Infinite Horizon Discrete-Time Optimal Control

by

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Abstract

We show that if a trajectory is optimal for an infinite horizon discrete-time optimal control problem with a continuously differentiable objective function that is additively separable over time time periods, then it optimal for an for an infinite horizon discrete-time optimal control problem with the same constraints and a linear objective function which is determined by the trajectory.

Keywords: infinite horizon discrete-time optimal control, continuously differentiable objective function, additively separable over time periods, absolutely convergent objective function sequence, linear objective function

AMS Subject Classification(s): 49N05, 90C46

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1. Introduction: The purpose of this note is considerably more modest than what its title may convey.

In Lahiri (2025) a model of a linear control problem with linear constraints is presented and for a certain proposition in that work (proposition 6.1), it is required that there is a solution which satisfies all its (linear) inequality constraints as strict inequalities. In that work, there is no example to show that the proposition is not vacuously true. The purpose of this note is to try to address this problem in a "round about" manner.

It is reasonable to expect that the chances of a solution that satisfies all its linear inequality constraints as strict inequalities will increase if we relax the linearity assumption on the objective function and allow the objective function to be continuously differentiable and additively separable over time periods. However, in order to be useful for the purposes of proposition 6.1 in Lahiri (2025), we would need to show that any such solution for an infinite horizon optimal control problem with a continuously differentiable objective function that is additively separable over time periods, would also be a solution for the optimal control problem with some linear objective function and the same inequality constraints. This is what we prove here.

If the optimal control problem has a finite horizon, then it follows from the result in Lahiri (2024), that a trajectory solves the optimal control problem with linear constraints and any continuously differentiable and concave objective function if and only if it solves a related optimal control problem with the same constraints and a linear objective function. For the case of infinite horizon problems we are not aware of any such result. In fact, in this note we do not prove any "if and only if" result although that may be possible if we invoke concavity. We simply prove that under appropriate convergence conditions a solution to an infinite horizon optimal control problem with continuously differentiable objective function that is separable over time periods solves a related infinite horizon optimal control problem with a linear objective function, where the objective function depends on the trajectory that solves the original problem. No originality for the result is being claimed, although having it documented seems desirable, so that concerns such as those related to proposition 6.1 in Lahiri (2025) may be "partially" resolved.

2. Notations: For any function G: $[0, 1]\rightarrow\mathbb{R}$ and h $\in[0, 1]$ we will denote the derivative of G with respect to the independent variable at h by DG(h).

For any function $g:X\times X\rightarrow\mathbb{R}$ and $(x, u)\in X\times X$, we will denote the partial derivative of g with respect to the first variable (coordinate) at (x, u) by $D_1g(x, u)$ and the partial derivative of g with respect to the second variable (coordinate) at (x, u) by $D_2g(x, u)$. The definitions of derivatives and partial derivatives applicable in the present context are as in Lahiri (2024).

3. Framework of Analysis:

Let $X = [0, b] \subset \mathbb{R}$ (the set of real numbers), with $b > 0$ be such that **set of available alternatives** at any time period is a non-empty subset of $X\times X$.

With $\mathbb{N}$ denoting the set of natural number (i.e., the set of strictly positive integers) let $\mathbb{N}^0$ denote $\mathbb{N}\cup\{0\}$, i.e., the set of non-negative integers. Time is measured in discrete periods $t\in \mathbb{N}^0$. Beginning with an initial state variable value in X, in each period $t\in\mathbb{N}^0$, a control variable value is chosen from X as a result of which an alternative denoted by $(x_t, u_t)\in X\times X$ is realized. While at all time periods x_t is an "inheritance" in the current period, u_t is chosen during the current period. This pair determines the value of the state variable, denoted by x_{t+1}, that will be realized in the next period.

At each time-period $t\in \mathbb{N}^0$, there exists a non-empty, closed and convex subset $\Omega_t \subset X\times X\times X$ satisfying the following properties: (i) For all $x\in X$, there exists $(u, y)\in X\times X$

such that $(x, u, y)\in\Omega_t$; and (ii) $(x, u, y), (x, u, z)\in\Omega_t$ implies $y = z$. Ω_t is the **two-period constraint set at time-period** t.

