Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Theorem : Convergence of Collatz (3n+1) Sequence to the Trivial Cycle Proved

© Dr.(Prof.) Keshava Prasad Halemane,
Professor - retired (2017JAN31) from
Department of Mathematical And Computational Sciences,
National Institute of Technology Karnataka Surathkal,
Srinivasnagar, Mangaluru - 575025, India.
https://orcid.org/0000-0003-3483-3521

Independent Researcher
(no affiliations; no funding; no data used; no conflict of interests)
SASHESHA, 8-129/12 Sowjanya Road, Naigara Hills,
Bikarnakatte, Kulshekar Post,
Mangaluru-575005. Karnataka State, India
https://www.linkedin.com/in/keshavaprasadahalemane/
k.prasad.h@gmail.com

Abstract

This paper presents the *Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani* (CTUHSK) theorem, which asserts the convergence of the Collatz (3n+1) Sequence to the trivial cycle, thereby proving the Collatz Conjecture, a long-standing *unsolved problem*. The proof is in two parts. The necessary condition is provided by the *order-preserving isomorphism* (along with an invariant-base-element) established between the relevant component H^s of a structured system framework H and the set of positive integers. The structured system framework H itself has been designed by a two-stage bijective mapping: (1) from the Collatz-domain to

BELnet, that is the network of *binary exponential ladders* defined on the set of positive odd numbers; and (2) from BELnet to the structured system framework H. The sufficient condition is provided by a reductio-ad-absurdum argument (along with a uniquely special modular arithmetic characteristic property of the Collatz system) that is used to demonstrate *domain exhaustion*; having already captured all the modular residue classes; logically excluding the existence of any extraneous elements or objects or sub-systems such as disjoint loops/cycles H^{∞} and/or divergent chains $H^{\&}$ or even any/all non-standard objects, in H. The proof uses the most fundamental Dedekind-Peano axioms and modulus arithmetic properties of the Collatz (3n+1) System; just enough mathematics, without any unnecessary sophistications.

Some directions for possible future research work on algorithmic, computational and/or dynamic characteristics of the Collatz system have also been presented. A situation has been identified wherein the *emergence of global system properties through persistent local subsystem characteristics* can be clearly demonstrated; with {(31←41←27)} and {(1←5←3)} as exceptional and limiting cases.

Keywords: Binary-Exponential-Ladder; Modular-Periodicity; Arborescence;
CTUHSK-Theorem; CTUHSK-Generative-Parameters;
Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Sequence;
Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture;
Order-Preserving Isomorphism; Dedekind-Peano Axioms; Convergence.

AMS MSC Mathematics Subject Classification: 11B83; 11Y55; 93A05; 93A99.

1. Introduction

The Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture, simply referred to as the *Collatz Conjecture* [1] states that the Collatz Sequence, also referred to as the Collatz (3n+1) Sequence, converges to the trivial cycle, that is, {(4->2->1)}, starting from any positive integer.

Lagarias [3]&[4] gives an exhaustive annotated bibliography, whereas Lagarias [5]&[6] gives an elaborate overview of the Collatz Problem, also referred to as the 3x+1 problem, referring to it as "The Ultimate Challenge". Guy [2] has been somewhat pessimistic in advising researchers "Don't Try to Solve These Problems". Lagarius [5]&[6] refers to the statement by the most revered Mathematician and Number Theory Expert Paul Erdos, who said that - "Mathematics is not yet ready for such problems" - about the Collatz Conjecture. The Collatz (3n+1) Sequence is indeed a highly complex chaotic system defined or generated with one of the simplest deterministic well-defined equations, that too, just two in number.

We will not get into any more of the reported literature except the problem definition; but will simply focus on a rather shockingly simple approach to solve this amazing long-standing unsolved problem. This paper presents the *Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani* (CTUHSK) Theorem, asserting the convergence of the Collatz (3n+1) Sequence to the trivial cycle. The proof is two-fold. First is the application of the most fundamental *Dedekind-Peano Axioms* and *Modulus Arithmetic* to a meticulously designed *Structured System Framework* of *Binary-Exponential-Ladder*s defined on the set of positive odd numbers, and establishing an *order-preserving isomorphism* between the set of natural numbers and the *structured system framework* of *Binary Exponential Ladders*. Secondly a reductio-ad-absurdum argument (along with a uniquely special modular arithmetic characteristic property of the Collatz system) ensures the non-existence of any extraneous elements.

2. Problem Description

We define the *Collatz Function* C(n) with a positive integer n as its input argument, in terms of a 'pull-Down' operator D(n) and a 'push-Up' operator U(n) as follows:

if (n is even) C(n) := D(n) = (n/2); else C(n) := U(n) = (3*n+1); [Eqn.1]

where the '*pull-Down*' D(n) operator takes only an even number as its input argument whereas the '*push-Up*' operator U(n) takes only an odd number as its input argument and gives an output that is an even number.

For convenience in our study of the Collatz Sequence, we define the *Compact Collatz Function* T(m) by the repeated application of the 'pull-down' operator D(m) wherever applicable, say, (p≥1) times, that is, $D^*(m) := D^p(m)$ so as to get an output $D^\#(m)$ that is an odd number:

if (m is even) $T(m) := D^*(m) := D^p(m) = (m / 2^p) := D^\#(m)$;
else (m is odd) $T(m) := U(m) = (3*m + 1) := U^\#(m)$; [Eqn.2]

where $D^\#(m)$ is called the "D-floor number" associated with the input argument m; and $U^\#(m)$ is called the "U-ceiling number" associated with the input argument m.

The *Compact Collatz Function* T(m) may as well be considered to have been redefined with these newly introduced two operators, the "D-floor operator" $D^{\#}(m)$ and the "U-ceiling operator" $U^{\#}(m)$ as given in [Eqn.2] above.

This new definition for the *Compact Collatz Function* T(m) facilitates our study of the corresponding *Compact Collatz Sequence*; which is no different from its equivalent standard Collatz Sequence, once we understand that the repeated application, say, (p≥1) times, of the 'pull-Down' operator D(m) has now been collapsed into an equivalent single "D-floor operator" $D^{\#}(m)$ giving the D-floor number $D^{\#}(m)$ as its output. The push-Up operator U has been simply redefined as the "U-ceiling operator" $U^{\#}$ for uniformity and elegant completeness.

The *Compact Collatz Sequence* is obtained by the repeated sequential application of the *Compact Collatz Function* T(m) starting with the given initial input number m - represented by an alternating series of $D^{\#}$ number and $U^{\#}$ number - except possibly the starting initial 'seed' number m and the final terminating number, which as per the Collatz Conjecture, is anyway a $D^{\#}$ number that is unity.

