Collatz-Conjecture Proved and Halemane-Conjecture Proposed

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Abstract

This paper presents a proof of the Collatz Conjecture, that is also known as the Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture, asserting the convergence of the Collatz (3x+1) sequence to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

An algebraic system framework designed as a bounded finite small-sized ideal-based graded-algebraic-filtration-structure defined on a modulo-2 quotient-semiring generated by the pair of roots -1 and 3; is an exact mathematical model to represent the inverse Collatz (3x+1) system.

The trivial-cycle of the Collatz-map is bypassed through an initialization-phase for this graded-algebraic-filtration-structure, starting directly with a 5-layered structure with the modulo-16 coprime-layer as its topmost layer. This also

facilitates a clear distinction among the six distinct possible combinations of the modulo-4 residue-classes and the modulo-6 residue-classes that are associated with any positive odd number; which is blurred in smaller structures.

A layer-shifting global-affine-transformation, defined by the condensed compact inverse Collatz (odd-to-odd) function; results in a Euclidean shift to the topmost coprime-layer of this filtration structure; avoiding the modulo-multiple-layer (zero-layer) and all the intermediate nilpotent-layers with nilpotent-elements (dead-end zero-divisors).

This system design of this ideal-based graded-algebraic-filtration-structure, establishes that the transitive closure of the subset {1,5,3} under the inverse Collatz function is the entire set of all positive integers; covering all the relevant (modulo-3 & modulo-6) modular-residue-classes and also covering all the possible valid triplet-combinations of (1) input-values (2) operations and (3) output-values; governed by the modular-periodicity characteristic (resulting in the self-similar symmetry structure) of the Collatz (3x+1) system; asserting that the Collatz sequence starting from any given positive integer converges to the trivial-cycle.

A deterministic discrete finite-state autonomous system (DDFSAS) model for the Collatz system leads to the Halemane-Assertion that there exist exactly six distinctly different possible classes of odd (3x+1) operations in the Collatz system.

Halemane-Conjecture states that the maximum number of odd (3x+1) operations required to reach the trivial-cycle {(1⇐2⇐4)}; starting from any given positive integer greater than one and moving along the Collatz sequence; is limited by that given number itself; the triad {(31⇐41⇐27)} is an exceptional limiting case.

Keywords: Binary-Exponential-Ladder; Modular-Periodicity;

Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) System;

Deterministic Discrete Finite-State Autonomous System (DDFSAS) Model;

Halemane-Assertion; Halemane-Conjecture;

Halemane's Theory of the CTUHSK System.

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1. Introduction

The Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture, simply referred to as the *Collatz Conjecture* [§1] states that the Collatz Sequence, also referred to as the Collatz (3x+1) Sequence, converges to the trivial-cycle, that is, $\{(1 \Leftarrow 2 \Leftarrow 4)\}$, starting from any positive integer.

The Collatz (3x+1) sequence, starting with any given positive integer is the sequence of numbers obtained by the repeated application of the Collatz function defined on the set of positive integers, given by [Eqn.1] as –

$$\begin{aligned}&\text{if } (x \equiv 0 (\text{mod} 2)) \text{ then } S(x) \coloneqq S_0(x) \coloneqq (x/2);\\ &\text{elseif } (x \equiv 1 (\text{mod} 2)) \text{ then } S(x) \coloneqq S_1(x) \coloneqq (3x+1);\end{aligned} \qquad \text{[Eqn.1]}$$

Note that the Collatz successor function S(x) is a unique one-to-one mapping. Lagarias [§3]&[§4] gives an exhaustive annotated bibliography, whereas Lagarias [§5]&[§6] gives an elaborate overview of the Collatz Problem, also referred to as the "3x+1 problem", referring to it as "The Ultimate Challenge". Guy [§2] has been somewhat pessimistic in advising researchers "Don't Try to Solve These Problems". Lagarius [§5]&[§6] refers to the statement by the most revered Mathematician and Number Theory Expert Paul Erdos, who said that - "Mathematics is not yet ready for such problems" - about the Collatz Conjecture. The Collatz (3x+1) Sequence is indeed a highly complex chaotic system defined or generated with one of the simplest deterministic well-defined equations, that too, just two in number. We will not get into any more of the reported literature except the problem definition; but will simply focus on a rather shockingly simple approach to solve this amazing long-standing unresolved problem.

This paper presents an algebraic model for the Collatz (3x+1) system; asserting the convergence of the Collatz sequence to the trivial-cycle {(1⇐2⇐4)}. Also, a deterministic discrete finite-state autonomous system (DDFSAS) model is presented, having six distinct classes of odd (3x+1) operations.

