Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Theorem : Convergence of Collatz (3n+1) Sequence to the Trivial-cycle Proved

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Abstract

This paper presents the *Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani* (CTUHSK) theorem, which asserts the convergence of the Collatz (3n+1) Sequence to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$; thereby proving the Collatz Conjecture, a long-standing unsolved problem. The proof is in two parts. The necessary condition is provided by the order-isomorphism established between the relevant component H^s (with an invariant-base-element) of a structured system framework H and the set of positive integers. The structured system framework H itself has been designed by a two-stage bijective mapping: (1) from the Collatz-domain to BELnet, that is the network of binary exponential ladders defined on the set of positive odd numbers; and (2) from BELnet to the structured system framework H. The

sufficient condition is provided by a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) that is used to demonstrate domain exhaustion; having already captured all the modular-residue-classes in H^s; logically excluding the existence of any extraneous elements or objects or sub-systems such as disjoint loops/cycles H^∞ and/or divergent chains $H^\&$ or even any/all non-standard objects, in H.

An independent definitive proof is presented based on an exact representation of the Collatz system dynamics; using a dynamically evolving graded algebraic structure of ideal based filtration scheme with divisibility classes for modulo-3 quotient semiring generated by the primitive root 2; along with a filtration shifting global affine transformation f(x)=(3x+1) when x is a positive odd number; for achieving a Euclidean expansion shift to the coprime layer; avoiding the modulo-multiple-layer and also all the nilpotent layers; the trivial-cycle {(1⇐2⇐4)} being bypassed by an initialization-phase for the filtration scheme, starting with directly the modul-9 coprime-layer.

Some directions for possible future research work on algorithmic, computational characteristics of the Collatz system have also been presented. A situation has been identified wherein the *emergence of global system properties through persistent local subsystem characteristics* can be clearly demonstrated; Halemane-Conjecture states that the maximum number of odd (3n+1) operations required to reach the trivial-cycle {(1⇐2⇐4)} starting from any given positive integer and moving along the Collatz sequence, is limited by that given number itself; with the triad {(31⇐41⇐27)} as an exceptional limiting case.

Keywords: Binary-Exponential-Ladder; Modular-Periodicity; Arborescence; CTUHSK-Theorem; CTUHSK-Generative-Parameters; Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Sequence; Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture; Order-Isomorphism; Dedekind-Peano Axioms; Convergence; Halemane-Conjecture.

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1. Introduction

The Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture, simply referred to as the *Collatz Conjecture* [§1] states that the Collatz Sequence, also referred to as the Collatz (3n+1) Sequence, converges to the trivial-cycle, that is, $\{(1 \Leftarrow 2 \Leftarrow 4)\}$, starting from any positive integer.

Lagarias [§3]&[§4] gives an exhaustive annotated bibliography, whereas Lagarias [§5]&[§6] gives an elaborate overview of the Collatz Problem, also referred to as the "3x+1 problem", referring to it as "The Ultimate Challenge". Guy [§2] has been somewhat pessimistic in advising researchers "Don't Try to Solve These Problems". Lagarius [§5]&[§6] refers to the statement by the most revered Mathematician and Number Theory Expert Paul Erdos, who said that - "Mathematics is not yet ready for such problems" - about the Collatz Conjecture. The Collatz (3n+1) Sequence is indeed a highly complex chaotic system defined or generated with one of the simplest deterministic well-defined equations, that too, just two in number.

We will not get into any more of the reported literature except the problem definition; but will simply focus on a rather shockingly simple approach to solve this amazing long-standing unsolved problem. This paper presents the *Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani* (CTUHSK) Theorem, asserting the convergence of the Collatz (3n+1) Sequence to the trivial-cycle. The proof is two-fold. First is the application of the most fundamental *Dedekind-Peano Axioms* and *Modular Arithmetic* to a meticulously designed *Structured System Framework* of *Binary-Exponential-Ladder*s defined on the set of positive odd numbers, and establishing an *order-isomorphism* between the set of natural numbers and the *structured system framework* of *Binary Exponential Ladder*s. Secondly a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) ensures the non-existence of any extraneous elements.

2. Problem Description

We define the *Collatz Function* C(n) with a positive integer n as its input argument, in terms of a 'pull-Down' operator D(n) and a 'push-Up' operator U(n) as follows:

$$\text{if (n is even) } C(n) \coloneqq D(n) = (n/2); \;\; \text{else } C(n) \coloneqq U(n) = (3*n+1); \qquad \text{[Eqn.1]}$$

where the '*pull-Down*' D(n) operator takes only an even number as its input argument whereas the '*push-Up*' operator U(n) takes only an odd number as its input argument and gives an output that is an even number.

For convenience in our study of the Collatz Sequence, we define the *Compact Collatz Function* T(m) by the repeated application of the 'pull-down' operator D(m) wherever applicable, say, ($p \geq 1$) times, that is, $D^*(m) \coloneqq D^p(m)$ so as to get an output $D^\#(m)$ that is an odd number:

if (m is even) $T(m) \coloneqq D^*(m) \coloneqq D^p(m) = (m / 2^p) \coloneqq D^\#(m)$;

else (m is odd) $T(m) \coloneqq U(m) = (3*m + 1) \coloneqq U^\#(m)$; [Eqn.2]

where $D^\#(m)$ is called the "D-floor number" associated with the input argument m; and $U^\#(m)$ is called the "U-ceiling number" associated with the input argument m.

The *Compact Collatz Function* T(m) may as well be considered to have been redefined with these newly introduced two operators, the "D-floor operator" $D^\#(m)$ and the "U-ceiling operator" $U^\#(m)$ as given in [Eqn.2] above.

This new definition for the *Compact Collatz Function* T(m) facilitates our study of the corresponding *Compact Collatz Sequence*; which is no different from its equivalent standard Collatz Sequence, once we understand that the repeated application, say, ($p \geq 1$) times, of the 'pull-Down' operator D(m) has now been collapsed into an equivalent single "D-floor operator" $D^\#(m)$ giving the D-floor number $D^\#(m)$ as its output. The push-Up operator U has been simply redefined as the "U-ceiling operator" $U^\#$ for uniformity and elegant completeness.

The *Compact Collatz Sequence* is obtained by the repeated sequential application of the *Compact Collatz Function* T(m) starting with the given initial input number m - represented by an alternating series of $D^\#$ number and $U^\#$ number - except possibly the starting initial 'seed' number m and the final terminating number, which as per the Collatz Conjecture, is anyway a $D^{\#}$ number that is unity.

3. Observations on the Pull-Down Operator

The pull-Down operator D *always* takes only an even number n as its input argument. Every application of this pull-down operator results in an alternating (toggling) effect on the nMOD3 property of the input argument number; that is, a 1MOD3 input gives a 2MOD3 output and a 2MOD3 input gives a 1MOD3 output; whereas a 0MOD3 input gives a 0MOD3 output. Repeated application of D, in case applicable, results in a final output that is an odd number and therefore becomes an input for the push-Up operator. In such a case, we call it a "D-floor operator" $D^{\#}$ as defined in [Eqn.2] above, and its output a "D-floor number" $D^{\#}(n)$ characterized by being a odd number; $D^{\#}(n)$ may be in any one of the three possible types:

(1) a 1MOD6 odd number, being a 1MOD3 odd number of the type (6m-5);

(2) a 5MOD6 odd number, being a 2MOD3 odd number of the type (6m-1);

(3) a 3MOD6 odd number, being a 0MOD3 odd number of the type (6m-3).