3. Observations on the Pull-Down Operator

The pull-Down operator D *always* takes only an even number n as its input argument. Every application of this pull-down operator results in an alternating (toggling) effect on the nMOD3 property of the input argument number; that is, a 1MOD3 input gives a 2MOD3 output and a 2MOD3 input gives a 1MOD3 output; whereas a 0MOD3 input gives a 0MOD3 output. Repeated application of D, in case applicable, results in a final output that is an odd number and therefore becomes an input for the push-Up operator. In such a case, we call it a "D-floor operator" $D^{\#}$ as defined in [Eqn.2] above, and its output a "D-floor number" $D^{\#}(n)$ characterized by being a odd number; $D^{\#}(n)$ may be in any one of the three possible types:

(1) a 1MOD6 odd number, being a 1MOD3 odd number of the type (6m-5);

(2) a 5MOD6 odd number, being a 2MOD3 odd number of the type (6m-1);

(3) a 3MOD6 odd number, being a 0MOD3 odd number of the type (6m-3).

4. Observations on the Push-Up Operator

The push-Up operator U *always* takes only an odd number m as its input argument, and *always* gives an output that is a 4MOD6 number, being a 1MOD3 even number that is of the type (6m-2); irrespective of whether the input is a 1MOD6 odd number or a 3MOD6 odd number or a 5MOD6 odd number. Note that one single application of the 'push-Up' operator U transforms any input odd number m into a 4MOD6 even number that becomes an input to the "D-floor operator $D^{\#}$". That is why we may as well consider the push-Up operator U as the "U-ceiling operator" $U^{\#}$ as defined in [Eqn.2] above.

5. Observations on the Compact Collatz Function

Start with any positive integer. (1) If the starting initial number n is even, then we apply the D-floor operator $D^{\#}$ operator giving an output that is the D-floor number $D^{\#}(n)$ which is given as input to the U-ceiling operator. Of course, if the starting number is a power of 2 we terminate at unity. Else, we have a D-floor number $D^{\#}$(n) that is an odd number greater than unity, in any non-trivial case; as the initial $D^{\#}$ node in the Compact Collatz Sequence. (2) If on the other hand the starting initial seed number n is an odd number, we treat that itself as the initial $D^{\#}$ node in the Compact Collatz Sequence.

Having thus obtained the initial $D^{\#}$ node in the Compact Collatz Sequence, we apply the U-ceiling operator $U^{\#}$ to get the U-ceiling number $U^{\#}$ that is a 4MOD6 even number. That in turn is given as input to the D-floor operator $D^{\#}$. Now the process continues.

Note that the *Compact Collatz Sequence* can therefore be defined by a *trajectory* generated by an *alternating sequence* of a "D-floor number" $D^{\#}$ and a "U-ceiling number" $U^{\#}$, *with its starting initial node being a $D^{\#}$ number.* The Compact Collatz Function as presented in [Eqn.2] defines the unique link (directed arc) from any given D-floor number $D^{\#}$ as the predecessor node to its corresponding unique U-ceiling number $U^{\#}$ as the successor node and also the unique link (directed arc) from any given U-ceiling number $U^{\#}$ as the predecessor node to its corresponding unique D-floor number $D^{\#}$ as the successor node. The unique link (directed arc) from a starting initial even "seed" number leading to the first node (D-floor number $D^{\#}$) in the *trajectory* is similarly defined.

As mentioned earlier, the application of the D-floor operator $D^{\#}$ on a U-ceiling number $U^{\#}$ that is a 4MOD6 even number of the form (6m-2) can lead to a D-floor number $D^{\#}$ that is an odd number that can be either: (1) a 1MOD6 odd number, being a 1MOD3 odd number that is of the type (6m-5); (2) a 5MOD6 odd number, being a 2MOD3 odd number that is of the type (6m-1); but can never be (3) a

3MOD6 odd number, that is a 0MOD3 odd number of the type (6m-3). The only exception, when the D-floor operator $D^{\#}$ gives an output D-floor number $D^{\#}$ that is a 3MOD6 odd number of the type (6m-3) is the situation when its input is a 0MOD6 even number, which is impossible for any U-ceiling number $U^{\#}$; although such an input may come in those special cases wherein the starting initial 'seed' number itself is a 0MOD6 even number that is of the form $(6m\text{-}3).2^{p}$ leading to an output $D^{\#}$ that is a 3MOD6 odd number of the form (6m-3).

6. Analysis of the Compact Collatz Sequence

From the above observations, it is clear that corresponding to every positive integer n as the starting initial 'seed' number, there is a starting initial node in the trajectory representing the *Compact Collatz Sequence*, that is a $D^{\#}$ number in exactly one of the three possible forms as mentioned above - that can be an input argument to the U-ceiling operator $U^{\#}$ giving exactly one unique output $U^{\#}$ which itself can be an input to the D-floor operator $D^{\#}$ so that the process continues. Successive application of each of these two operators ($U^{\#}$ and $D^{\#}$) wherever applicable, traces a unique *trajectory*, wherein each node is represented by the unique output number of the appropriate operation applied to the input number represented by the preceding node in the trajectory.

The anticipated terminating trivial cycle {(4->2->1)} can be obtained only through a final application of the D-floor operator $D^{\#}$ on a 4MOD6 even number that is of the form (6m-2).

7. Binary-Exponential-Ladder with its Defining-Base-Rung $D^{\#}$

Here, we present a meticulously designed *Structured System Framework* that *partition*s the *set of positive integers* to facilitate a *general systems analysis* of the *Compact Collatz Sequence*.

Let every positive odd number be associated with a *Binary-Exponential-Ladder*, denoted by BEL(2m-1) and defined as a sequence $\{(2m\text{-}1).2^u \mid (u \geq 0)\}$; with its *defining-base-rung* (u=0) given by the odd number (2m-1).

$$\text{BEL}(2m\text{-}1) := \{(2m\text{-}1).2^u \mid (u \geq 0)\}; \qquad \text{[Eqn.3]}$$

Thus, we establish an exact one-to-one correspondence (bijection) between the *set of positive odd numbers* that form the $D^{\#}$ value for the *defining-base-rung* and the corresponding *Binary Exponential Ladder* BEL($D^{\#}$).