For $t\in\mathbb{N}^0$, $(x, u, y)\in\Omega_t$ can be interpreted in the following manner: given that $x\in X$ is the realization of the state variable at time-period t, it is possible to choose the pair $(u, y)\in X\times X$ at time-period t.

For all $(x, t)\in X\times\mathbb{N}^0$, let $\Omega_t(x) = \{(u, y)\in X\times X|\ (x, u, y)\in \Omega_t\}$.

Clearly for all $(x, t)\in X\times\mathbb{N}^0$, $\Omega_t(x)$ is a non-empty, closed and convex subset of $X\times X$.

For $x\in X$, let $\mathcal{F}(x) = \{<(x_t, u_t)|t\in \mathbb{N}^0>|\ (x_t, u_t, x_{t+1})\in\Omega_t, t\in\mathbb{N}^0, x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $<(x_t, u_t)|t\in \mathbb{N}^0 >\in\mathcal{F}(x)$ as a **trajectory starting at (from)** x.

Clearly, $\mathcal{F}(x)$ is non-empty for all $x\in X$.

For each $t\in\mathbb{N}^0$, let $f^{(t)}: X\times X\rightarrow\mathbb{R}$ be a function that is continuously differentiable on $X\times X$.

The concept of a function that is continuously differentiable in the sense that we invoke here has been discussed in Lahiri (2024).

We will refer to the ordered triplet $(<f^{(t)}|\ t\in\mathbb{N}^0>, <\Omega_t|t\in\mathbb{N}^0>)$ as a **infinite horizon discrete-time optimal control problem** hereafter referred to as an **optimal control (OC) problem**.

For $x\in X$, let $\mathcal{S}(x)$ is the set of solutions of the following optimization problem:

Maximize $\sum_{t=0}^{\infty} f^{(t)}(x_t,u_t)$ subject to the infinite sequence $<(x_t, u_t)|t\in \mathbb{N}^0>$ satisfying the constraints: $(x_t, u_t, x_{t+1})\in\Omega_t$, $t\in\mathbb{N}^0$, $x_0 = x$.

Thus, for $x\in X$, let $S(x) = \underset{<(x_t,u_t)|t\in\mathbb{N}^0>\in\mathcal{F}(x)}{\text{argmax}} \sum_{t=0}^{\infty} f^{(t)}(x_t,u_t)$.

Note 3.1: The exact mathematical interpretation of the expression (formula)

$\sum_{t=0}^{\infty} f^{(t)}(x_t,u_t)$ is $\lim_{T\to\infty} f^{(t)}(x_t,u_t)$.

4. Absolutely convergent objective function sequence:

For what follows we will assume the following for $<f^{(t)}|\ t\in\mathbb{N}^0>$:

Given any sequence $<(x_t, u_t)|t\in\mathbb{N}^0>$ in $X\times X$, $\sum_{t=0}^{\infty} |f^{(t)}(x_t,u_t)| < +\infty$, $\sum_{t=0}^{\infty} |D_i f^{(t)}(x_t,u_t)| < +\infty$, for $i\in\{1, 2\}$.

We will refer to any $<f^{(t)}|\ t\in\mathbb{N}^0>$ satisfying this property as an **absolutely convergent objective function sequence**.

Given sequences $<(x_t^{(1)}, y_t^{(1)})|t\in\mathbb{N}^0>$, $<(x_t^{(2)}, y_t^{(2)})|t\in\mathbb{N}^0>$, let F:[0, 1]→$\mathbb{R}$ be the function such that for all h$\in$[0,1], F(h) =

$\sum_{t=0}^{\infty} \mathrm{f}^{(\mathrm{t})}(x_t^{(1)}+\mathrm{h}(x_t^{(2)}-x_t^{(1)}), u_t^{(1)}+\mathrm{h}(u_t^{(2)}-u_t^{(1)}))$.

Since, $<\mathrm{f}^{(\mathrm{t})}| \mathrm{t}\in\mathbb{N}^0>$ is an absolutely convergent objective function sequence, F is well defined.