4. Observations on the Push-Up Operator

The push-Up operator U *always* takes only an odd number m as its input argument, and *always* gives an output that is a 4MOD6 number, being a 1MOD3 even number that is of the type (6m-2); irrespective of whether the input is a 1MOD6 odd number or a 3MOD6 odd number or a 5MOD6 odd number. Note that one single application of the 'push-Up' operator U transforms any input odd number m into a 4MOD6 even number that becomes an input to the "D-floor operator $D^{\#}$". That is why we may as well consider the push-Up operator U as the "U-ceiling operator" $U^{\#}$ as defined in [Eqn.2] above.

5. Observations on the Compact Collatz Function

Start with any positive integer. (1) If the starting initial number n is even, then we apply the D-floor operator $D^{\#}$ giving an output that is the D-floor number $D^{\#}(n)$ which is given as input to the U-ceiling operator. Of course, if the starting number is a power of 2 we terminate at unity. Else, we have a D-floor number $D^{\#}(n)$ that is an odd number greater than unity, in any non-trivial case; as the initial $D^{\#}$ node in the compact Collatz sequence. (2) If on the other hand the starting initial seed number n is an odd number, we treat that itself as the initial $D^{\#}$ node in the compact Collatz sequence.

Having thus obtained the initial $D^{\#}$ node in the Compact Collatz Sequence, we apply the U-ceiling operator $U^{\#}$ to get the U-ceiling number $U^{\#}$ that is a 4MOD6 even number. That in turn is given as input to the D-floor operator $D^{\#}$. Now the process continues.

Note that the compact Collatz sequence can therefore be defined by a *trajectory* generated by an *alternating sequence* of a "D-floor number" $D^{\#}$ and a "U-ceiling number" $U^{\#}$, *with its starting initial node being a $D^{\#}$ number.* The Compact Collatz Function as presented in [Eqn.2] defines the unique link (directed arc) from any given D-floor number $D^{\#}$ as the predecessor node to its corresponding unique

U-ceiling number $U^{\#}$ as the successor node and also the unique link (directed arc) from any given U-ceiling number $U^{\#}$ as the predecessor node to its corresponding unique D-floor number $D^{\#}$ as the successor node. The unique link (directed arc) from a starting initial even "seed" number leading to the first node (D-floor number $D^{\#}$) in the *trajectory* is similarly defined.

As mentioned earlier, the application of the D-floor operator $D^{\#}$ on a U-ceiling number $U^{\#}$ that is a 4MOD6 even number of the form (6m-2) can lead to a D-floor number $D^{\#}$ that is an odd number that can be either: (1) a 1MOD6 odd number, being a 1MOD3 odd number that is of the type (6m-5); (2) a 5MOD6 odd number, being a 2MOD3 odd number that is of the type (6m-1); but can never be (3) a 3MOD6 odd number, that is a 0MOD3 odd number of the type (6m-3). The only exception, when the D-floor operator $D^{\#}$ gives an output D-floor number $D^{\#}$ that is a 3MOD6 odd number of the type (6m-3) is the situation when its input is a 0MOD6 even number, which is impossible for any U-ceiling number $U^{\#}$; although such an input may come in those special cases wherein the starting initial 'seed' number itself is a 0MOD6 even number that is of the form $(6m-3).2^p$ with $p>0$; leading to an output $D^{\#}$ that is a 3MOD6 odd number of the form (6m-3).

6. Analysis of the Compact Collatz Sequence

From the above observations, it is clear that corresponding to every positive integer n as the starting initial 'seed' number, there is a starting initial node in the trajectory representing the compact Collatz sequence, that is a $D^{\#}$ number in exactly one of the three possible forms as mentioned above - that can be an input argument to the U-ceiling operator $U^{\#}$ giving exactly one unique output $U^{\#}$ which itself can be an input to the D-floor operator $D^{\#}$ so that the process continues. Successive application of each of these two operators ($U^{\#}$ and $D^{\#}$) wherever applicable, traces a unique trajectory, wherein each node is represented by the unique output number of the appropriate operation applied to the input number represented by the preceding node in the trajectory.

The anticipated terminating trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ can be obtained only through a final application of the D-floor operator $D^{\#}$ on a 4MOD6 even number that is of the form $(6m-2).2^p$ with m=1 and $0 \leq p$.

7. Binary-Exponential-Ladder with its Defining-Base-Rung $D^{\#}$

Here, we present a meticulously designed *Structured System Framework* that *partition*s the *set of positive integers* to facilitate a *general systems analysis* of the *Compact Collatz Sequence*.

Let every positive odd number be associated with a *Binary-Exponential-Ladder*, denoted by BEL(2m-1) and defined as a sequence $\{(2m-1).2^u \mid (u \geq 0)\}$; with its *defining-base-rung* (u=0) given by the odd number (2m-1).

$$\text{BEL}(2m-1) \coloneqq \{(2m-1).2^u \mid (u \geq 0); (m \geq 0)\}; \qquad \text{[Eqn.3]}$$

Thus, we establish an exact one-to-one correspondence (bijection) between the *set of positive odd numbers* that form the $D^{\#}$ value and the *defining-base-rung* of the corresponding *Binary Exponential Ladder* $\text{BEL}(D^{\#})$.

Every *positive even number* in the form $\{(2m\text{-}1).2^u \mid (u>0)\}$; for which there exists its corresponding $D^{\#}$ value, $D^{\#}((2m\text{-}1).2^u) = (2m\text{-}1)$; there exists exactly one corresponding *Binary-Exponential-Ladder* BEL(2m-1) that contains the given even number $(2m\text{-}1).2^u$ as B((2m-1),u) as one of the higher rungs in BEL(2m-1).

$$B((2m\text{-}1), u) \coloneqq \{(2m\text{-}1).2^u \mid (u > 0)\}; \qquad \text{[Eqn.4]}$$

Therefore, *the set of all Binary-Exponential-Ladders form a partition of the set of all positive integers*; with an *exact one-to-one correspondence* (*bijection*) between each positive odd number $D^{\#}$ and the corresponding *Binary-Exponential-Ladder* for which it is the *defining-base-rung* $D^{\#}$; whereas each of the positive even numbers correspond to exactly one of the higher rungs of a specific Binary-Exponential-Ladder identified by the D-floor number $D^{\#}$ associated with that given positive even number.

This partitioned framework of positive integers goes another step deeper because of the fact that the *defining-base-rung* $D^{\#}$ of a Binary-Exponential-Ladder BEL($D^{\#}$) can itself be in one of the three possible forms 1MOD6 or 5MOD6 or 3MOD6; whereas all the upper rungs of the Binary-Exponential-Ladder are either (1) alternatingly 2MOD6 and 4MOD6; or (2) all being 0MOD6 numbers.

The Collatz Conjecture states that every Collatz Sequence, starting from any positive integer, converges to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ which is in BEL(1) that is uniquely identified by its defining-base-rung $D^{\#}$ value that is unity. Therefore, our focus will be the Binary-Exponential-Ladders BEL(1) and its relationship with every other Binary-Exponential-Ladder BEL($D^{\#}$).

Note that $D^{\#}$ can be a positive odd number in any one of the three possible forms: either (1) a 1MOD6 number of the form (6m-5); or (2) a 5MOD6 number of the form (6m-1); or (3) a 3MOD6 number of the form (6m-3). BEL(6m-5) contains the output of $U^{\#}$ at a higher rung $(6m-5)2^{W}$ with w being a positive even exponent; the input of $U^{\#}$ being given by $[\{(6m-5).2^{W} -1\}/3]$. BEL(6m-1) contains the output of $U^{\#}$ at a higher rung $(6m-1)2^{V}$ with v being a positive odd exponent; the input of $U^{\#}$ being given by $[\{(6m-1).2^{V} -1\}/3]$. However, BEL(6m-3) cannot contain any such output of the U-ceiling operator $U^{\#}$ irrespective of any input argument.