Every *positive even number* in the form $\{(2m\text{-}1).2^u \mid (u>0)\}$; for which there exists its corresponding $D^{\#}$ value, $D^{\#}((2m\text{-}1).2^u) = (2m\text{-}1)$; for which there exists exactly one corresponding *Binary-Exponential-Ladder* BEL(2m-1) that contains the given even number $(2m\text{-}1).2^u$ as B((2m-1),u) as one of the higher rungs in BEL(2m-1).

$$B((2m\text{-}1), u) := \{(2m\text{-}1).2^u \mid (u > 0)\}; \qquad \text{[Eqn.4]}$$

Thus, we establish that *the set of all Binary-Exponential-Ladders form a partition of the set of all positive integers*; with an *exact one-to-one correspondence* (*bijection*) between each positive odd number $D^{\#}$ and the corresponding *Binary-*

Exponential-Ladder for which it is the *defining-base-rung* D$^{\#}$; whereas each of the positive even numbers correspond to exactly one of the higher rungs of a specific Binary-Exponential-Ladder identified by the D-floor number D$^{\#}$ associated with that given positive even number.

This partitioned framework of positive integers goes another step deeper because of the fact that the *defining-base-rung* D$^{\#}$ of a Binary-Exponential-Ladder BEL(D$^{\#}$) can itself be in one of the three possible forms 1MOD6 or 5MOD6 or 3MOD6; whereas all the upper rungs of the Binary-Exponential-Ladder are either (1) alternatingly 2MOD6 and 4MOD6; or (2) all being 0MOD6 numbers.

The Collatz Conjecture states that every Collatz Sequence, starting from any positive integer, converges to the trivial cycle {(4->2->1)} which is in BEL(1) that is uniquely identified by its defining-base-rung D$^{\#}$ value that is unity. Therefore, our focus will be the Binary-Exponential-Ladders BEL(1) and its relationship with every other Binary-Exponential-Ladder BEL(D$^{\#}$).

As seen above, D$^{\#}$ can be a positive odd number in any one of the three possible forms: (1) a 1MOD6 number of the form (6m-5); (2) a 5MOD6 number of the form (6m-1); (3) a 3MOD6 number of the form (6m-3). BEL(6m-5) contains the output of U$^{\#}$ at (6m-5)2^{W} with w being an even exponent that is of the form (2k) wherein the input of U$^{\#}$ is given by [{(6m-5).2^{W} -1}/3]. BEL(6m-1) contains the output of U$^{\#}$ at (6m-1)2^{V} with v being an odd exponent that is of the form (2k-1) wherein the input of U$^{\#}$ is given by [{(6m-1).2^{V} -1}/3]. However, BEL(6m-3) cannot contain any such output of the U-ceiling operator U$^{\#}$ irrespective of any input argument.

8. Immediate Neighborhood of a Binary-Exponential-Ladder

The relationship between a pair of Binary-Exponential-Ladders BEL(m) and BEL(n) can be considered to be defined and characterized by the relationship between the corresponding pair of the *defining-base-rung* $D^{\#}$ values m and n along with the corresponding pair $U^{\#}(m)$ and $U^{\#}(n)$.

The immediate-neighborhood of a given Binary-Exponential-Ladder BEL($D^{\#}$) is defined by the *immediate-predecessors* and *immediate-successors*, considering the U-ceiling operator $U^{\#}$; since the D-floor operator $D^{\#}$ is applicable only within a given Binary-exponential-Ladder and not between a pair of them.

8.1 Single Unique Immediate Successor

It turns out that the only *one single unique immediate successor* of BEL(m) is BEL($D^{\#}(U^{\#}(m))$) that contains $U^{\#}(m)$ as one of its higher rungs, with n := $D^{\#}(U^{\#}(m))$ as its identifying characteristic D-floor number being its defining-base-rung.

$$S(BEL(m)) = BEL(D^{\#}(U^{\#}(m))) := BEL(n); \text{ with } n := D^{\#}(U^{\#}(m)); \quad \text{[Eqn.5]}$$

Note that [Eqn.5] describes the forward tracing movement (transfer function) from one BEL to its immediate successor BEL defined by the very same Compact Collatz Function [Eqn.2] above; wherein the downward movement ($D^{\#}$ operator) from a higher rung to the defining base rung within the same BEL is compressed and left as implicit; whereas the upward movement from the defining base rung of one BEL to a higher rung of its immediate successor BEL is kept explicit; thus redefining the Collatz Sequence by its equivalent, a *condensed compact Collatz sequence*; which is simply a hop from one BEL to its immediate successor BEL. Here, [Eqn.5] can therefore be considered as the condensed compact Collatz function; representing one hop from a positive odd number to its immediate successor odd number; thus defining the condensed compact Collatz sequence, that is equivalent to skipping the intermediate even numbers in the standard Collatz sequence. Our study will focus on the characteristics of this condensed compact Collatz sequence, or equivalently the characteristics of the network of binary exponential ladders.

8.2 Multiple Immediate Predecessors

There exists a *set of immediate-predecessors* for each BEL($D^{\#}$) of the form BEL(6m-5) and BEL(6m-1) although none for BEL(6m-3). Note that if S(BEL(m)) is BEL(n) than BEL(m) is one of the predecessors of BEL(n).

The *set of immediate-predecessors* for a given BEL(n) is defined by considering the *inverse of the immediate-successor relationship*; as the set of all BELs each of which having its single unique immediate-successor as BEL(n).

$$\{P(BEL(n))\} := \{BEL(m) \mid BEL(n) = S(BEL(m)\}; \qquad [Eqn.6]$$

BEL(1MOD6) or equivalently BEL(6m-5) has, as its set of immediate-predecessors, {BEL([(1MOD6).2^{W} -1]/3)} or equivalently {BEL([(6m-5).2^{W} -1]/3)} with w being an positive even exponent of the form (2k), wherein the input of $U^{\#}$ is given by

[{(1MOD6).2^W -1}/3] or equivalently [{(6m-5).2^W -1}/3] and the output of $U^\#$ being {(1MOD6).2^W} or equivalently {(6m-5).2^W} that is contained in BEL(1MOD6). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(1MOD6).

{P(BEL(6m-5))} = {BEL([(6m-5).2^W -1]/3)}; [Eqn.7]

BEL(5MOD6) or equivalently BEL(6m-1) has, its set of immediate-predecessors, {BEL([(5MOD6).2^V -1]/3)} or equivalently {BEL([(6m-1).2^V -1]/3)} with v being a positive odd exponent of the form (2k-1), wherein the input of $U^\#$ is given by [{(5MOD6).2^V -1}/3] or equivalently [{(6m-1).2^V -1}/3] and the output of $U^\#$ being {(5MOD6).2^V} or equivalently {(6m-1).2^V} that is contained in BEL(5MOD6) or equivalently BEL(6m-1). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(5MOD6).

