Infinite Horizon Linear Programming With One-Dimensional Control Variable

by

Somdeb Lahiri

ORCID: https://orcid.org/0000-0002-5247-3497

(Formerly with) PD Energy University, Gandhinagar (EU-G), India.

January 7, 2026.

This version: January 26, 2026.

Abstract

We provide a set of sufficient conditions for the optimal value of an infinite horizon linear programming problem with one-dimensional control variable to be equal to the optimal value of its implied infinite horizon dual linear programming problem. The simplicity of this characterization is entirely due to the fact that in each period, there is only one inequality constraint that the control variable is required to satisfy. We show this, by introducing a generalized version of the model. In this more general framework the duality gap problem can be partially resolved via a limiting argument. However, if the solution values of the dual linear programming problems of the truncated "fixed end-point linear programming problems" solved by the optimal solution are bounded above, and a transversality condition is satisfied, then the duality gap problem can be resolved.

Keywords: infinite horizon linear programming, discrete-time, linear constraints, duality, infinite horizon dual linear programming problem, transversality condition

AMS Subject Classification: 49K05, 90C05, 90C46

JEL Codes: C44, C61.

1. Introduction: At the outset, let us make it clear that we are concerned with infinite "horizon" linear programming problems or what we later generalize and refer to as "discrete-time infinite horizon" linear programming problems and not infinite linear programming problems, the latter embracing a much larger class of problems than what the former does. Hence, a counter example applicable for infinite linear programming problems need not be applicable for infinite horizon linear programming problems or its generalized version.

The study of infinite horizon linear programming problems originates in Hopkins (1969). The model discussed in Hopkins (1969) allows the chosen variables at any time period, to be constrained by choices made in all previous time periods. This model is discussed further in Grinold (1971). Among the earliest work on infinite horizon linear programming problems that restricts the chosen variable at any time period by just the chosen variables at the immediately previous period- and no further back in time- is the work in Evers (1973). The model in Evers (1973) assumes that the parameters determining the constraints for the choice variables are invariant over time. Some additional work based on the model developed by Evers (1973) is available in Grinold (1977). Romeijn, Smith and Bean (1992) along with its sequel- Romeijn and Smith (1998)- generalize the model in Evers (1973) and allow these parameters to vary with the time period. All works cited so far have one feature in common, i.e., they allow the control variables or chosen alternatives to be points in a finite dimensional Euclidean space, where the dimension could be greater than one.

There is a considerable body of literature on infinite horizon dynamic optimization with one dimensional control variable that is discussed in depth in Mitra (2000). The model discussed in Mitra (2000) allows for non-linear objective functions that are separable over time periods and constraint sets that allow for non-linearity. This model, referred to in Mitra (2000) as the reduced form model, has been studied with linear objective functions in Lahiri (2025a). In Lahiri (2025b), we discuss infinite horizon dynamic optimization with one dimensional control variables and linear objective function, where the constraint set in each period is determined by exactly one linear inequality. The model in Lahiri (2025b) is the framework of analysis that we start out with in this paper.

The purpose of Lahiri (2025a, 2025b) was to sharpen the results about the reduced form model in Mitra (2000) in the contexts of the two former papers respectively. Thus issues concerning Euler equations and transversality conditions are discussed in Lahiri (2025b) and there is no attempt made in that paper to relate the model to linear programming.

In Lahiri (2025c), there is a model of infinite horizon optimal control with linear objective functions and with linear equations and linear inequalities constraining the evolution of a one-dimensional state variable and a one-dimensional control variable. That paper deals with the problem concerning "duality gap" that Grinold (1971), Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) are concerned with. However, the model discussed in Lahiri (2025c) is conceptually different from the other works we have mentioned so far, since in Lahiri (2025c) there is a control variable as well as a state variable, each evolving in its own way and both contribute to the objective function. There is a way to "mathematically" reconcile the optimal control model discussed in Lahiri (2025c) with the framework that Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) are concerned with as we shall show later in this paper. Such a reconciliation would require the non-trivial assumption that *in every time period, the coefficient of the control variable in the equation of motion of the state variable is "strictly positive"*. This could be a way to make the model discussed in Lahiri (2025c) a one-dimensional control variable version of the multi-dimensional control variable model pursued in Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998). We will refer to this one-dimensional control variable version of the model in Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) as a "Discrete-Time Infinite Horizon Linear Programming Problem". In such problems, at each time period, the upper bound function for the set of available values for the control variable in the next period is a continuous, piece-wise affine and concave function on the closed interval from which the current value of the control variable can be conceivably chosen.