8. Immediate Neighborhood of a Binary-Exponential-Ladder

The relationship between a pair of Binary-Exponential-Ladders BEL(m) and BEL(n) can be considered to be defined and characterized by the relationship between the corresponding pair of the *defining-base-rung* $D^{\#}$ values m and n along with the corresponding pair $U^{\#}(m)$ and $U^{\#}(n)$.

The immediate-neighborhood of a given Binary-Exponential-Ladder $BEL(D^{\#})$ is defined by the *immediate-predecessors* and *immediate-successors*, considering the U-ceiling operator $U^{\#}$; since the D-floor operator $D^{\#}$ is applicable only within a given Binary-exponential-Ladder and not between a pair of them.

8.1 Single Unique Immediate Successor – CTUHSK function

It turns out that the only *one single unique immediate successor* of BEL(m) is $BEL(D^{\#}(U^{\#}(m)))$ that contains $U^{\#}(m)$ as one of its higher rungs; its identifying characteristic D-floor number associated with its defining-base-rung being given by $n \coloneqq D^{\#}(U^{\#}(m))$. That is,

$$S(BEL(m)) = BEL(D^{\#}(U^{\#}(m))) \coloneqq BEL(n); \text{ with } n \coloneqq D^{\#}(U^{\#}(m)); \quad [Eqn.5]$$

Note that [Eqn.5] describes the forward tracing movement (transfer function) from one BEL to its immediate successor BEL defined by the very same Compact Collatz Function [Eqn.2] above; wherein the downward movement ($D^{\#}$ operator) from a higher rung to the defining-base-rung within the same BEL is compressed and left as implicit; whereas the upward movement from the defining-base-rung of one BEL to a higher rung of its immediate-successor BEL is kept explicit; thus redefining the compact Collatz sequence by its equivalent, a *condensed compact Collatz* (CTUHSK) *sequence*; which is simply a hop from one BEL to its immediate successor BEL. Here, [Eqn.5] can therefore be considered as the condensed compact Collatz (CTUHSK) function; representing one hop from a positive odd number to its immediate successor odd number; thus defining the condensed compact Collatz (CTUHSK) sequence, that is equivalent to skipping the intermediate even numbers in the standard Collatz sequence. Our study will focus on the characteristics of this condensed compact Collatz (CTUHSK) sequence defined by the condensed compact Collatz (CTUHSK) function; or equivalently the characteristics of BELnet, that is, the network of binary exponential ladders.

8.2 Multiple Immediate Predecessors – inverse CTUHSK function

There exists a *set of immediate-predecessors* for each BEL($D^{\#}$) of the form BEL(6m-5) and BEL(6m-1) although none for BEL(6m-3). Note that if S(BEL(m)) is BEL(n) than BEL(m) is one of the predecessors of BEL(n).

The *set of immediate-predecessors* for a given BEL(n) is defined by considering the *inverse of the immediate-successor relationship*; as the set of all BELs each of which having its single unique immediate-successor as BEL(n).

$$\{P(BEL(n))\} \coloneqq \{BEL(m) \mid BEL(n) = S(BEL(m)\}; \qquad [Eqn.6]$$

BEL(1MOD6) or equivalently BEL(6m-5) has, as its set of immediate-predecessors, {BEL([(1MOD6).2^W -1]/3)} or equivalently {BEL([(6m-5).2^W -1]/3)} with w being a positive even exponent. Here, the input of $U^{\#}$ is given by [{(1MOD6).2^W -1}/3] or equivalently [{(6m-5).2^W -1}/3] and the output of $U^{\#}$ is {(1MOD6).2^W} or equivalently {(6m-5).2^W} that is contained in BEL(1MOD6) or equivalently BEL(6m-5). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(1MOD6).

$$\{P(BEL(6m\text{-}5))\} = \{BEL([(6m\text{-}5).2^W \text{-}1]/3)\}; \qquad [Eqn.7]$$

BEL(5MOD6) or equivalently BEL(6m-1) has, its set of immediate-predecessors, {BEL([(5MOD6).2^V -1]/3)} or equivalently {BEL([(6m-1).2^V -1]/3)} with v being a positive odd exponent. Here, the input of $U^{\#}$ is given by [{(5MOD6).2^V -1}/3] or equivalently [{(6m-1).2^V -1}/3] and the output of $U^{\#}$ is {(5MOD6).2^V} or equivalently {(6m-1).2^V} that is contained in BEL(5MOD6) or equivalently BEL(6m-1). Each of the three possible classes of BEL, namely, BEL(1MOD6) and BEL(5MOD6) and BEL(3MOD6) can be the immediate-predecessor of BEL(5MOD6).

$\{P(BEL(6m\text{-}1))\} = \{BEL([(6m\text{-}1).2^{V} \text{-}1]/3)\}$; [Eqn.8]

BEL(3MOD6) or equivalently BEL(6m-3) has no immediate-predecessors.

$\{P(BEL(6m\text{-}3))\} = \phi$; [Eqn.9]

Note that [Eqn.6] or equivalently, [Eqn.7]&[Eqn.8]&[Eqn.9] together define the algebra for the inverse of the condensed compact Collatz (CTUHSK) function.

The CTUHSK function in [Eqn.5] along with its inverse in [Eqn.6] or equivalently in [Eqn.7]&[Eqn.8]&[Eqn.9] together provide a complete mathematical model for the network of binary exponential ladders.

8.3 Quaternary-Exponential-Ladder

The above-mentioned property, that *only* the alternating rungs, defined by $\{(1MOD6).4^{u} \mid u>0\}$ of BEL(1MOD6) or $\{(5MOD6).2.4^{u} \mid u\geq 0\}$ of BEL(5MOD6) are the ‘active’ higher rungs of BEL forming the nodes in the CTUHSK-Sequence; makes it convenient to define a system of *Quaternary-Exponential-Ladder*s QEL wherein every rung of QEL becomes an active node in the CTUHSK-Sequence. This concept is not directly needed for proving the convergence of the Collatz Sequence, and therefore we will not take up this line of study in this paper.

8.4 BEL(1) as the Central Focus

Considering BEL(1) as our central focus of interest, which itself belongs to the type BEL(1MOD6) or equivalently BEL(6m-5); it is interesting to note that it has its *single unique immediate-successor*; as S(BEL(1)) = BEL($D^{\#}$($U^{\#}$(1))) = BEL(1); that is, BEL(1) itself is its single unique immediate-successor, and that it has no other immediate-successor distinct from itself; because of the fact that the trivial-cycle {(1⇐2⇐4)} contained within BEL(1).

8.5. BELnet : Network of Binary-Exponential-Ladders

The above discussion about the successor predecessor relationship among the binary-exponential-ladders and its neighborhood leads to the observation that the *network of binary-exponential-ladders*, BELnet, has countably infinite number of each of the three classes/types of nodes: (1) BEL(1MOD6) or equivalently BEL(6m-5); (2) BEL(5MOD6) or equivalently BEL(6m-1); and (3) BEL(3MOD6) or equivalently BEL(6m-3). Each BEL being a node of the BELnet has a single unique outward directed arc that points towards its single unique immediate-successor, specifically linking onto some higher rung. Multiple (countably infinite number of) inward directed arcs, each linked onto some specific higher rung of a given BEL, emanate from each of its immediate-predecessors. BEL(1) is an invariant-base-element or equivalently a sink node in BELnet, the network of binary exponential ladders.

Thus, the set of binary-exponential-ladders is an exact representation of the Collatz-domain and the network of binary-exponential-ladders BELnet is an exact representation of the Collatz-map.

The *connectedness* of the network of binary-exponential-ladders BELnet will be analyzed from the *design of a structured system framework* consisting of the entire set of binary-exponential-ladders, merely as a re-organized *condensation* of the very same BELnet, as presented below.