{P(BEL(6m-1))} = {BEL([(6m-1).2^V -1]/3)}; [Eqn.8]

BEL(3MOD6) or equivalently BEL(6m-3) has no immediate-predecessors.

{P(BEL(6m-3))} = ϕ; [Eqn.9]

Note that [Eqn.6] or equivalently, [Eqn.7]&[Eqn.8]&[Eqn.9] together define the algebra for the inverse of the condensed compact Collatz function.

8.3 Quaternary-Exponential-Ladder

The above property, that *only* the alternating rungs, defined by {(1MOD6).4^u | u>0} of BEL(1MOD6) or {(5MOD6).2.4^u | u≥0} of BEL(5MOD6) are the 'active' nodes in the CTUHSK-Sequence; makes it convenient to define a system of *Quarternary-Exponential-Ladder*s QEL wherein every rung of QEL becomes an 'active' node in the CTUHSK-Sequence. This concept is not directly needed for proving the convergence of the Collatz Sequence, and therefore we will not take up this line of study in this paper.

8.4 BEL(1) as the Central Focus

Considering BEL(1) as our central focus of interest, which itself belongs to the type BEL(1MOD6) or equivalently BEL(6m-5); it is interesting to note that it has its *single unique immediate-successor*; as S(BEL(1)) = BEL($D^\#$($U^\#$(1))) = BEL(1); that is, BEL(1) itself is its single unique immediate-successor, and that it has no other immediate-successor distinct from itself; because of the fact that the trivial cycle {(4->2->1)} contained within BEL(1).

8.5. BELnet : Network of Binary-Exponential-Ladders

The above discussion about the successor predecessor relationship among the binary-exponential-ladders and its neighborhood leads to the observation that the *network of binary-exponential-ladders*, BELnet, has countably infinite number of each of the three classes/types of nodes: (1) BEL(1MOD6) or equivalently

BEL(6m-5); (2) BEL(5MOD6) or equivalently BEL(6m-1); and (3) BEL(3MOD6) or equivalently BEL(6m-3). Each BEL being a node of the BELnet has a single unique outward directed arc that points towards its single unique immediate-successor, specifically linking onto some higher rung. Multiple (countably infinite number of) inward directed arcs, each linked onto some specific higher rung of a given BEL, emanate from its immediate-predecessor. BEL(1) is an invariant-base-element or equivalently a sink node in BELnet, the network of binary exponential ladders.

Thus, the set of binary-exponential-ladders is an exact representation of the Collatz-domain and the network of binary-exponential-ladders BELnet is an exact representation of the Collatz-map.

The *connectedness* of the network of binary-exponential-ladders BELnet will be analyzed from the *design of a structured system framework* consisting of the entire set of binary-exponential-ladders, merely as a re-organized *condensation* of the very same BELnet, as presented below.

9. Structured System Framework H

From the above discussion we find that it is convenient for our study to consider a *Structured System Framework* H as an infinite (well-ordered) sequence of terms each of which being a set of BELs; that is, $H := \{\ldots H_{k-1} < H_k < H_{k+1} \ldots\}$ wherein the *well-ordering relationship* between the adjacent terms of the sequence is derived from the successor predecessor relationships among the BELs that form the member elements of these adjacent terms in the sequence.

Specifically, H_k is defined as the set formed by the unique immediate-successor of each BEL belonging to H_{k+1} and also the set of immediate-predecessors of each BEL belonging to H_{k-1}; that is,

$$H := \{. . . H_{k-1} < H_k < H_{k+1} . . .\}; \qquad \text{[Eqn.10]}$$

and

$$H_k := \{S(BEL(m)) \mid BEL(m) \in H_{k+1}\} \cup \{BEL(m) \mid S(BEL(m)) \in H_{k-1}\}; \qquad \text{[Eqn.11]}$$

Note that the second part of [Eqn.11] here is required to ensure that BELs of the class/type BEL(3MOD6) can be included in each term H_k since each of them have immediate-successor in H_{k-1} although none of them have any predecessors in H_{k+1}.

Now, we may as well define the predecessor relationship as the inverse of the above defined successor relationship, as –

$$H_{k-1} := S(H_k) \quad \text{and} \quad H_k := S(H_{k+1}); \qquad \text{[Eqn.12]}$$

and

$$P(H_{k-1}) := H_k \quad \text{and} \quad P(H_k) := H_{k+1}; \qquad \text{[Eqn.13]}$$

The multiplicity of the *immediate-predecessor* relationship among the BELs requires that the set of all immediate-predecessors of every element of H_{k-1} form the elements of the set H_k so as to guarantee the strict and complete ordering relation $H_{k-1} < H_k < H_{k+1}$ among these sets, in spite of only a partial ordering relationship among the BELs; and also to guarantee that the entire set of all the BELs are present in H thus making it as a *re-organized structure* for BELnet.

9.1 Closed Chains and Unbounded Chains and Sink Nodes in H

The design of the structured system framework H can *in general* allow for the existence of sink nodes (invariant-base-elements) and/or unbounded open chains and/or closed chains (loops). That is, the structured system framework H can in general be *partitioned into three mutually disjoint and independent components*,

$$H := H^{s} \cup H^{\&} \cup H^{\infty} \qquad \text{[Eqn.14]}$$

where (1) H^{s} corresponds to the set of all possible terms in H connected with sink nodes; (2) $H^{\&}$ corresponds to the set of all possible terms in H connected with unbounded open chains; and (3) H^{∞} corresponds to the set of all possible terms in H connected with closed chains (loops). In such a situation, each of these components, H^{s} and $H^{\&}$ and H^{∞} needs to satisfy the well-ordering conditions expressed above in [Eqn.10], [Eqn.11], [Eqn.12] & [Eqn.13].

9.2 A Sink Node H_0 in H^{s}

We have observed earlier that BEL(1) itself is its single unique immediate-successor and does not have any immediate-successor distinct from itself, although it has multiple immediate-predecessors. That is, BEL(1) is an invariant-base-element or equivalently a sink node in BELnet. Therefore, the component H^{s} must necessarily have a term H_0 as its *invariant-base-element* or equivalently a *sink node*, that is, $H_0 := \{BEL(1)\}$; with BEL(1) being its singleton member element. That is,

$H^s := \{H_0, H_1, H_2, . . .\};$ [Eqn.15]

From the above discussion we observe that $H^s := \{H_0, H_1, H_2, . . .\}$ is, by its very design, an infinite (well-ordered) sequence of terms, each term being a countably infinite set of BELs with an exception that the 'root' $H_0 := \{BEL(1)\}$ is a singleton set. The set of k^{th} immediate predecessors of BEL(1) form the set H_k at tier-k level in the hierarchy, if one wishes to consider it as a hierarchy.

The Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Theorem is presented and proved below, which establishes an *order-preserving isomorphism* between H^s and the set of positive integers, thus proving the convergence of the Collatz Sequence starting from any given positive integer contained in any BEL that is in H^s to the trivial cycle {(4->2->1)} at the base of BEL(1) which itself is in H^s.

10. Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani Theorem

STATEMENT OF THE CTUHSK THEOREM

The Collatz Sequence converges to the trivial cycle {(4->2->1)}.

PROOF

There are two parts to the proof – a *necessary condition* and a *sufficient condition*. Note that the entire Collatz Domain has been mapped to the structured system framework H through a two-stage bijective mapping. The necessary condition is provided by establishing an order-isomorphism between H^s and the set of natural numbers, by the application of the Dedekind-Peano axioms. The sufficient condition is provided by a reductio-ad-absurdum argument (along with a uniquely special modular arithmetic characteristic property of the Collatz system) that establishes domain exhaustion and therefore guarantees the non-existence of any extraneous elements.

10.1 Proof of CTUHSL Theorem - Necessary Condition

We show that H^s satisfies the Dedekind-Peano's axioms (replacing the 'successor' by the 'predecessor') and therefore H^s is order-isomorphic with the set of natural numbers; and satisfies the necessary condition for the above convergence statement.

DEDEKIND-PEANO AXIOM : Existence of 1 as the invariant-base-element.

$H_0 \in H^s$. H_0 is the invariant-base-element of H^s.

The trivial cycle {(4->2->1)} $\in$ BEL(1) is contained in $H_0 \in H^s$.

DEDEKIND-PEANO AXIOM : Existence of a *successor function.*

By the very design of $H^s := \{H_0, H_1, H_2, \ldots\}$, for every positive integer k,

$H_k \in H^s$ is the *predecessor* of $H_{k-1} \in H^s$.

Application of the Compact Collatz Function with the input from numbers contained in some BEL that is a member of H_k yields the single unique output number contained in some immediate-successor BEL that is a member of H_{k-1}; because of the definition of the successor predecessor relationship between H_k and H_{k-1}.

DEDEKIND-PEANO AXIOM : 1 is not a successor; 1 has no predecessor; 1 is a source node in the sequence of natural numbers.

H_0 is not a predecessor to any other H_k. There *does not exist any* $H_k \in H^s$, $k \neq 0$; that is distinct from H_0; with $H_k \neq H_0$; such that H_0 is the predecessor of H_k.

H_0 does not have any successor distinct from itself.

H_0 is a sink node in the sequence $H^s := \{H_0, H_1, H_2, \ldots\}$.

Once the Collatz Sequence reaches the trivial cycle (sink) there is no exit from it.

DEDEKIND-PEANO AXIOM : Successor function is a unique one-to-one mapping.

If H_u is the predecessor of H_v and also H_u is the predecessor of H_w;

then it necessarily implies $H_v = H_w$ by the very design of H^s;

and,

If H_v is the predecessor of H_u and also H_w is the predecessor of H_u;

then it necessarily implies $H_v = H_w$ by the very design of H^s;

This is because the predecessor relation in H^s is a unique one-to-one mapping.

Also, note that for each positive integer k there corresponds a unique set $H_k \in H^s$, and for each $H_k \in H^s$ there corresponds a unique positive integer k; thus, establishing a one-to-one mapping (*bijection*) between H^s and the set of positive integers.

This guarantees the Compact Collatz Sequence to be a linear directed path (chain) with no forking or merging in H^s (although merging is observed deeper at the level of the BELs) and the path traces through . . . H_{k+1} onto H_k onto H_{k-1} . . . etc in that order, wherein each of these terms in H^s correspond to a node (either a $D^{\#}$ node in H_{k+1} followed by a $U^{\#}$ node in H_k followed by a $D^{\#}$ node in H_k followed by a $U^{\#}$ node in H_{k-1} and so on) in the Compact Collatz Sequence which is itself an alternating sequence of these $D^{\#}$ nodes and $U^{\#}$ nodes as observed earlier.

DEDEKIND-PEANO AXIOM : Principle of induction.

Collatz Sequence starting with any number from BEL(1) $\in H_0$ converges in the trivial cycle {(4->2->1)} $\in$ BEL(1) $\in H_0$.

Collatz Sequence starting with any positive number that passes through H_k must necessarily pass through H_{k-1} because by design $H_{k-1} := S(H_k)$.

Therefore, the Collatz Sequence starting with any positive integer being contained in some $H_k \in H^s$, $k \geq 0$; must necessarily reach H_0 and therefore converge in the trivial cycle.

Thus, we establish a direct *order-preserving isomorphism* between H^s and the set of Natural Numbers N; and the proof of the necessary condition for the convergence of the Collatz Sequence; that the Collatz sequence starting with any positive odd number that is the defining base rung of a BEL within H^s converges to the trivial cycle {(4->2->1)}.

10.2 Proof of CTUHSL Theorem - Sufficient Condition

The proof for the sufficient condition is provided in two parts; that is, by establishing the non-existence of any extraneous elements/objects/sub-systems as (1) disjoint loops/cycles H^{∞}; and/or (2) divergent chains $H^{\&}$ in the structured system framework H; so that $H^s = H$.

10.2.1 Non-existence of Disjoint Loops/Cycles H^{∞} in H

The non-existence of any loop/cycle H^{∞} (finite or infinite) other than the trivial cycle {(4->2->1)} in BELnet can be proved here by a *reductio-ad-absurdum* argument. If indeed there exists such a loop/cycle, it must necessarily be a sink loop with no exit, because of the characteristic property that there exists only a single unique immediate-successor for every BEL in BELnet. Also, the non-existence of any immediate-predecessor for BEL(6m-3) implies that BEL(6m-3) cannot be in any such loop. Being a sink loop consisting of only BEL(6m-5) and BEL(6m-1) - each with multiple immediate-predecessors; implies that such a loop can be collapsed into an element, say, X_{θ} in H^{∞} (that is similar to H_0 as in H^s); and a sequence of predecessors can be developed similar to [Eqn.15] as in [Eqn.16];

$$H^{\infty} := \{X_{\theta}, X_1, X_2, \ldots\}; \qquad \text{[Eqn.16]}$$

Applying the same arguments as in the necessary condition of the theorem presented above, we can establish the order-isomorphism between the set of natural numbers and H^{∞}; with H^{∞} itself having the loop/cycle as a sink in the super-node X_{θ}. This implies that there is a parallel system that is equivalent to H^s that is also order-isomorphic to the set of natural numbers. As explained above, X_{θ} can contain only {BEL(6m-5)} and {BEL(6m-1)} but cannot contain {BEL(6m-3)}.