Duality gap is about the possibility of the optimal value of an infinite horizon linear programming problem being different from the optimal value of its infinite horizon dual linear programming problem. In linear programming, primal and dual linear programming problems are defined in the context of finite number of unknown variables. Hence, we refer to the dual linear programming problem in the infinite horizon context as "implied infinite horizon dual".

It needs to be noted that the model discussed in Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) is more general than the model discussed in Lahiri (2025b) and hence the one discussed here. The major contribution of Romeijn, Smith and Bean (1992) that is reproduced in Romeijn and Smith (1998), is a set of sufficient conditions that includes a transversality condition under which the optimal value of the infinite horizon linear programming problem is equal to the optimal value of its implied infinite horizon dual linear programming problem.

In this paper, our first significant result is a set of sufficient conditions that also includes a transversality condition under which the optimal value of the infinite horizon linear programming problem is equal to the optimal value of its implied infinite horizon dual linear programming problem. Our sufficient conditions require that in every period the linear inequality determining the constraint for the control variable for the next period is expressed in terms of a "non-constant" function of the current value of the control variable, the control variable along the optimal trajectory is always strictly positive beginning with time period one and is strictly less than its upper-bound in “at least one” time period. In addition, the transversality condition we invoke is that the product in each period, of the control variable and the dual variable for the inequality constraining the control variable in that period, converges to zero. Our sufficient conditions are considerably simpler than the ones required in the earlier papers mentioned here, for the purpose of obtaining similar conclusions. There does not seem to be any connection between our simplicity and the fact that unlike the “predecessor parers”, we focus our attention on one-dimensional control variables. In fact, our sufficient conditions here are also less restrictive than the sufficient conditions invoked in Lahiri (2025c) to address the duality gap problem.

The requirement that the control variable is always strictly positive beginning with the first period, leads to a difference equation for the evolution of the dual variable. The requirement that the control variable is strictly less than its upper-bound in “at least one” time period implies that the corresponding value of the dual variable is zero. This leads to a unique optimal solution of the implied infinite horizon dual linear programming problem. If the objective function is a discounted sum of the instantaneous values of the control variables and the slopes of the function determining the upper bounds of the control variable converge to a real number that is not equal to one, then from the difference equation governing the evolution of the dual variables it follows that the transversality condition must be satisfied.

In a final section of this paper we show that the major stumbling block towards defining the implied infinite horizon dual linear programming problem is the “multiplicity of constraints” in the more general context of a discrete-time infinite horizon linear programming problem. The possibility of multiple constraints in each period complicates the resolution of the duality gap problem. However, we are able to resolve the duality gap problem to some extent, without invoking any additional assumption on the model or the optimal trajectory, other than the ones that are used in the definitions for the general framework. This result says that the optimal value of the dual of the truncated “free end-point” linear programming problem with the initial value of the control variable in the latter being the same as the initial value of the control variable in the optimal trajectory (i.e., the kind of linear programming problems in the “approximation result”) converges to the optimal value of the discrete-time infinite horizon linear programming problem.

In a concluding section we note that if for a discrete-time infinite horizon linear programming problem, the solution values of the dual linear programming problems of the truncated "fixed end-point linear programming problems" solved by the optimal solution are bounded above, then by successive applications of the Bolzano-Weirstrass's theorem we can obtain an infinite sequence that solves the "implied" infinite horizon dual linear programming problem, provided a transversality condition is satisfied. This would resolve the duality gap problem in the general context. It is really "a matter of opinion" (like much else in science!) whether this result contributes substantially to infinite horizon linear programming.

2. The Framework of Analysis: Our framework of analysis presented below is "almost" identical to the one in Lahiri (2025b).

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers and $\mathbb{R}_{++}$ the set of strictly positive real numbers. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$.

Let $X = [0, b] \subset \mathbb{R}_+$ and let $b \in \mathbb{R}_{++}$ denote the set of available alternatives. Time is measured in discrete periods $t \in \mathbb{N}^0$. At each time period 't' an alternative is chosen, and the chosen alternative is denoted by $x_t \in X$.