9. Structured System Framework H

From the above discussion we find that it is convenient for our study to consider a *Structured System Framework* H as an ordered infinite linear sequence of terms each of which being a set of BELs; that is, $H \coloneqq \{...H_{k-1} < H_k < H_{k+1}...\}$ wherein the *ordering relationship* between the adjacent terms of the sequence is derived from the successor predecessor relationships among the BELs that form the member elements of these adjacent terms in the sequence.

Specifically, H_k is defined as the set formed by the unique immediate-successor of each BEL belonging to H_{k+1} and also the set of immediate-predecessors of each BEL belonging to H_{k-1}; that is,

$$H \coloneqq \{...H_{k-1} < H_k < H_{k+1}...\}; \quad \text{[Eqn.10]}$$

and

$$H_k \coloneqq \{S(BEL(m)) \mid BEL(m) \in H_{k+1}\} \cup \{BEL(m) \mid S(BEL(m)) \in H_{k-1}\}; \quad \text{[Eqn.11]}$$

Note that the second part of [Eqn.11] here is required to ensure that BELs of the class/type BEL(3MOD6) can be included in each term H_k since each of them have immediate-successor in H_{k-1} although none of them have any predecessors in H_{k+1}.

Now, we may as well define the predecessor relationship as the inverse of the above defined successor relationship, as –

$H_{k-1} \coloneqq S(H_k)$ and $H_k \coloneqq S(H_{k+1})$; [Eqn.12]

and

$P(H_{k-1}) \coloneqq H_k$ and $P(H_k) \coloneqq H_{k+1}$; [Eqn.13]

The multiplicity of the *immediate-predecessor* relationship among the BELs requires that the set of all immediate-predecessors of every element of H_{k-1} form the elements of the set H_k; so as to guarantee that the structured system framework H is ordered infinite linear sequence; that is, $H_{k-1} < H_k < H_{k+1}$ among these sets; in spite of only a partial ordering relationship among the BELs; and also to guarantee that the entire set of all the BELs are present in H thus making it as merely a *re-organized structure* for BELnet.

9.1 Closed Chains and Unbounded Chains and Sink Nodes in H

The design of the structured system framework H can *in general* allow for the existence of sink nodes (invariant-base-elements) and/or unbounded open chains and/or closed chains (loops). That is, the structured system framework H can in general be *partitioned into three mutually disjoint and independent components*,

$H \coloneqq H^s \cup H^\& \cup H^\infty$ [Eqn.14]

where (1) H^s corresponds to the set of all possible terms in H connected with sink nodes; (2) $H^\&$ corresponds to the set of all possible terms in H connected with unbounded open chains; and (3) H^∞ corresponds to the set of all possible terms in H connected with closed chains (loops). In such a situation, each of these components, H^s and $H^\&$ and H^∞ needs to satisfy the ordering conditions expressed above in [Eqn.10], [Eqn.11], [Eqn.12] & [Eqn.13].

9.2 A Sink Node H_0 in H^s

We have observed earlier that BEL(1) itself is its single unique immediate-successor and does not have any immediate-successor distinct from itself, although it has multiple immediate-predecessors. That is, BEL(1) is an invariant-base-element or equivalently a sink node in BELnet. Therefore, the component H^s must necessarily have a term H_0 as its *invariant-base-element* or equivalently a *sink node*, that is, $H_0 \coloneqq \{BEL(1)\}$; with BEL(1) being its singleton member element. That is,

$$H^s \coloneqq \{H_0, H_1, H_2, \ldots\}; \qquad \text{[Eqn.15]}$$

From the above discussion we observe that $H^s \coloneqq \{H_0, H_1, H_2, \ldots\}$ is, by its very design, a well-ordered infinite linear sequence of terms (with a distinct smallest/least element H_0) as will be established in CTUHSK theorem. Each term $H_k \in H^s$ is a countably infinite set of BELs with an exception that the invariant base element or the 'root' $H_0 \coloneqq \{BEL(1)\}$ is a singleton set. The set of k^{th} immediate predecessors of BEL(1) form the set H_k at tier-k level in the hierarchy, if one wishes to consider it as a hierarchy.

Note that the entire Collatz Domain has been mapped to the structured system framework H through a two-stage bijective mapping.

10. Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani Theorem

STATEMENT OF THE CTUHSK THEOREM

The Collatz Sequence converges to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

PROOF

There are two parts to the proof – a *necessary condition* and a *sufficient condition*. The necessary condition is provided by establishing an order-isomorphism between H^s and the set of natural numbers, by the application of the Dedekind-Peano axioms; thus proving the convergence of the CTUHSK sequence, starting from any given positive integer contained in any BEL that is in H^s to the trivial-cycle at the base of BEL(1) which itself is in $H_0 \in H^s$. The sufficient condition is provided by a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) that establishes *domain exhaustion*; having already captured all the modular (modulo-3) residue classes in H^s; and guarantees the non-existence of any extraneous elements.

10.1 Proof of CTUHSK Theorem - Necessary Condition

We show that H^s satisfies the Dedekind-Peano's axioms (replacing the 'successor' by the 'predecessor') and therefore H^s is order-isomorphic with the set of natural numbers; and satisfies the necessary condition for the above convergence statement.

DEDEKIND-PEANO AXIOM : Existence of 1 as the invariant-base-element.

$H_0 \in H^s$. H_0 is the invariant-base-element of H^s.

The trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\} \in BEL(1)$ is contained in $H_0 \in H^s$.

DEDEKIND-PEANO AXIOM : Existence of a *successor function*.

By the very design of $H^s \coloneqq \{H_0, H_1, H_2, \ldots\}$, for every positive integer k,

$H_k \in H^s$ is the *predecessor* of $H_{k-1} \in H^s$.

Application of the CTUHSK function with the input from numbers contained in some BEL that is a member of H_k yields the single unique output number contained in some immediate-successor BEL that is a member of H_{k-1}; because of the definition of the successor predecessor relationship between H_k and H_{k-1}.

DEDEKIND-PEANO AXIOM : 1 is not a successor; 1 has no predecessor; 1 is a source node in the sequence of natural numbers.

H_0 is not a predecessor to any other H_k. There *does not exist any* $H_k \in H^s$, $k \neq 0$; that is distinct from H_0; with $H_k \neq H_0$; such that H_0 is the predecessor of H_k.

H_0 does not have any successor distinct from itself.

H_0 is a sink node in the sequence $H^s \coloneqq \{H_0, H_1, H_2, \ldots\}$.

Once the CTUHSK Sequence reaches the trivial-cycle/sink there is no exit from it.

DEDEKIND-PEANO AXIOM : Successor function is a unique one-to-one mapping.

If H_u is the predecessor of H_v and also H_u is the predecessor of H_w;

then it necessarily implies $H_v = H_w$ by the very design of H^s;

and, also,

If H_v is the predecessor of H_u and also H_w is the predecessor of H_u;

then it necessarily implies $H_v = H_w$ by the very design of H^s;

This is because the predecessor relation in H^s is a unique one-to-one mapping.

Also, note that for each positive integer k there corresponds a unique set $H_k \in H^s$, and for each $H_k \in H^s$ there corresponds a unique positive integer k; thus, establishing a bijection between H^s and the set of positive integers.

DEDEKIND-PEANO AXIOM : Principle of induction.

Collatz sequence starting with any number from BEL(1) $\in H_0$ converges in the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\} \in$ BEL(1) $\in H_0$.

