / \partial x \partial y \end{array} \right) \Big|_{z=0} \quad (12)$$

and $e_{..}$ with subscripts are the corresponding strain components. The x and y force flows applied at the part Γ_1 of the contour are $\bar{N}_{x\nu}, \bar{N}_{y\nu}$. Further we keep the notations of (Selyugin 2019a).

Now we consider the static variational principle for the post-buckled plates. As it was shown in (Selyugin 2019a), the principle is a stationary one; the solutions resulting from it may be non-unique.

In the paper of Stumpf (1979) it is shown that the total complementary energy principle (for non-linear von Karman plate) dual to the minimal total potential energy principle is of maximum type. This conclusion follows from the convexity (under some assumptions) of the total plate potential energy. The complementary energy principle functional in the case of the composite plate (Selyugin 2019a) is written as follows:

$$I_c = - \int_S \Pi_c dS + \int_{\Gamma_2} [N_{x\nu} \bar{u} + N_{y\nu} \bar{v} + N_{z\nu} \bar{w} - M_{\nu} w_{,n}] dS \quad (13)$$

where Γ_2 is the part of the boundary contour with prescribed displacements $\bar{u}, \bar{v}, \bar{w}$; M_v is the bending moment acting along the border tangential line, $w_{,n}$ is the derivative of the deflection along the direction normal to the contour. Hereinafter the subscript after comma means the differentiation w.r.t. the proper coordinate. The complementary energy density per unit undeformed plate area is given by the formula:

$$\begin{aligned} \Pi_c = & \frac{1}{2} N_{xx} \left(u_{,x} - \frac{1}{2} w_{,x}^2 \right) + \frac{1}{2} N_{yy} \left(v_{,y} - \frac{1}{2} w_{,y}^2 \right) + \frac{1}{2} N_{xy} (u_{,y} + v_{,x} - w_{,x} w_{,y}) + \\ & + N_{xz} w_{,x} + N_{yz} w_{,y} + \frac{1}{2} M_{xx} (-w_{,xx}) + \frac{1}{2} M_{yy} (-w_{,yy}) + \frac{1}{2} M_{xy} (-2w_{,xy}) \end{aligned} \quad (14)$$

Hereinafter N and M with subscripts means the first Piola stress tensor components (multiplied to z in the case of M) integrated over the plate thickness. The following relations are also used above:

$$w_{,x} = \frac{N_{yy} N_{xz} - N_{xy} N_{yz}}{N_{xx} N_{yy} - N_{xy}^2} = \gamma_1(N_{ij}) \quad (15)$$

$$w_{,y} = \frac{N_{xx} N_{yz} - N_{xy} N_{xz}}{N_{xx} N_{yy} - N_{xy}^2} = \gamma_2(N_{ij}) \quad (16)$$

With the below notations (T means the transposition)

$$\vec{N} = (N_{xx}, N_{yy}, N_{xy})^T \quad (17)$$

$$\vec{M} = (M_{xx}, M_{yy}, M_{xy})^T \quad (18)$$

after some elementary but cumbersome transformations, we write (14) in the form:

$$\Pi_c = \frac{1}{2} \vec{N}^T A^{-1} \vec{N} + \frac{1}{2} \vec{M}^T D^{-1} \vec{M} + \frac{1}{2} N_{xz} \gamma_1(N_{ij}) + \frac{1}{2} N_{yz} \gamma_2(N_{ij}) \quad (19)$$

where the superscript (-1) means the inversion of the matrix. The solution of the latter principle is minimal (see Stumpf 1979) and unique. The both principles give the upper and lower bounds of the solution values for both variational principles.

Applying the results of Stumpf (1979) to our case, we obtain that near the bifurcation point the dual extremum-type variational principles (for the total potential energy and for the total complementary energy) occur. The result of every principle is unique.

3. STABILITY IN COMPOSITE PLATE POSTBUCKLING

In this Section we consider the stability of composite plate in postbuckling near the single-modal bifurcation point.

As it is said in previous Section, the total potential energy of the post-buckled composite plate is a convex functional (under the assumptions made). Due to that the second variation of the energy w.r.t. the displacements is positively defined and the plate behavior is stable near the single-modal bifurcation point.

4. DISCUSSION ON ENERGY VALUES

Using the above-indicated duality and extremum-type of the variational principles, we may write the following equality for the resulting variational functionals:

$$\Pi - W = -\Pi - \Omega \quad (20)$$

where W is the second integral (the external work potential) in (11), and

$$\Omega = \frac{1}{2} N_{xz}^2 \frac{N_{yy}}{N_{xx} N_{yy} - N_{xy}^2} + \frac{1}{2} N_{yz}^2 \frac{N_{xx}}{N_{xx} N_{yy} - N_{xy}^2} - \frac{N_{xz} N_{yz} N_{xy}}{N_{xx} N_{yy} - N_{xy}^2} \quad (21)$$

The rhs of (20) corresponds to the complementary energy functional (13). The value of Ω in the postbuckling in the case of two-directional compression is obviously negative. Then the relations (20) and (21) are transformed to

$$2\Pi - W > 0 \quad (22)$$

As we see, in the considered case due to the non-linearity the total strain potential energy is larger than one half of the external work potential. In the infinitesimal case without the postbuckling we have an equality in (22) instead of a non-equality.

5. LAY-UP OPTIMALITY VIA COMPLEMENTARY ENERGY APPROACH

We discuss here the following item. The total complementary energy functional (13) of the post-buckled composite plate may be considered as a measure of the structural compliance. We formulate the following optimization problem: to choose the ply orientation angles of the symmetric lay-up for minimizing the structural compliance.

The complementary energy density is given by (19). Keeping in mind the known equality on the differentiation of the inverted square symmetric matrix (say, G) as the function of the argument g (Korn and Korn, 2003):

$$\frac{d}{dg} G^{-1} = -G^{-1} \frac{dG}{dg} G^{-1} \quad (23)$$

we make the necessary transformations (similar to the approach of Selyugin 2019b) for performing the proper variations of the forces, the moments, and the ply angles. After that, we obtain the same optimality conditions for the ply orientations as indicated in (Selyugin 2019b).

6. CONCLUSIONS

Near the single-modal bifurcation point:

- The dual variational principles of the paper are of the extremum type;
- The plate is stable in postbuckling;

- Due to the geometric non-linearity, the value of the total strain potential energy is larger than the one-half of the external work potential;
- The use of the total complementary energy as a measure of the minimized structural compliance leads to the same lay-up optimality conditions of the first order, like the use of the total potential energy value as a measure of the structural stiffness.

ACKNOWLEDGEMENT

The author is thankful to colleagues for fruitful discussions.

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