Let $\langle c^{(t)} | t \in \mathbb{N}^0 \rangle$ be a sequence in $\mathbb{R}_+$ and let $\langle a^{(t)} | t \in \mathbb{N}^0 \rangle$ be a sequence in $\mathbb{R}$ such that $c^{(t)} \in [0, b]$ and $c^{(t)} + a^{(t)}b \in [0, b]$ for all $t \in \mathbb{N}^0$.

Thus, $\langle c^{(t)} | t \in \mathbb{N}^0 \rangle$ is a bounded sequence in $\mathbb{R}_+$.

Note 2.1: $\langle c^{(t)} | t \in \mathbb{N}^0 \rangle$ is a bounded sequence in $\mathbb{R}_+$ combined with $b > 0$, $-c^{(t)} \leq a^{(t)}b \leq b - c^{(t)}$ for all $t \in \mathbb{N}^0$ implies $\inf\{-\frac{c^{(t)}}{b} | t \in \mathbb{N}^0\} \leq a^{(t)} \leq \sup\{1 - \frac{c^{(t)}}{b} | t \in \mathbb{N}^0\}$, where both $|\inf\{-\frac{c^{(t)}}{b} | t \in \mathbb{N}^0\}| < +\infty$ and $|\sup\{1 - \frac{c^{(t)}}{b} | t \in \mathbb{N}^0\}| < +\infty$.

Thus, $\langle a^{(t)} | t \in \mathbb{N}^0 \rangle$ must also be a bounded sequence.

Since for all $x \in [0, b]$, $x = \frac{x}{b}b + (1 - \frac{x}{b})0$ with $\frac{x}{b} \in [0, 1]$ it must be the case that $c^{(t)} + a^{(t)}x \in X = [0, b]$ for all $(x, t) \in X \times \mathbb{N}^0$.

For $t \in \mathbb{N}^0$, let $\Omega_t = \{(x, y) \in X \times X | \; y \leq c^{(t)} + a^{(t)}x\} = \{(x, y) | 0 \leq y \leq c^{(t)} + a^{(t)}x, x \in [0, b]\}$.

Is it easy to see that for all $t \in \mathbb{N}^0$, Ω_t is a trapezium with its extreme points being (0, 0), $(0, c^{(t)})$, $(b, c^{(t)} + a^{(t)}b)$, (b, 0).

For $t \in \mathbb{N}^0$, Ω_t is **the two-period linearly constrained set at (time-period)** t.

For $(x, t) \in X \times \mathbb{N}^0$, let $\Omega_t(x) = \{y | 0 \leq y \leq c^{(t)} + a^{(t)}x\} = [0, c^{(t)} + a^{(t)}x]$.

Note that for all $t \in \mathbb{N}^0$, Ω_t is a non-empty, closed and bounded subset of X×X and for each (x, t) $\in X \times \mathbb{N}^0$, the set $\Omega_t(x)$ is a non-empty, closed and bounded interval in X, though the interval $\Omega_t(x)$ may be a singleton (i.e., degenerate) as for instance when $a^{(t)} = 0$.

For (x, t) $\in X \times \mathbb{N}^0$, the set $\Omega_t(x)$ is said to be **the transition set from** x **at (time-period)** t.

For $x \in X$, let $\mathcal{F}(x) = \{\langle x_t | t \in \mathbb{N}^0 \rangle | \ x_{t+1} \leq c^{(t)} + a^{(t)}x_t$ for all $t \in \mathbb{N}^0$, $x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$ as **a trajectory starting at (from)** x.

Let $\langle p^{(t)} | t \in \mathbb{N}^0 \rangle$ be a sequence in $\mathbb{R}$ satisfying $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$.

Note 2.2: $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$ implies $\lim_{t \to \infty} |p^{(t)}| = 0$ and for all sequences $\langle x_t | t \in \mathbb{N}^0 \rangle$, it must be the case that $|\sum_{t=0}^{\infty} p^{(t)} x_t| \leq \sum_{t=0}^{\infty} |p^{(t)}| x_t < +\infty$.

We shall refer to the sequence $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ as a **infinite horizon linear programming problem with one-dimensional control variable (1-IHLP problem)**.