The Collatz sequence starting with any positive integer that passes through H_k must necessarily pass through H_{k-1} because by design $H_{k-1} \coloneqq S(H_k)$. Therefore, the CTUHSK sequence starting from any positive integer that is contained in some BEL in H^s passes through a linear directed path (chain) with no forking or merging in H^s (although merging is observed deeper at the level of the BELs); moving from H_{k+1} to H_k and from H_k to H_{k-1} and so on to H_0 in that order; and converging in the trivial-cycle.

Thus, we establish an *order-isomorphism* between H^s and the set of natural numbers; that provides the necessary condition as mentioned above.

10.2 Proof of CTUHSK Theorem - Sufficient Condition

The proof for the sufficient condition is provided by establishing the non-existence of any extraneous elements/objects like disjoint loops/cycles H^∞; and/or divergent chains $H^\&$ in the structured system framework H; so that $H^s = H$.

First, the proof utilizes the fact that the Collatz domain being the set of natural numbers, is a well-ordered infinite linear sequence, with a distinct smallest (least) element, 1; containing all the three modular (modulo-3) residue classes; and that any non-empty sub-domain must also be a well-ordered subset, having a distinct smallest (least) element. Same is the case with H^s as already established above in the first part of the proof as the necessary condition.

Secondly, the relative magnitudes of any two numbers in the Collatz sequence is governed by the total number nD of applications of the pull-down operator D as compared to the total number nU of applications of the push-up operator U that is required in order to move from the earlier node in the sequence to reach the later node in the sequence. Note that every application of D pulls the value downwards by a factor of 2 whereas every application of U pushes the value upwards by a factor of about 3 ($3+\delta$); the cutoff being around $nD/nU \cong (\log 3/\log 2)$. Also, it is clear that every application of push-up operator U must necessarily be followed by an immediate application of at least one instance of the pull-down operator D.

We establish that as we move along the Collatz sequence, starting with any given number $n>1$, passing through the successor nodes belonging to each of the three modulo-3 modular-residue-classes; eventually we must necessarily encounter some node associated with a number m that is less than that starting number; $m<n$.

The relative magnitudes of any two adjacent positive odd numbers in the condensed compact Collatz sequence, from odd number to odd number outside of the trivial-cycle; or equivalently from one BEL to the next BEL in BELnet, is always a strictly decreasing step (subsequence) whenever the Collatz sequence passes through: either

(1) any non-base (first or higher) rung in the successor BEL of the form (6m-5); or
(2) any non-base non-first higher rung in the successor BEL of the form (6m-1).

The concern arises only in the exceptionally unique situations wherein the path traverses through the lowest non-base (first) rung of the successor BEL having a positive odd number of the form (6m-1) as its defining-base-rung, and the predecessor BEL has its defining-base-rung being a positive odd number of the form (4m-1); leading to the case where (6m-1)>(4m-1); that is, a strictly increasing step (subsequence). However, this increase from (4m-1) to (6m-1) can get immediately compensated in the later steps in the condensed compact Collatz sequence; observing that the very next node is (9m-1) which may be either odd or even; requiring further deeper study as described below. This is an exceptionally unique situation represented by the triad {(9m-1)⇐(6m-1)⇐(4m-1)}; corresponding to a subsequence of $(U^1D^1)U^1D^p$ with $1\leq p$.

DYNAMICALLY EVOLVING ALGEBRAIC FILTRATION SYSTEM

If (9m-1) is a positive even number, it gets into what is called as the coprime loop (principal orbit) of the modulo-3^2 quotient semiring generated by the primitive root (generator) 2, having its orbit defined by the six out of the nine modulo-9 residue classes, that is, $\{[1]_9, [5]_9, [7]_9, [8]_9, [4]_9, [2]_9, [1]_9\}$. This orbit size 6 is given by the Euler Totient Function for $\phi(3^k)=(3^k–3^{k-1})=6$ for k=2. However, exactly one out of the two sets of alternating nodes in this principal orbit corresponds to active entry doors, because of the characteristic property that the output of the push-up operator U is always of the form (6m-2) and neither (6m-4) nor (6m-0) irrespective of its input. In either case, these three possible entry doors work in taking the input number (9m-1) and pulling it down by the pull-down operator D. Here, k=nU, the number of applications of the push-up operator U so far in the Collatz sequence. Note that the subsequence {(8⇐5⇐3)} exhibits such phenomenon.

On the other hand, if (9m-1) is a positive odd number, it gets thrown into the next larger coprime loop (principal orbit) of the modulo-27 (that is, from modulo-3^k to modulo-3^{k+1}) quotient semiring generated by the primitive root (generator) 2, having its orbit defined by the defined by the 18 out of 27 modulo-27 residue classes, that is,

$\{[1]_{27},[14]_{27},[7]_{27},[17]_{27},[22]_{27},[11]_{27},[19]_{27},[23]_{27},[25]_{27},[26]_{27},[13]_{27},[20]_{27},[10]_{27},[5]_{27},[16]_{27},[8]_{27},[4]_{27},[2]_{27},[1]_{27}\}$.

This means that there are now 9 possible entry doors to this principal orbit, for bringing the number down, relative to the starting number, say (27n-1). Note that the numerical value of n in (27n-1) will be about half that of m in (9m-1) that we started with above. Also note that every division by the primitive root 2; achieved by a multiplication by the modular inverse of 2 in the quotient semiring; that being 5 modulo-9 and 14 modulo-27; brings down the quotient associated with the coset representative; for example, the value of n in (27n-1) gets reduced by about a factor of two in every application of either (U^1D^1) or (D) operation.

As we go through these divisions (pull-down operations D), if & when we hit upon an odd number greater than 1, we go to the next larger modulo-3^{k+1} coprime loop with orbit size increased three-fold. Note that the subsequence {(17⇐11⇐7)} exhibits such phenomenon with two consecutive push-up operations, that is, $(UD)^2$.

Suppose we are at an intermediate node in the compact Collatz sequence with the positive odd number m and having the coprime layer associated with the corresponding quotient semiring be of modulo-3^P; with the principal orbit size given by the Euler Totient Function for $\phi(3^P)=(3^P-3^{P-1})=2.3^{P-1}$. So, the number of active entry doors is 3^{P-1} which makes this algebraic filtration system capable of pulling down any input number. However, let us study the worst-case scenario wherein at this stage we encounter a situation which takes us through the next consecutive k steps, every step being a (U^1D^1) step. This is similar to a repeated application of the case that we discussed above, namely, moving along a repeated subsequence of the form $\{[8]_9 \Leftarrow [5]_6 \Leftarrow [3]_4\}$ with all the intermediate numbers of the form $[8]_9$ being positive odd numbers. At every step, it creates a new coprime layer with a higher modulus as discussed earlier. This leads to a stepwise expansion/refinement of the algebraic filtration system from modulo-3^P all the way to modulo-3^Q where Q=(P+k); with a far larger orbit size of 2.3^{Q-1}. The resulting larger coprime layer has the enhanced capability to trap any input number and pull it down until we hit a positive odd number n as the resultant. Comparing the two positive odd numbers m and n with a sequence of k consecutive (U^1D^1) steps in between them shows that $\{(n+1)/(m+1)=(3/2)^k\}$.