2. The Inverse Collatz Function

The inverse of the condensed compact Collatz (odd-to-odd) function P(x), defined on the set of positive odd numbers, is given by:

if $(n \equiv 1 (\text{mod} 6))$ then $\{P(n)\} \coloneqq \{P_1(n, w) \coloneqq (n.2^w - 1)/3\}$;

elseif ($m \equiv 5 (mod 6)$) then $\{P(m)\} \coloneqq \{P_5(m, v) \coloneqq (m.2^v - 1)/3\}$; [Eqn.2]

Note that the Collatz predecessor function $\{P(x)\}$ is a one-to-many mapping. Here, w is a positive even integer exponent and v is a positive odd integer exponent; and $\{P(x)\}$ correspond to the set of all immediate-predecessors of x in the condensed compact Collatz (odd-to-odd) sequence (which is a sequence of positive odd integers; the intervening nodes corresponding to the positive even integers the standard canonical Collatz sequence being suppressed or rather absorbed into the immediate-successor node).

3. Modular-Periodicity of the Collatz (3x+1) System

Observe that the immediate-predecessor relation given by $P_1(n, w)$ and $P_5(m, v)$ in [Eqn.2] above satisfies the modular-periodicity property (characteristic of the CTUHSK system; governing the self-similar symmetry in the arborescence structure) of the Collatz-Map; given by [Eqn.3] here below.

$$[P_1((6k+1), (2j))]_3 = [j]_3 - [k]_3;$$

and

$$[P_5((6k+5), (2j-1))]_3 = [j]_3 -1 + [k]_3; \qquad \text{[Eqn.3]}$$

Note that [Eqn.3] describes the exceptionally unique modular (modulo-3 and modulo-6) periodicity relation among the three parameters defining the predecessor relation, as represented by the inverse Collatz function given in [Eqn.2] above; namely, (1) the input value (2) the binary exponent value and (3) the output value.

4. Algebraic System Framework for the Inverse Collatz Function

We have designed an elegant algebraic system framework that provides an exact representation of the inverse Collatz function with the following characteristics.

4.1 BASE SEMIRING AND IDEAL

Start with the semiring of non-negative integers, R = ($Z_{\geq 0}$, +, .);
The modulo-2 quotient semiring is constructed by considering the ideal I = $\langle 2 \rangle$ defined by the modulo-base 2.

4.2 PAIR OF ROOTS AS GENERATORS

The modulo-2^k quotient semiring, for any k>1, has -1 and 3 as the pair of roots serve as generators; creating two distinct orbits (of exactly equal size) one being a mirror image of the other with respect to the modulus-base 2^k. The division-by-3 operation (as required by the inverse Collatz function) within any the semiring structure, is achieved through multiplication by the modular inverse of 3 in the specific quotient-semiring; for example, multiplying by 11 in modulo-16 semiring.

4.3 DIVISIBILITY-CLASSES

The divisibility-classes based on the modulo-2 base (used for the creation of the quotient semiring) defines the subring layers in the filtration-structure.

4.4 IDEAL-BASED FILTRATION STRUCTURE

To define a hierarchical filtration structure, we create a tower of subrings or ideals, corresponding to the divisibility-order with respect to the modulo-base 2; the bottommost being the zero-layer or the all-absorbing-layer. Each higher layer is created to accommodate for the possible valid values of the binary exponents w and v in [Eqn.2] above as required in the definition of the inverse Collatz function; the topmost layer being the units or the coprime-layer forming the principal-orbit. The quotient semirings defined with respect to modulus base 2^k characterize the layer-k with divisibility-order k in this filtration structure. The root 3 generates the multiplicative structure (corresponding to the division-by-3-operation in the inverse Collatz function) within any given subring; and the elements transition down the filtration hierarchy by exact division by 2 (although the inverse Collatz function does not have any operation of division by 2).

4.5 EUCLIDEAN EXPANSION SHIFT TO THE TOPMOST COPRIME LAYER

The inverse Collatz operators P(x) in [Eqn.2] above, suggest the use of an appropriate filtration layer-shifting global-affine-transformations $f(x) \coloneqq (2^w.x\text{-}1)$

and/or f(x) ≔ (2^v.x-1); maybe even considering at least the lowest possible valid values of w=2 and v=1 to begin with; to achieve a Euclidean-expansion-shift to a higher layer; while also avoiding the modulo-multiple-layer (zero-layer) and all the intermediate nilpotent-layers with nilpotent-elements (dead-end zero-divisors) like (4m-2) and (4m-0).

4.6 INITIALIZATION PHASE TO BYPASS THE TRIVIAL CYCLE

The trivial-cycle {(1⇐2⇐4)} can be bypassed by an initialization-phase for this filtration structure, starting directly with the modul-16 coprime-layer.

5. An Elegant Proof of the Collatz Conjecture

Observe that both the domain as well as the range of the condensed compact inverse Collatz (odd-to-odd) function (a one-to-many mapping); given in [Eqn.2] above, is the set of all positive odd integers; both the domain and the range covers every relevant modular (modulo-3 and modulo-6) residue class.