For T$\in\mathbb{N}$, consider the function F^{T}: [0, 1]→$\mathbb{R}$ defined as follows: for all h$\in$[0, 1], $\mathrm{F}^{\mathrm{T}}(\mathrm{h})$ $= \sum_{t=0}^{T} \mathrm{f}^{(\mathrm{t})}(x_t^{(1)}+\mathrm{h}(x_t^{(2)}-x_t^{(1)}), u_t^{(1)}+\mathrm{h}(u_t^{(2)}-u_t^{(1)}))$.

Clearly, for all T$\in\mathbb{N}$, F^{T} is continuously differentiable on [0, 1], with $\mathrm{DF}^{\mathrm{T}}(\mathrm{h})$ =

$\sum_{t=0}^{T} [\mathrm{D}_1\mathrm{f}^{(\mathrm{t})}(x_t^{(1)}+\mathrm{h}(x_t^{(2)}-x_t^{(1)}), u_t^{(1)}+\mathrm{h}(u_t^{(2)}-u_t^{(1)}))(x_t^{(2)}-x_t^{(1)}) + \mathrm{D}_2\mathrm{f}^{(\mathrm{t})}(x_t^{(1)}+\mathrm{h}(x_t^{(2)}-x_t^{(1)}), u_t^{(1)}+\mathrm{h}(u_t^{(2)}-u_t^{(1)}$
for all h$\in$[0,1].

For all h$\in$(0, 1]: $|\frac{F(\mathrm{h})-F(0)}{\mathrm{h}} - \frac{F^T(\mathrm{h})-F^T(0)}{\mathrm{h}}|$ =

$|\frac{\sum_{t=\mathrm{T}+1}^{\infty} \mathrm{f}^{(\mathrm{t})}(x_t^{(1)}+\mathrm{h}(x_t^{(2)}-x_t^{(1)}), u_t^{(1)}+\mathrm{h}(u_t^{(2)}-u_t^{(1)})) - \sum_{t=\mathrm{T}+1}^{\infty} \mathrm{f}^{(\mathrm{t})}(x_t^{(1)}, u_t^{(1)})}{\mathrm{h}}|$.

Thus. for all h$\in$ (0, 1]: $\lim_{T\to\infty} |\frac{F(\mathrm{h})-F(0)}{\mathrm{h}} - \frac{F^T(\mathrm{h})-F^T(0)}{\mathrm{h}}|$ =

$\lim_{T\to\infty} |\frac{\sum_{t=\mathrm{T}+1}^{\infty} \mathrm{f}^{(\mathrm{t})}(x_t^{(1)}+\mathrm{h}(x_t^{(2)}-x_t^{(1)}), u_t^{(1)}+\mathrm{h}(u_t^{(2)}-u_t^{(1)})) - \sum_{t=\mathrm{T}+1}^{\infty} \mathrm{f}^{(\mathrm{t})}(x_t^{(1)}, u_t^{(1)})}{\mathrm{h}}| = 0$.

By the mean value theorem for differentiable functions for each T$\in\mathbb{N}$, and h$\in$(0, 1], there exists θ(T, h)$\in$ (0, 1) such that $\mathrm{F}^{\mathrm{T}}(\mathrm{h}) = \mathrm{F}^{\mathrm{T}}(0) + \mathrm{DF}^{\mathrm{T}}(\theta(\mathrm{T}, \mathrm{h})\mathrm{h})\mathrm{h}$.

Thus. for all h$\in$ (0, 1]: $\lim_{T\to\infty} |\frac{F(\mathrm{h})-F(0)}{\mathrm{h}} - DF^T(\theta(\mathrm{T}, \mathrm{h})\mathrm{h})| = 0$.