This scheme of algebraic process describes the Collatz system dynamics. This process produces a sequence of possibly countably infinite refinements, every time with larger and larger quotient semirings having coprime loops with larger span of input (entry/trap) doors because of the three-fold increase in the orbit size, from $\phi(3^k)$ to $\phi(3^{k+1})$. Every application of the (U^1D^1) operator or the (D) operator leads to a reduction in the value of the quotient associated with the coset representative, by about a factor of two in every step. First crucial observation – (1) at every stage in this stage-wise evolution of algebraic filtration system, the three-fold expansion/refinement of the coprime layer, that is, from 2.3^{k-1} to 2.3^k and then to 2.3^{k+1} etc., far exceeds the increase in the numerical value of the operands involved, that is, $\{(n+1)/(m+1)=(3/2)\}$. Second crucial observation – (2) the active input (entry/trap) doors in the coprime layer, 3^{k-1} in number; are almost equally distributed, w belonging to the modulo-4 residue class $[0]_4$ and v belonging to the modulo-4 residue class $[2]_4$; (w being an even binary exponent of 2 and v being an odd binary exponent of 2 in the binary-exponential-ladder, as per [Eqn.7] & [Eqn.8] respectively); thus enabling w and/or v number of repeated applications of U, the pull-down operator; the specific numbers w and v are such that $w+v=3^{k-1}$; and if w=v+1 for a specific value of k, then v=w+1 for the next value k+1, etc. Therefore, we observe that the exceptionally unique worst-case scenario of repeated applications of the (U^1D^1) operator is compensated for by the multiple applications of the (D) operator following a possible repeated application of the (U^1D^1) operator. These two crucial observations about the characteristic property of the Collatz system dynamics and the specific design of the dynamically evolving algebraic filtration system, together establishes that sequence must necessarily lead to a node with n<m, and eventually to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

Note that there also exists a self-similar structural symmetry of the Collatz (3n+1) system governed by a modular-periodicity (triadic/hexadic) relation among (1) the defining-base-rung value of a BEL; (2) the binary-exponent value corresponding to the higher rung of the BEL corresponding to the input (entry/trap) door of the coprime layer discussed above; and (3) the positive odd number corresponding to the defining-base-rung of the predecessor BEL in the Collatz sequence; as defined by [Eqn.7] & [Eqn.8]; that is further studied in a later section of this paper. It also establishes a closure condition covering all the relevant modular-residue-classes, among the various system parameters.

INDEPENDENT AND DEFINITIVE PROOF

The entire dynamics of the Collatz (3n+1) system can be exactly represented by the above explained algebraic framework. It is a dynamically-evolving graded-algebraic-structure of ideal-based-filtration scheme; with divisibility-classes for modulo-3 quotient-semiring generated by the primitive-root 2. We use a filtration-shifting global-affine-transformation, f(x)=(3x+1) when x is a positive odd number; achieving a Euclidean-expansion-shift to the topmost coprime-layer of that filtration scheme; while also avoiding the modulo-multiple-layer (zero-layer) and also all the nilpotent-layers with nilpotent-elements (dead-end zero-divisors) like (6m-3) and (6m-0). The trivial-cycle {(1⇐2⇐4)} is bypassed by an initialization-phase for the filtration scheme, starting with directly the modul-9 coprime-layer. This design for this system-dynamics-framework by itself provides an independent definitive proof of the Collatz conjecture, establishing that the entire map of the Collatz system dynamics is exactly represented by this novel algebraic-structure defined by the set of all positive integers; neither missing any natural number nor including any extraneous elements; and therefore, providing both the necessary-&-sufficient condition simultaneously in one single sweep.

However, let us now apply the reductio-ad-absurdum argument.

As we move along the Collatz sequence, starting with any given number n>1, passing through the successor nodes belonging to each of the three modulo-3 modular-residue-classes; eventually we must necessarily encounter some node associated with a number m that is less than that starting number n; guaranteed by the above described ideal-based filtration process using divisibility classes, etc.

Assume that there exists some non-empty H^{∞} (loop/cycle) or some non-empty $H^{\&}$ (unbounded chain) that is, some non-empty H^{D} that is mutually disjoint with and complementary to H^{S}; that is, $H = H^{S} \cup H^{D}$ and $H^{S} \cap H^{D} = \phi$.

Note that, since H^D contains the complementary sub-set of BELs not contained in H^S and the entire set of BELs has a bijective mapping with the set of natural numbers, H^D must have a smallest (least) element, corresponding to a natural number, say, d belonging to one of three modular-residue-classes similar to what we have seen above. Now, using the above reasoning, by repeated application of the Collatz function, we can find a successor node in the Collatz sequence starting from d having its numerical value lower than d.

However, this is a direct contradiction to the assumption that we started with. Therefore, we assert that $H^S = \phi$; and conclude the non-existence of any such sub-systems complementary to H^S in the structured system framework H.

10.3 Existence of any/every/all Non-standard Rogue Elements in H

Note that the Collatz-domain does not allow for the existence of any non-standard *rogue elements* etc. The study of non-standard extraneous objects (of whatever kind) is immaterial to the present proof, because the Collatz (3n+1) Conjecture is stated strictly within the domain of natural numbers.

Having established that $H^\infty = \phi$ and $H^\& = \phi$ in [Eqn.14] above; we get $H = H^s$; therefore, asserting the Collatz Conjecture, that starting from any positive integer the Collatz (3n+1) Sequence converges to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

END OF PROOF

11. Some Explicit Forms for the BEL Neighborhood

We can perform some simple algebraic manipulation to get the parametric relation [Eqn.16] that gives a generic form for the set H_k that is the set of k^{th} predecessors of $H_0 = \{BEL(1)\}$; that is, the set H_k corresponds to the set of tier-k level of the hierarchy with the set of Binary-Exponential-Ladders {BEL(m)} each with its defining-base-rung m being a positive odd number m>1.

$$m = [2^z - \{3^0.2^{z0} + 3^1.2^{z1} + 3^2.2^{z2} + \ldots + 3^{k-1}.2^{zk-1}\}] / 3^k; \qquad [Eqn.16]$$

wherein k>0 is the tier-level and $z > z_0$ whereas the k-tuple $(z_0, z_1, z_2, \ldots z_{k-1})$ form the set of strictly decreasing non-negative integer exponents values in [Eqn.16] each of which takes a unique value corresponding to each positive odd number m>1. That is, each positive odd number m>1 can be considered to be defined by the corresponding unique set of these parameters. The set of values for the k-tuple $(z_0, z_1, z_2, \ldots z_{k-1})$ are of strictly decreasing non-negative integer exponent values, all less than z, that is, $z > z_0 > z_1 > z_2 > \ldots > z_{k-1}$; ($z_k \coloneqq 0$; $z_{k-1} = 0$ for positive odd number m>1).

Now, define $p_0 \coloneqq (z - z_0)$; $p_j \coloneqq (z_{j-1} - z_j)$; where p_j corresponds to the number of rungs in BEL{H_j} above the *defining-base-rung* of BEL{H_j} for the node located in BEL{H_j} that the Collatz sequence/trajectory passes through; BEL{H_j} being the *Binary-Exponential-Ladder* at tier-j with j=0,1,2, . . . k. Thus, we may as well redefine the set of (k+1) parameters as a tuple (GPT) $\coloneqq (p_0, p_1, p_2, \ldots p_k)$ the set of (k+1) CTUHSK generative parameters that generate each positive integer n as per the parametric relation [Eqn.16] given above ($p_k = 0$ for positive odd number m).

For any k>0, the above set of exponents $z, z_0, z_1, z_2, z_3, \ldots z_k$, can be redefined in terms of the newly defined CTUHSK generative parameters, by rewriting the above definition as $z \coloneqq (z_0 + p_0)$; $z_{j-1} \coloneqq (z_j + p_j)$; $z_k \coloneqq 0$; $p_k = 0$ for any positive odd number m.

Table-1 gives some of the possible set of valid values for the CTUHSK generative parameters and therefore the corresponding valid values of the exponents in [Eqn.16] above along with their resultant $n(GPT) \coloneqq n(p_0, p_1, p_2, \ldots p_k)$ values.