Also, observe that by the very design of the algebraic system framework, the transitive closure of the subset {1,5,3} under the inverse Collatz function is the entire set of all positive integers (neither missing any positive integer nor including any extraneous-elements); covering all the relevant (modulo-3 and modulo-6) modular-residue-classes; also covering all the possible valid triplet-combinations of (1) input-values (2) operations and (3) output-values; governed by the exceptionally unique modular-periodicity characteristic requirements/ constraints (also resulting in the self-similar symmetry structure) of the Collatz (3x+1) system. This is a definitive proof of the Collatz conjecture.

6. Deterministic Discrete Finite-State Autonomous System Model

The Collatz (3x+1) system is a deterministic discrete finite-state autonomous system (DDFSAS). The set of distinct possible system state transitions in this DDFSAS is uniquely defined and identified by a combination of the appropriate modular-residue-classes of the four parameters that govern the condensed compact inverse Collatz (odd-to-odd) function P(x) as in [Eqn.2] above which itself has two distinct operators P_1(n, w) and P_5(m, v) for the generic function P(x).

The first parameter-class is the modulo-6 residue class of the positive odd number input argument value x to P(x); that is, exactly one of the two possibilities, either $[1]_6$ or $[5]_6$; associated with the corresponding operators P_1(n, w) and P_5(m, v).

The second parameter-class is the modulo-3 residue class of the quotient associated with the modulo-6 coset-representative of the positive odd number input argument value x to P(x); that is, exactly one of the three possibilities, either $[0]_3$ or $[1]_3$ or $[2]_3$.

The third parameter-class is the modulo-6 residue class of the appropriate valid binary exponent (positive integer) w in P_1(n, w) or v in P_5(m, v); which itself is solely dependent on the first parameter, that is, from exactly one of the two sets of exponents, either $\{[1]_6; [3]_6; [5]_6\}$ or $\{[0]_6; [2]_6; [4]_6\}$.

The fourth parameter-class is the modulo-3 residue class of the output value of P(x); that is, exactly one of the three possibilities, either $[0]_3$ or $[1]_3$ or $[2]_3$.

Thus, the number of distinct possible combinations of these four discrete finite sets of parameter-classes is 2x3x3x3 = 54. However, these combinations are subject to the two constraints in [Eqn.3]; with a threefold multiplication factor, arising from the modulo-3 mapping on its dependent variable, on the left-hand-side of each of these two constraints. Therefore, the effective number of distinct classes of the system-state-transitions in the DDFSAS reduced to 2x3 = 6. That is, with a 6x6 state-transition-matrix having exactly one non-zero binary (0 or 1) entry in each row and each column. Observe that the six distinct classes of state transitions are: $\{([1]_6$ to $[1]_6)$; $([1]_6$ to $[5]_6)$; $([1]_6$ to $[3]_6)$; $([5]_6$ to $[3]_6)$; $([5]_6$ to $[1]_6)$; $([5]_6$ to $[5]_6)\}$.

7. Halemane-Assertion and Halemane-Conjecture

Halemane-Assertion states that there exist exactly six distinctly different possible classes of odd (3x+1) operations in the Collatz system, arising from the exceptionally unique modular (modulo-3 and modulo-6) periodicity characteristic (also resulting in the self-similar symmetry structure) of the Collatz system.

Halemane-Conjecture [§7] states that the upper bound on the number of odd (3x+1) operations required to reach the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$; starting from any given positive integer greater than one and moving along the Collatz sequence; is the given number itself; the triad $\{(31 \Leftarrow 41 \Leftarrow 27)\}$ is an exceptional limiting case.

8. Summary

We used an algebraic system framework designed as a bounded finite 5-layered ideal-based graded-algebraic-filtration-structure that is defined on a modulo-2 quotient-semiring generated by the pair of roots -1 and 3; as an exact mathematical model to represent the inverse Collatz (3x+1) system. By this very design it is observed that the transitive closure of the subset {1,5,3} under the inverse Collatz function is the entire set of all positive integers; providing a definitive proof of the Collatz conjecture.

The Halemane-Assertion that there exist exactly six distinctly different possible classes of odd (3x+1) operations in the Collatz system, is directly linked to the design of a deterministic discrete finite-state autonomous system (DDFSAS) model for the Collatz system.

Halemane-Conjecture [§7] states that the upper bound on the number of Collatz odd (3x+1) operations required to reach the trivial-cycle {(1⇐2⇐4)}; starting from any natural number greater than one and moving along the Collatz sequence; is given number itself; the triad {(31⇐41⇐27)} is an exceptional limiting case.

9. Recommended Reading

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10. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an Independent Researcher, and that I have not received any funding from any source other than my own savings, and I do not have any obligations or encumbrances of any kind, neither financial nor legal nor of any other kind, regarding the contents of the manuscript - of which I am the original author and creator. Also, I hereby declare there is no conflict of interests.

11. Author Rights Retention Notice

12. Dedication

To my **ಅಜ್ಜ**(ajja) Karinja Halemane Keshava Bhat & **ಅಜ್ಜಿ**(ajji) Thirumaleshwari, **ಅಪ್ಪ**(appa) Shama Bhat & **ಅಮ್ಮ**(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.