For h$\in$ (0, 1] and T$\in\mathbb{N}$, $|\frac{F(\mathrm{h})-F(0)}{\mathrm{h}}$ -

$\sum_{t=0}^{\infty} [\mathrm{D}_1\mathrm{f}^{(\mathrm{t})}(x_t^{(1)},u_t^{(1)})(x_t^{(2)}-x_t^{(1)}) + \mathrm{D}_2\mathrm{f}^{(\mathrm{t})}(x_t^{(1)},u_t^{(1)})(u_t^{(2)}-u_t^{(1)})]| = |\frac{F(\mathrm{h})-F(0)}{\mathrm{h}}$ -

$DF^T(\theta(\mathrm{T}, \mathrm{h})\mathrm{h}) + DF^T(\theta(\mathrm{T}, \mathrm{h})\mathrm{h})$-

$\sum_{t=0}^{\infty} [\mathrm{D}_1\mathrm{f}^{(\mathrm{t})}(x_t^{(1)},u_t^{(1)})(x_t^{(2)}-x_t^{(1)}) + \mathrm{D}_2\mathrm{f}^{(\mathrm{t})}(x_t^{(1)},u_t^{(1)})(u_t^{(2)}-u_t^{(1)})]| \leq |\frac{F(\mathrm{h})-F(0)}{\mathrm{h}}$ -

$DF^T(\theta(\mathrm{T}, \mathrm{h})\mathrm{h})| + |DF^T(\theta(\mathrm{T}, \mathrm{h})\mathrm{h})$-

$\sum_{t=0}^{\infty} [\mathrm{D}_1\mathrm{f}^{(\mathrm{t})}(x_t^{(1)},u_t^{(1)})(x_t^{(2)}-x_t^{(1)}) + \mathrm{D}_2\mathrm{f}^{(\mathrm{t})}(x_t^{(1)},u_t^{(1)})(u_t^{(2)}-u_t^{(1)})]|$.

Since for each h$\in$(0, 1], the sequence $<\theta(\mathrm{T},\mathrm{h})| \mathrm{T} \in\mathbb{N}>$ lies in the closed and bounded interval [0, 1], it must have a convergent subsequence converging to some θ(h)$\in$[0, 1]. Since as far as 'T' is concerned we shall be concerned only with what happens as "T→∞" without loss of generality suppose that for each h$\in$(0, 1], $\lim_{T\to\infty} \theta(T,\mathrm{h}) = \theta(\mathrm{h})$.

Since, for all hϵ(0, 1]: $\lim\limits_{T\to\infty} |\frac{F(\mathrm{h})-F(0)}{\mathrm{h}} - DF^T(\theta(\mathrm{T},\mathrm{h})\mathrm{h})| = 0$, it must be the case that

for all hϵ(0, 1], $|\frac{F(\mathrm{h})-F(0)}{\mathrm{h}} - \sum_{t=0}^{\infty} [\mathrm{D_1 f^{(t)}(x_t,u_t)}(x_t^{(2)}-x_t^{(1)}) + \mathrm{D_2 f^{(t)}(x_t,u_t)}(u_t^{(2)}-u_t^{(1)})]| \leq$

$\lim\limits_{T\to\infty} |DF^T(\theta(\mathrm{T},\mathrm{h})\mathrm{h}) - \sum_{t=0}^{\infty} [\mathrm{D_1 f^{(t)}}(x_t^{(1)},u_t^{(1)})(x_t^{(2)}-x_t^{(1)}) + \mathrm{D_2 f^{(t)}}(x_t^{(1)},u_t^{(1)})(u_t^{(2)}-u_t^{(1)})]|$
$=$

$\lim\limits_{T\to\infty} |\sum_{t=0}^{T} [\mathrm{D_1 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{T},\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{T},\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))(x_t^{(2)}-x_t^{(1)}) + \mathrm{D_2 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{T},\mathrm{h})\mathrm{h}($

$- \sum_{t=0}^{\infty} [\mathrm{D_1 f^{(t)}(x_t,u_t)}(x_t^{(2)}-x_t^{(1)}) + \mathrm{D_2 f^{(t)}(x_t,u_t)}(u_t^{(2)}-u_t^{(1)})]| =$

$|\sum_{t=0}^{\infty} [(\mathrm{D_1 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))-\mathrm{D_1 f^{(t)}(x_t,u_t)})(x_t^{(2)}-x_t^{(1)}) +$