Suppose among set of BELs of the form BEL(6m-1)$\in H^{\infty}$; we assume that there exists a BEL(6k-1)$\in H^{\infty}$; that is the BEL with the minimum numerical value associated with its defining base rung of lowest denomination. Either {BEL(6k-1)}$\in X_{\theta}$; or {BEL(6m-1)}$\in (H^{\infty} \setminus X_{\theta})$. In either case, applying [Eqn.8] with v=1 leads to a situation wherein we get the set of its immediate predecessors of lowest denomination as -

$$\{P(BEL(6k\text{-}1))\} = \{BEL([(6m\text{-}1).2^1 \text{-}1]/3)\} = \{BEL(4k\text{-}1)\}; \qquad [Eqn.17]$$

establishing that BEL(6k-1)$\in H^\infty$.

This is a direct contradiction to the above assumption, since for any natural number k, (4k-1)<(6k-1); whereas the above assumption is that BEL(6k-1)$\in H^\infty$ is the BEL with the minimum numerical value associated with its defining base rung of lowest denomination.

Therefore, we conclude the non-existence of any such loops/cycles in BELnet whether finite or infinite, and hence the non-existence of H^∞ in H.

10.2.2 Non-existence of Divergent Chains $H^{\&}$ in H

Suppose among set of BELs of the form BEL(6m-1)$\in H^{\&}$; we assume that there exists a BEL(6k-1)$\in H^{\&}$; that is the BEL with the minimum numerical value associated with its defining base rung of lowest denomination. We can use exactly the similar reductio-ad-absurdum argument as used above, banking on [Eqn.17] in establishing the non-existence of H^∞; to show that BEL(6k-1)$\in H^{\&}$ must necessarily implies that BEL(4k-1)$\in H^{\&}$. Because k is necessarily a natural number belonging to the Collatz domain; although $H^{\&}$ is assumed to be a divergent chain; such a possibility leads to the situation wherein k can take its least possible value, k=1.

This is a contradiction, because k=1 corresponds to BEL(5) which is already in $H_1 \in H^s$; and because $H^{\&}$ and H^s are in mutually disjoint components of H. Therefore, there cannot exist any divergent chain (infinite threads); and $H^{\&} = \phi$.

10.2.3 Existence of any/every/all Non-standard Rogue Elements in H

Note that the Collatz-domain does not allow for the existence of any non-standard *rogue elements* etc. The study of non-standard extraneous objects (of whatever kind) is immaterial to the present proof, because the Collatz (3n+1) Conjecture is stated strictly within the domain of natural numbers.

Having established that $H^{\infty} = \phi$ and $H^{\&} = \phi$ in [Eqn.14] above; we get $H = H^s$; therefore, asserting the Collatz Conjecture, that starting from any positive integer the Collatz (3n+1) Sequence converges to the trivial cycle {(4->2->1)}.

END OF PROOF

11. Some Explicit Forms for the BEL Neighborhood

We can perform some simple algebraic manipulation to get the parametric relation [Eqn.18] that gives a generic form for the set H_k that is the set of k^{th} predecessors of $H_0 = \{BEL(1)\}$; that is, the set H_k corresponds to the set of tier-k level of the hierarchy with the set of Binary-Exponential-Ladders {BEL(m)} each with its defining-base-rung m being a positive odd number m>1.

$$m = [2^z - \{3^0.2^{z0} + 3^1.2^{z1} + 3^2.2^{z2} + \ldots + 3^{k-1}.2^{zk-1}\}] / 3^k; \qquad \text{[Eqn.18]}$$

wherein $k>0$ is the tier-level and $z>z_0$ whereas the k-tuple $(z_0, z_1, z_2, \ldots z_{k-1})$ form the set of strictly decreasing non-negative integer exponents values in [Eqn.18] each of which takes a unique value corresponding to each positive odd number $m>1$. That is, each positive odd number $m>1$ can be considered to be defined by the corresponding unique set of these parameters. The set of values for the k-tuple $(z_0, z_1, z_2, \ldots z_{k-1})$ are of strictly decreasing non-negative integer exponent values, all less than z, that is, $z > z_0 > z_1 > z_2 > \ldots > z_{k-1}$; ($z_k := 0$; $z_{k-1} = 0$ for positive odd number $m>1$).

Now, define $p_0 := (z - z_0)$; $p_j := (z_{j-1} - z_j)$; where p_j corresponds to the number of rungs in BEL{H_j} above the *defining-base-rung* of BEL{H_j} for the node located in BEL{H_j} that the Collatz sequence/trajectory passes through; BEL{H_j} being the *Binary-Exponential-Ladder* at tier-j with $j=0,1,2, \ldots k$. Thus, we may as well redefine the set of $(k+1)$ parameters as a tuple (GPT) $:= (p_0, p_1, p_2, \ldots p_k)$ the set of $(k+1)$ CTUHSK generative parameters that generate each positive integer n as per the parametric relation [Eqn.18] given above ($p_k = 0$ for positive odd number m).

For any $k>0$, the above set of exponents $z, z_0, z_1, z_2, z_3, \ldots z_k$, can be redefined in terms of the newly defined CTUHSK generative parameters, by rewriting the above definition as $z := (z_0 + p_0)$; $z_{j-1} := (z_j + p_j)$; $z_k := 0$; $p_k = 0$ for any positive odd number m.

Table-1 gives some of the possible set of valid values for the CTUHSK generative parameters and therefore the corresponding valid values of the exponents in [Eqn.18] above along with their resultant $n(\text{GPT}) := n(p_0, p_1, p_2, \ldots p_k)$ values.