Table-1 : Some typical CTUHSK generative parameter tuples GPT(n)

p0	p1	p2	p3	p4	p5	p6	p7	p8	n	z	z0	z1	z2	z3	z4	z5	z6	z6	z7
0									1	2	0								
1									2										
2									4										
3									8										
4									16										
4	0								5	4	0	0							
4	1	0							3	5	1	0	0						
4	3	0							13	7	3	0	0						
4	3	2	0						17	9	5	2	0	0					
4	3	2	1	0					11	10	6	3	1	0	0				
4	3	2	1	1	0				7	11	7	4	2	1	0	0			
4	3	2	1	1	2	0			9	13	9	6	4	3	2	0	0		
6	0								21	6	0	0							
4	3	2	1	3	0				29	13	9	6	4	3	0	0			
4	3	2	1	3	1	0			19	14	10	7	5	4	1	0	0		
4	3	2	1	3	1	2	0		25	16	12	9	7	6	3	2	0	0	
4	3	2	1	3	1	2	2	0	33	18	14	11	9	8	5	4	2	0	0
p0	p1	p2	p3	p4	p5	p6	p7	p8	n	z	z0	z1	z2	z3	z4	z5	z6	z6	z7

Table-1 : Some typical CTUHSK generative parameter tuples GPT(n)

12. A Challenge to the Cool-Headed Brave-Hearts

If you can prove that corresponding to every positive odd number m>1 there exists a unique valid set of CTUHSK generative parameters GPT(m) = $\{p_0, p_1, p_2, \ldots p_k\}$ and therefore the corresponding valid set of exponents $\{z, z_0, z_1, z_2, z_3, \ldots z_{k-1}, z_k\}$ in the parametric equation [Eqn.16] given above that generates every positive odd number m>1, then you can directly prove the Collatz Conjecture establishing the convergence of the Collatz Sequence to the trivial-cycle {(1⇐2⇐4)}.

12.1 Restrictions on the CTUHSK Generative Parameters

Note that the set of valid values for the CTUHSK generative parameters and therefore for the exponents in [Eqn.16] above, are governed by certain rules as can be seen from the earlier observations, regarding the matching relationship between the $((D^{\#})MOD3)$ of the predecessor and the $((U^{\#})MOD3)$ of the successor in the Collatz Sequence.

Specifically, [Eqn.17] states the relationship satisfied among –
(i) the x(MOD3) value of the exponent x for $U^{\#} = \{(6m-5).4^x\}$ at some higher rung in QEL(6m-5);
with (ii) with its defining-base-rung at (6m-5);
and (iii) its predecessor $D^{\#} = [\{(6m-5).4^x - 1\}/3]$.
Similarly, [Eqn.18] states the relationship satisfied among –
(i) the y(MOD3) value of the exponent y for $U^{\#} = \{(6m-1).2.4^y\}$ at some higher rung in QEL(6m-1);

with (ii) with its defining-base-rung at (6m-1);

and (iii) its predecessor $D^{\#} = [\{(6m\text{-}1).2.4^y - 1\}/3]$.

$$[\{(6m\text{-}5).4^x - 1\}/3]MOD3 = [x(MOD3) - m(MOD3) + 1]MOD3; \qquad [Eqn.17]$$

and

$$[\{(6m\text{-}1).2.4^y - 1\}/3]MOD3 = [y(MOD3) + m(MOD3) - 1]MOD3; \qquad [Eqn.19]$$

Rewriting [Eqn.17]&[Eqn.18] for the Binary-Exponential-Ladders, we get the equivalent set of equations as:

$$[\{(6m\text{-}5).2^w - 1\}/3]MOD3 = [[w/2]MOD3 - m(MOD3) + 1]MOD3; \qquad [Eqn.19]$$

and

$$[\{(6m\text{-}1).2^v - 1\}/3]MOD3 = [[(v\text{-}1)/2]MOD3 + m(MOD3) - 1]MOD3; \qquad [Eqn.20]$$

12.2 Most Intriguing and Reassuring Observation

Suppose we move forward from H_{k-1} to H_k and then onto H_{k+1}; that is, trace backwards along the Collatz Sequence, in the reverse direction; moving from $BEL(n_{k-1})$ to $BEL(n_k)$ and then onto $BEL(n_{k+1})$; with $BEL(n_{k-1}) = S(BEL(n_k)$ and $BEL(n_k) = S(BEL(n_{k+1})$; every time selecting the immediate-predecessor BEL of the *lowest denomination* with the *lowest value of the defining-base-rung*, that is, selecting $BEL(n_{k-1}) \in H_{k-1}$; as represented by the lowest value of 2 for w in [Eqn.7] or the lowest value of 1 for v in [Eqn.8] above.

We find that [Eqn.7] with w=2 leads to a situation wherein $n_{k-1} < n_k$; that is,

$$BEL(6m\text{-}5) = S(BEL([(6m\text{-}5).2^2 \text{ -}1]/3)) = S(BEL(8m\text{-}7)); \qquad [Eqn.21]$$

whereas [Eqn.8] with v=1 leads to a situation wherein $n_{k-1} > n_k$; that is,

$$BEL(6m\text{-}1) = S(BEL([(6m\text{-}1).2^1 \text{ -}1]/3)) = S(BEL(4m\text{-}1)); \qquad [Eqn.22]$$

Thus, we find that moving forward from H_{k-1} to H_k and then onto H_{k+1}; every time selecting the immediate-predecessor BEL of the lowest denomination, we encounter the three distinct situations:– (1) BEL(6m-3)$\in H_{k-1}$ being a leaf node of the BELnet, does not have any predecessors; (2) BEL(6m-5)$\in H_{k-1}$ has its immediate-predecessor BEL of lowest denomination being BEL(8m-7)$\in H_k$; (3) BEL(6m-1)$\in$ H_{k-1} has its immediate-predecessor BEL of lowest denomination being BEL(4m-1)$\in H_k$.

This establishes the fact that apart from the one-third of the BELs that lead to the leaf-nodes, the remaining two-thirds of the BELs are equally distributed between the two cases - BEL(6m-5) leads to the situation wherein the arborescence grows forward with increasing values for the defining-base-rung; whereas BEL(6m-1) leads to the situation wherein the arborescence grows forward with decreasing values for the defining-base-rung, thus *reaching-down* to some positive odd number of the form (4m-1) that was *left out in the earlier stages/tier-levels* of the hierarchy.

Note that there can be only three types of positive odd numbers corresponding to the three modular (nMOD3) residue classes; odd multiples of three that are of the form (6m-3) or numbers of the form (6m-5) or numbers of the form (6m-1); of which the odd multiples of three correspond to the defining-base-rung of the leaf nodes in the BELnet arborescence. Of the remaining two cases, it is of utmost significance that every positive odd number of the form (6m-1) forming the defining-base-rung of BEL(6m-1)∈H_{k-1} having its immediate-predecessor BEL of lowest denomination that is of the form BEL(4m-1)∈H_k. This is how as we move from H_{k-1} to H_k and then onto H_{k+1}; we encounter some positive odd numbers of the form (4m-1) that *could not be reached in the earlier stages/tier-levels*; as indicated by the presence of 1 in the CTUHSK generative parameter tuple.

The CTUHSK generative parameter tuple corresponding to every positive odd integer of the form (4m-1) must necessarily end with the subsequence (..., 1, 0); as for example, GPT(11) = (4, 3, 2, 1, 0); whereas the presence of a positive odd integer of the form (4m-1) anywhere in the Collatz Sequence is indicated by the presence of a 1 in the corresponding position in its CTUHSK generative parameter tuple; as for example, GPT(9) = (4, 3, 2, 1, 1, 2, 0); with the Collatz Sequence for 9 is 9->7->11->17->13->5->1; the two repeated occurrence of 1s corresponding to the subsequence 7->11->17 in the trajectory. It is indeed intriguing to observe that this specific process of reaching-down to such positive odd numbers of the form (4m-1) can indeed be a repeated contiguous operation at times, as can be seen by the presence of repeated 1's in the CTUHSK generative parameter tuple.