$(\mathrm{D_2 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))-\mathrm{D_2 f^{(t)}(x_t,u_t)})(u_t^{(2)}-u_t^{(1)})]| \leq$

$\sum_{t=0}^{\infty} [|\mathrm{D_1 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))-\mathrm{D_1 f^{(t)}}(x_t^{(1)},u_t^{(1)})||x_t^{(2)}-x_t^{(1)}| + |\mathrm{D_2 f^{(t)}}(x_t^{(1)}+$
$||u_t^{(2)}-u_t^{(1)}||]$Thus, $|\frac{F(\mathrm{h})-F(0)}{\mathrm{h}} -$

$\sum_{t=0}^{\infty} [\mathrm{D_1 f^{(t)}(x_t,u_t)}(x_t^{(2)}-x_t^{(1)}) + \mathrm{D_2 f^{(t)}(x_t,u_t)}(u_t^{(2)}-u_t^{(1)})]| \leq$

$\sum_{t=0}^{\infty} [|\mathrm{D_1 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))-\mathrm{D_1 f^{(t)}}(x_t^{(1)},u_t^{(1)})||x_t^{(2)}-x_t^{(1)}| + |\mathrm{D_2 f^{(t)}}(x_t^{(1)}+$
$u_t^{(1)}|]$.

Since, for all t$\epsilon\mathbb{N}^0$,

$\lim\limits_{\mathrm{h}\to 0} |\mathrm{D_1 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))-\mathrm{D_1 f^{(t)}(x_t,u_t)}| = 0$,

$\lim\limits_{\mathrm{h}\to 0} |\mathrm{D_2 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))-\mathrm{D_2 f^{(t)}(x_t,u_t)}| = 0$,

$|x_t^{(2)}-x_t^{(1)}| \epsilon\ [0, \mathrm{b}]$ and $|x_t^{(2)}-x_t^{(1)}| \epsilon [0, \mathrm{b}]$, it follows from this and the assumption that

$<\mathrm{f^{(t)}}|\ \mathrm{t}\epsilon\mathbb{N}^0>$ is an absolutely convergent objective function sequence,

$\lim\limits_{\mathrm{h}\to 0} \sum_{t=0}^{\infty} [|\mathrm{D_1 f^{(t)}}(x_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(x_t^{(2)}-x_t^{(1)}),\ u_t^{(1)}+\theta(\mathrm{h})\mathrm{h}(u_t^{(2)}-u_t^{(1)}))-\mathrm{D_1 f^{(t)}}(x_t^{(1)},u_t^{(1)})||x_t^{(2)}-x_t^{(1)}| + |\mathrm{D_2 f^{(t)}}$
$= 0$.

Thus,

$\lim\limits_{\mathrm{h}\to 0} |\frac{F(\mathrm{h})-F(0)}{\mathrm{h}} - \sum_{t=0}^{\infty} [\mathrm{D_1 f^{(t)}}(x_t^{(1)},u_t^{(1)})(x_t^{(2)}-x_t^{(1)}) + \mathrm{D_2 f^{(t)}}(x_t^{(1)},u_t^{(1)})(u_t^{(2)}-u_t^{(1)})]| = 0$.

Thus, DF(0) = $\sum_{t=0}^{\infty} [\mathrm{D_1 f^{(t)}}(x_t^{(1)},u_t^{(1)})(x_t^{(2)}-x_t^{(1)}) + \mathrm{D_2 f^{(t)}}(x_t^{(1)},u_t^{(1)})(u_t^{(2)}-u_t^{(1)})]$.

Lemma 4.1: If DF(0) > 0, then there exists δ ϵ (0, 1), such that for all hϵ(0, δ), F(h) > F(0).

Proof: Since DF(0) = $\lim\limits_{\mathrm{h}\to 0}\frac{F(\mathrm{h})-F(0)}{\mathrm{h}}$, DF(0) > 0 implies that there exists δ ϵ (0, 1), such that for all hϵ(0, δ), $\frac{F(\mathrm{h})-F(0)}{\mathrm{h}}$, i.e., there exists δ ϵ (0, 1), such that for all hϵ(0, δ), F(h) > F(0). Q.E.D.