Table-1 : Some typical CTUHSK generative parameter tuples GPT(n)

p0	p1	p2	p3	p4	p5	p6	p7	p8	n	z	z0	z1	z2	z3	z4	z5	z6	z6	z7
0									1	2	0								
1									2										
2									4										
3									8										
4									16										
4	0								5	4	0	0							
4	1	0							3	5	1	0	0						
4	3	0							13	7	3	0	0						
4	3	2	0						17	9	5	2	0	0					
4	3	2	1	0					11	10	6	3	1	0	0				
4	3	2	1	1	0				7	11	7	4	2	1	0	0			
4	3	2	1	1	2	0			9	13	9	6	4	3	2	0	0		
6	0								21	6	0	0							
4	3	2	1	3	0				29	13	9	6	4	3	0	0			
4	3	2	1	3	1	0			19	14	10	7	5	4	1	0	0		
4	3	2	1	3	1	2	0		25	16	12	9	7	6	3	2	0	0	
4	3	2	1	3	1	2	2	0	33	18	14	11	9	8	5	4	2	0	0
p0	p1	p2	p3	p4	p5	p6	p7	p8	n	z	z0	z1	z2	z3	z4	z5	z6	z6	z7

Table-1 : Some typical CTUHSK generative parameter tuples GPT(n)

12. A Challenge to the Cool-Headed Brave-Hearts

If you can prove that corresponding to every positive odd number m>1 there exists a unique valid set of CTUHSK generative parameters GPT(m) = $\{p_0, p_1, p_2, \ldots p_k\}$ and therefore the corresponding valid set of exponents $\{z, z_0, z_1, z_2, z_3, \ldots z_{k-1}, z_k\}$ in the parametric equation [Eqn.18] given above that generates every positive odd number m>1, then you can directly prove the Collatz Conjecture establishing the convergence of the Collatz Sequence to the trivial cycle {(4->2->1)}.

12.1 Restrictions on the CTUHSK Generative Parameters

Note that the set of valid values for the CTUHSK generative parameters and therefore for the exponents in [Eqn.18] above, are governed by certain rules as can be seen from the earlier observations, regarding the matching relationship between the $((D^{\#})MOD3)$ of the predecessor and the $((U^{\#})MOD3)$ of the successor in the Collatz Sequence.

Specifically, [Eqn.19] states the relationship satisfied among (i) the [x]MOD3 value of the exponent x for $U^{\#} = \{(6m-5).4^{x}\}$ at some higher rung in QEL(6m-5); (ii) with its defining-base-rung at (6m-5); and (iii) its predecessor $D^{\#} = [\{(6m-5).4^{x} - 1\}/3]$. Similarly, [Eqn.20] states the relationship satisfied among (i) the [y]MOD3 value of the exponent x for $U^{\#} = \{(6m-1).2.4^{y}\}$ at some higher rung in QEL(6m-1); (ii) with its defining-base-rung at (6m-1); and (iii) its predecessor $D^{\#} = [\{(6m-1).2.4^{y} - 1\}/3]$.

$$[\{(6m-5).4^{x} - 1\}/3]MOD3 = [[x]MOD3 - [m]MOD3 + 1]MOD3; \qquad [Eqn.19]$$

and

$$[\{(6m-1).2.4^{y} - 1\}/3]MOD3 = [[y]MOD3 + [m]MOD3 - 1]MOD3; \qquad [Eqn.20]$$

Rewriting [Eqn.19] & [Eqn.20] for the Binary-Exponential-Ladders, we get the equivalent set of equations as:

$$[\{(6m-5).2^{w} - 1\}/3]MOD3 = [[w/2]MOD3 - [m]MOD3 + 1]MOD3; \qquad [Eqn.21]$$

and

$$[\{(6m-1).2^{v} - 1\}/3]MOD3 = [[(v-1)/2]MOD3 + [m]MOD3 - 1]MOD3; \qquad [Eqn.22]$$

12.2 Most Intriguing and Reassuring Observation

Suppose we move forward from H_{k-1} to H_k and then onto H_{k+1}; that is, trace backwards along the Collatz Sequence, in the reverse direction; moving from $BEL(n_{k-1})$ to $BEL(n_k)$ and then onto $BEL(n_{k+1})$; with $BEL(n_{k-1}) = S(BEL(n_k)$ and $BEL(n_k) = S(BEL(n_{k+1})$; every time selecting the immediate-predecessor BEL of the *lowest denomination* with the *lowest value of the defining-base-rung*, that is, selecting $BEL(n_{k-1}) \in H_{k-1}$; as represented by the lowest value of 2 for w in [Eqn.7] or the lowest value of 1 for v in [Eqn.8] above.

We find that [Eqn.7] with w=2 leads to a situation wherein $n_{k-1} < n_k$; that is,

$$BEL(6m\text{-}5) = S(BEL([(6m\text{-}5).2^2 \text{ -}1]/3)) = S(BEL(8m\text{-}7)); \qquad [Eqn.23]$$

whereas [Eqn.8] with v=1 leads to a situation wherein $n_{k-1} > n_k$; that is,

$$BEL(6m\text{-}1) = S(BEL([(6m\text{-}1).2^1 \text{ -}1]/3)) = S(BEL(4m\text{-}1)); \qquad [Eqn.24]$$

which incidentally the same as [Eqn.17] as seen earlier.

Thus, we find that moving forward from H_{k-1} to H_k and then onto H_{k+1}; every time selecting the immediate-predecessor BEL of the lowest denomination, we encounter the three distinct situations:– (1) $BEL(6m\text{-}3) \in H_{k-1}$ being a leaf node of the BELnet, does not have any predecessors; (2) $BEL(6m\text{-}5) \in H_{k-1}$ has its immediate-predecessor BEL of lowest denomination being $BEL(8m\text{-}7) \in H_k$; (3) $BEL(6m\text{-}1) \in H_{k-1}$ has its immediate-predecessor BEL of lowest denomination being $BEL(4m\text{-}1) \in H_k$.

This establishes the fact that apart from the one-third of the BELs that lead to the leaf-nodes, the remaining two-thirds of the BELs are equally distributed between the two cases - BEL(6m-5) leads to the situation wherein the arborescence grows forward with increasing values for the defining-base-rung; whereas BEL(6m-1) leads to the situation wherein the arborescence grows forward with decreasing values for the defining-base-rung, thus *reaching-down* to some positive odd number of the form (4m-1) that was *left out in the earlier stages/tier-levels* of the hierarchy.

Note that there can be only three types of positive odd numbers corresponding to the three modular (nMOD3) residue classes; odd multiples of three that are of the form (6m-3) or numbers of the form (6m-5) or numbers of the form (6m-1); of which the odd multiples of three correspond to the defining-base-rung of the leaf nodes in the BELnet arborescence. Of the remaining two cases, it is of utmost significance that every positive odd number of the form (6m-1) forming the defining-base-rung of BEL(6m-1)$\in H_{k-1}$ having its immediate-predecessor BEL of lowest denomination that is of the form BEL(4m-1)$\in H_k$. This is how as we move from H_{k-1} to H_k and then onto H_{k+1}; we encounter some positive odd numbers of the form (4m-1) that *could not be reached in the earlier stages/tier-levels*; as indicated by the presence of 1 in the CTUHSK generative parameter tuple.