This is a situation wherein the emergence of global system properties through persistent local component subsystem characteristics can be clearly demonstrated! Banking on this global property, the author claims that the length of the CTUHSK generative parameter tuple |GPT(m)| required to generate a given positive odd number m is limited to be *no more than* m itself; whereas the triad {(31⇐41⇐27)} with |GPT(31)|=40 and |GPT(41)|=41 and |GPT(27)|=42 is an exception. Also, the triad {(1⇐5⇐3)} with |GPT(1)|=1 and |GPT(5)|=2 and |GPT(3)|=3 is another limiting case.

Halemane-Conjecture states that the maximum number of odd (3n+1) operations required to reach the trivial-cycle {(1⇐2⇐4)} starting from any given positive integer and moving along the Collatz sequence, is limited by that given number itself; with the triad {(31⇐41⇐27)} as an exceptional limiting case.

We will delve deeper into such algorithmic & computational details in a sequel to this research work, wherein a systematic (algorithmic) procedure for building the BELnet using the CTUHSK generative parameter tuples, corresponding to a systematic (algorithmic) procedure to produce a dictionary (lexicographic) of the CTUHSK generative parameter tuples, is planned to be presented.

13. Self-Similar Structural Symmetry of BELnet

From the above discussion one can notice that BELnet forms an arborescence in H^s with a self-similar structural symmetry. {BEL(1)} stands at the center, with its trivial-cycle at its defining-base-rung. At every tier-level k corresponding to the k^{th} term H_k in the sequence H^s, Binary-Exponential-Ladders of all the three modular (MOD3) residue classes are present in BELnet, each being countably infinite and equal in number; each having its single unique immediate-successor in H_{k-1}. Each of the {BEL(6m-5)} and each of the {BEL(6m-1)} has its immediate-predecessors in H_{k+1}; whereas of each the {BEL(6m-3)} remain as leaf-nodes since they can't have any immediate-predecessors. Thus, one-third of the BELs remain as leaf nodes; the other two-thirds become intermediate nodes that propagate the arborescence structure unboundedly to infinity. The self-similar structural symmetry in the network of binary-exponential-ladders BELnet is a self-similar derived from the modular-periodicity w.r.t both the defining-base-rung value among the neighboring BELs as well as the binary-exponent value within any given BEL, as defined by [Eqn.17] & [Eqn.18] or equivalently [Eqn.19] & [Eqn.20] above.

Note that each of the {BEL(6m-5)} with (6m-5) as its defining-base-rung, has its immediate-predecessor of lowest denomination being {BEL(8m-7)} as defined in [Eqn.21] where (8m-7)>(6m-5); facilitating the outward growth of the BELnet arborescence by pushing upward in the number-line. Also note that each of the {BEL(6m-1)} with (6m-1) as its defining-base-rung, has its immediate-predecessor of lowest denomination being {BEL(4m-1)} as defined in [Eqn.22] where (4m-1)<(6m-1); facilitating the inward growth of the BELnet arborescence by pulling downward in reaching-down the not-yet-reached in the number-line (positive odd numbers). Therefore, we observe that the specific instance of [Eqn.8] represented by [Eqn.24]; in the inverse of the CTUHSK function; acts as the precise mechanism that systematically fills the gaps in the number line (positive odd numbers) that otherwise gets left out by [Eqn.7] or even [Eqn21]; while constructing the lexicographic dictionary of CTUHSK generative parameter tuples for growing the BELnet arborescence.

This self-similar structural symmetry with the modular-periodicity (triadic/hexadic) both with respect to the defining-base-rung value and the binary-exponent value, along with the above mentioned unique feature of reaching-down the not-yet-reached numbers in the number-line, and using all the three modular (nMOD3) residue classes among the set of positive odd numbers as the defining-base-rung for the BELs, together guarantee the connectedness of the network of binary-exponential-ladders BELnet; which has of course been established above by showing the non-existence of any disjoint loops and/or divergent chains.

14. Conclusion

We established a two stage bijective mapping from the Collatz-domain to a meticulously designed *structured system framework* H with a careful reorganization & condensation :- in the first stage, a bijective mapping from the Collatz-domain to BELnet, that is the network of Binary-Exponential-Ladders defined on the set of positive odd numbers; and then in the second stage, a bijective mapping from BELnet to the structured system framework H. Then we established the *order-isomorphism* between the relevant component H^s (that is shown to be a well-ordered infinite linear sequence; or rather a hierarchy; or indeed an arborescence of the *Binary-Exponential-Ladders* having its root; with the invariant base element H_0 of the *structured system framework* H) and the set of positive integers; providing the necessary condition. A sufficient condition is provided through a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) that establishes *domain exhaustion*; having already captured all the modular (MOD3) residue classes in H^s; and guarantees the non-existence of any extraneous objects like disjoint loops/cycles and/or divergent chains. Since the Collatz (3n+1) Conjecture is restricted to the set of natural numbers, the existence of any non-standard 'rogue' and/or 'ghost' elements are of no concern here.

The entire dynamics of the Collatz (3n+1) system has been captured in a dynamically evolving graded algebraic structure of ideal based filtration scheme with divisibility classes for modulo-3 quotient semiring generated by the primitive root 2; along with a filtration shifting global affine transformation for a Euclidean expansion shift to the topmost coprime layer of that filtration scheme, avoiding the modulo-multiple-layer (zero layer) and also all the nilpotent layers with nilpotent elements (dead-end zero-divisors) like (6m-3) and (6m-0). The trivial-cycle {(1⇐2⇐4)} is bypassed by an initialization phase for the filtration scheme, starting with directly the modul-9 coprime layer. This design for the system dynamics framework by itself provides an independent definitive proof of the Collatz conjecture, establishing that the entire map of the Collatz system dynamics is exactly represented by this novel algebraic structure defined by the set of all positive integers; neither missing any natural number nor including any extraneous elements; and therefore, providing both the necessary & sufficient condition simultaneously in one single sweep.

As a direction towards future research work, we have also presented a possible alternative approach in proving this convergence, using modular arithmetic for the conditions to be satisfied by the CTUHSK generative parameters or equivalently the exponents in a closed-form expression for generating the reverse Collatz sequence corresponding to any given positive odd number.

It is observed that the consistent & persistent application of the modular arithmetic characteristics associated with the local neighborhood of BELnet imposed by the CTUHSK function gives rise to the emergence of the global system property associated with the structured system framework that forms an arborescence (hierarchy) of BELnet. This leads to an upper limit on the depth in the arborescence or the rank/level in the hierarchy that one needs to analyse in order to reach for the existence of a binary-exponential-ladder with the desired positive odd number as its defining-base-rung in BELnet. Halemane-Conjecture states that the maximum number of odd (3n+1) operations required to reach the trivial-cycle

{(1⇐2⇐4)} starting from any given positive integer and moving along the Collatz sequence, is limited by that given number itself; with the triad {(31⇐41⇐27)} as an exceptional limiting case.

15. Recommended Reading

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16. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an Independent Researcher, and that I have not received any funding from any source other than my own savings, and I do not have any obligations or encumbrances of any kind, neither financial nor legal nor of any other kind, regarding the contents of the manuscript - of which I am the original author and creator. Also, I hereby declare there is no conflict of interests.

17. Author Rights Retention Notice

18. Dedication

To my **ಅಜ್ಜ**(ajja) Karinja Halemane Keshava Bhat & **ಅಜ್ಜಿ**(ajji) Thirumaleshwari, **ಅಪ್ಪ**(appa) Shama Bhat & **ಅಮ್ಮ**(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection.*