5. Absolutely convergent OC problems and a necessary condition for optimality:

We shall refer to a OC problem ($<f^{(t)}| t\epsilon\mathbb{N}^0>$, $<\Omega_t |t\epsilon\mathbb{N}^0>$) in which $<f^{(t)}| t\epsilon\mathbb{N}^0>$ is an absolutely convergent sequence as an **absolutely convergent optimal control (AC-OC) problem**.

Proposition 5.1: Let ($<f^{(t)}| t\epsilon\mathbb{N}^0>$, $<\Omega_t |t\epsilon\mathbb{N}^0>$) be an AC-OC problem and for some $x\epsilon X$, let $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{F}(x)$. For $t\epsilon\mathbb{N}^0$ and $i\epsilon\{1, 2\}$, let $p_i^{(t)} = D_if^{(t)}(x_t,u_t)$. If $<(x_t, u_t)|t\in \mathbb{N}^0>\in S(x)$ <u>if and only if</u> $<(x_t, u_t)|t\in \mathbb{N}^0>\in \underset{<(y_t,v_t)|\, t\epsilon\mathbb{N}^0>\epsilon\mathcal{F}(x)}{\text{argmax}} \sum_{t=0}^{\infty}[p_1^{(t)}y_t+p_2^{(t)}v_t]$.

Proof: Suppose $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$ and towards a contradiction there exists $<(z_t, w_t)|t\in \mathbb{N}^0> \epsilon\ \mathcal{F}(x)$ such that $\sum_{t=0}^{\infty}[p_1^{(t)}z_t+p_2^{(t)}w_t] > \sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t]$.

Let $F:[0, 1]\rightarrow\mathbb{R}$ be the function such that for all $h\epsilon[0,1]$, $F(h) =$

$\sum_{t=0}^{\infty} f^{(t)}(x_t+h(z_t\text{-}x_t), u_t+h(w_t\text{-}u_t))$.

Thus, $F(0) = \sum_{t=0}^{\infty} f^{(t)}(x_t, u_t)$

From the discussion in section 4, we know that DF(0) exists and $DF(0) =$

$\sum_{t=0}^{\infty}[D_1f^{(t)}(x_t, u_t)(z_t - x_t) + D_2f^{(t)}(x_t, u_t)(w_t - u_t)] =$

$\sum_{t=0}^{\infty}[p_1^{(t)}(z_t - x_t) + p_2^{(t)}(w_t - u_t)]$.

Thus, $\sum_{t=0}^{\infty}[p_1^{(t)}z_t+p_2^{(t)}w_t] > \sum_{t=0}^{\infty}[p_1^{(t)}x_t+p_2^{(t)}u_t]$ implies $DF(0) =$

$\sum_{t=0}^{\infty}[p_1^{(t)}(z_t - x_t) + p_2^{(t)}(w_t - u_t)] > 0$.

By lemma 4.1, there exists $\delta\ \epsilon\ (0, 1)$, such that for all $h\epsilon(0, \delta)$, $F(h) > F(0)$.

Since $\mathcal{F}(x)$ is a convex set, $<(x_t, u_t)|t\in \mathbb{N}^0>\in\mathcal{F}(x)$ and $<(z_t, w_t)|t\in \mathbb{N}^0> \epsilon\ \mathcal{F}(x)$, $<(x_t+h(z_t\text{-}x_t), u_t+h(w_t\text{-}u_t))t\in \mathbb{N}^0>\in\mathcal{F}(x)$ for all $h\epsilon(0, \delta)$.

Thus, for $h\epsilon(0, \delta)$, $\sum_{t=0}^{\infty} f^{(t)}(x_t+h(z_t\text{-}x_t), u_t+h(w_t\text{-}u_t)) = F(h) > F(0) = \sum_{t=0}^{\infty} f^{(t)}(x_t, u_t)$, contradicts our assumption that $<(x_t, u_t)|t\in \mathbb{N}^0>\in \mathcal{S}(x)$ and proves the proposition.

Q.E.D.

References

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