The CTUHSK generative parameter tuple corresponding to every positive odd integer of the form (4m-1) must necessarily end with the sub-sequence (..., 1, 0); as for example, GPT(11) = (4, 3, 2, 1, 0); whereas the presence of a positive odd integer of the form (4m-1) anywhere in the Collatz Sequence is indicated by the presence of a 1 in the corresponding position in its CTUHSK generative parameter tuple; as for example, GPT(9) = (4, 3, 2, 1, 1, 2, 0); with the Collatz Sequence for

9 is 9->7->11->17->13->5->1; the two repeated occurrence of 1s corresponding to the sub-sequence 7->11->17 in the trajectory. It is indeed intriguing to observe that this specific process of reaching-down to such positive odd numbers of the form (4m-1) can indeed be a repeated contiguous operation at times, as can be seen by the presence of repeated 1's in the CTUHSK generative parameter tuple.

This is a situation wherein the emergence of global system properties through persistent local component subsystem characteristics can be clearly demonstrated! Banking on this global property, the author claims that the length of the CTUHSK generative parameter tuple |GPT(m)| required to generate a given positive odd number m is limited to be *no more than* m itself; whereas the triad |GPT(27)| = 42 and |GPT(31)| = 40 with |GPT(41)| = 41 is an exception. Note that the triad |GPT(3)| = 3 and |GPT(1)| = 1 with |GPT(5)| = 2 is another limiting case.

We will delve deeper into such algorithmic & computational details in a sequel to this research work, wherein a systematic (algorithmic) procedure for building the BELnet using the CTUHSK generative parameter tuples, corresponding to a systematic (algorithmic) procedure to produce a dictionary (lexicographic) of the CTUHSK generative parameter tuples, is planned to be presented.

13. Self-Similar Structural Symmetry of BELnet

From the above discussion one can notice that BELnet forms an arborescence in H^s with a self-similar structural symmetry. {BEL(1)} stands at the center, with its trivial cycle at its defining-base-rung. At every tier-level k corresponding to the k^{th} term H_k in the sequence H^s, Binary-Exponential-Ladders of all the three modular residue classes are present, each being countably infinite and equal in number; each having its single unique immediate-successor in H_{k-1}. Each of the {BEL(6m-5)} and each of the {BEL(6m-1)} has its immediate-predecessors in H_{k+1}; whereas of each the {BEL(6m-3)} remain as leaf-nodes since they can't have any immediate-predecessors. Thus, one-third of the BELs remain as leaf nodes; the other two-thirds become intermediate nodes that propagate the arborescence structure unboundedly to infinity. The self-similar structural symmetry in the network of binary-exponential-ladders BELnet is a self-similar derived from the modular-periodicity w.r.t both the defining-base-rung value among the neighboring BELs as well as the binary-exponent value within any given BEL, as defined by [Eqn.19] & [Eqn.20] or equivalently [Eqn.21] & [Eqn.22] above.

Note that each of the {BEL(6m-5)} with (6m-5) as its defining-base-rung, has its immediate-predecessor of lowest denomination being {BEL(8m-7)} as defined in [Eqn.23] where (8m-7)>(6m-5); facilitating the outward growth of the BELnet arborescence by pushing upward in the number-line. Also note that each of the {BEL(6m-1)} with (6m-1) as its defining-base-rung, has its immediate-predecessor of lowest denomination being {BEL(4m-1)} as defined in [Eqn.24] where (4m-1)<(6m-1); facilitating the inward growth of the BELnet arborescence by pulling downward in reaching-down the not-yet-reached in the number-line (positive odd numbers). Therefore, we observe that the specific instance of [Eqn.8] represented by [Eqn.24]; in the inverse of the condensed compact Collatz function; acts as the precise mechanism that systematically fills the gaps in the number line (positive odd numbers) that otherwise gets left out by [Eqn.7] or even [Eqn23]; while constructing the lexicographic dictionary of CTUHSK generative parameter tuples for growing the BELnet arborescence.

This self-similar structural symmetry with the modular-periodicity (triadic/hexadic) both with respect to the defining-base-rung value and the binary-exponent value, along with the above mentioned unique feature of reaching-down the not-yet-reached numbers in the number-line, and using all the three modular (nMOD3) residue classes among the set of positive odd numbers as the defining-base-rung for the BELs, together guarantee the connectedness of the network of binary-exponential-ladders BELnet; which has of course been established above by showing the non-existence of any disjoint loops and/or divergent chains.

14. Conclusion

We have established a two stage bijective mapping from the Collatz-domain to a meticulously designed *structured system framework* H with a careful reorganization & condensation :- in the first stage, a bijective mapping from the Collatz-domain to BELnet, that is the network of Binary-Exponential-Ladders defined on the set of positive odd numbers; and then in the second stage, a bijective mapping from BELnet to the structured system framework H. Then we establish the *order-preserving isomorphism* between the relevant component H^s (that is shown to be an infinite well-ordered sequence or rather a hierarchy or indeed an arborescence of the *Binary-Exponential-Ladders* having its root at H_0) of the *structured system framework* H and the set of positive integers; proving the necessary condition. A sufficient condition is provided through a reductio-ad-absurdum argument (along with a uniquely special modular arithmetic characteristic property of the Collatz system) that is used to rule out the existence of any extraneous objects like disjoint loops/cycles and/or divergent chains. Since the Collatz (3n+1) Conjecture is restricted to the set of natural numbers, the existence of any non-standard ‘rogue’ and/or ‘ghost’ elements are of no concern here.

As a direction towards future research work, we have also presented a possible alternative approach in proving this convergence, using modulus arithmetic for the conditions to be satisfied by the CTUHSK generative parameters or equivalently the exponents in a closed-form expression for generating the reverse Collatz sequence corresponding to any given positive odd number.

It is observed that the consistent & persistent application of the modulus arithmetic characteristics associated with the local neighborhood of BELnet imposed by the Compact Collatz (CTUHSK) function gives rise to the emergence of the global system property associated with the structured system framework that forms an arborescence (hierarchy) of BELnet – this leads to an upper limit on the depth in the arborescence or the rank/level in the hierarchy that one needs to analyse in order to reach for the existence of a binary-exponential-ladder with the desired positive odd number as its defining base rung in BELnet.

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16. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an Independent Researcher, and that I have not received any funding from any source other than my own savings, and I do not have any obligations or encumbrances of any kind, neither financial nor legal nor of any other kind, regarding the contents of the manuscript - of which I am the original author and creator. Also, I hereby declare there is no conflict of interests.

17. Author Rights Retention Notice

18. Dedication

To my **ಅಜ್ಜ**(ajja) Karinja Halemane Keshava Bhat & **ಅಜ್ಜಿ**(ajji) Thirumaleshwari, **ಅಪ್ಪ**(appa) Shama Bhat & **ಅಮ್ಮ**(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.