Given a 1-IHLP problem $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ and $x \in X$ we shall be concerned with the following optimization problem denoted by **P1**(x):

Maximize $\sum_{t=0}^{\infty} p^{(t)} x_t$, subject to $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$.

If for some $c \in \mathbb{R}_+$ and $a \in \mathbb{R}$ the 1-IHLP $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ satisfies $c^{(t)} = c$ and $a^{(t)} = a$ for all $t \in \mathbb{N}^0$, then we may refer to the 1-IHLP problem as a **quasi time-invariant 1-IHLP problem**.

We will denote a quasi time-invariant 1-IHLP problem such as above by $(\langle p^{(t)} | t \in \mathbb{N}^0 \rangle, c, a)$.

Example 2.1: The discounted 1-IHLP problem: There exists $\delta \in (0, 1)$ such that for all $t \in \mathbb{N}^0$, $p^{(t)} = \delta^t$.

Note 2.3: If a "discounted 1-IHLP problem", is at the same time a quasi time-invariant 1-IHLP such that there exists a real numbers $c \in [0, b]$ and a real number a such that $c + ab \in [0, b]$ satisfying $c^{(t)} = c$ and $a^{(t)} = a$ for all $t \in \mathbb{N}^0$, then we have a model that closely resembles a one-dimensional control variable version of the infinite horizon linear programming discussed in Grinold (1977). We may refer to such a problem as a **time-invariant 1-IHLP problem** and denote it by (δ, c, a).

3. Some Preliminary results about trajectories: Given a 1-IHLP problem $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ for each $x\in X$, let $\mathcal{S}(x) = \underset{<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)}{\text{argmax}} \sum_{t=0}^{\infty} p^{(t)} x_t$. $\mathcal{S}(x)$ is the **set of solutions for P1**(x) **starting from** x **(for the 1-IHLP problem).**

The following result is proposition 4 in Lahiri (2025b).

Proposition 3.1: For all $x\in X$, $\mathcal{S}(x) \neq \phi$.

An immediate consequence of proposition 3.1, is that there exists a function $V: X \to \mathbb{R}$ such that for all $x\in X$: $V(x) = \sum_{t=0}^{\infty} p^{(t)} x_t$, where $<x_t|t\in\mathbb{N}^0>\in \mathcal{S}(x)$.

In note 2.3 we have defined a time-invariant 1-IHLP problem, It is well known that for a time invariant 1-IHLP problem (δ, c, a) for all $x\in X$: $V(x) = x + \delta \max_{y\in[0,\, c+ax]} V(y)$.

This equation is known as Bellman's fundamental equation of dynamic programming for the 1-IHLP problem.

For more on the fundamental equation of dynamic programming one may refer to Mitra (2000).

The following proposition provides a "solution procedure" for a large class of 1-IHLP problems.

Proposition 3.2: Suppose $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ is a 1-IHLP problem satisfying $p^{(t)} \geq 0$ for all $t\in\mathbb{N}$ and $a^{(t)} \geq 0$ for all $t\in\mathbb{N}^0$. Let $x\in X$ and let $<x_t|t\in\mathbb{N}^0>$ be the sequence with $x_0 = x$, and $x_{t+1} = c^{(t)} + a^{(t)}x_t$ for all $t\in\mathbb{N}^0$. Then, $<x_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$.

Further, if $<y_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$, then for all $T\in\{t\in\mathbb{N}|\ p^{(t)} > 0\}$ it must be the case that $y_T = x_T$.

Proof: Clearly $<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)$. Let $<y_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)$. Clearly $y_1\in [0, c^{(0)} + a^{(0)}x] = [0, x_1]$ and hence $y_1 \leq x_1$. Suppose that for some $T\in\mathbb{N}$ it is the case that $y_T \leq x_T$. Since, $a^{(T)} \geq 0$, $y_{T+1} \in [0, c^{(T)} + a^{(T)}y_T] \subset [0, c^{(T)} + a^{(T)}x_T] = [0, x_{T+1}]$, i.e., $y_{T+1} \leq x_{T+1}$. Thus, $y_T \leq x_T$ implies $y_{T+t} \leq x_{T+t}$ for all $t\in\mathbb{N}^0$.

Since $y_1 \leq x_1$, it must be the case that $y_t \leq x_t$ for all $t\in\mathbb{N}$.

Thus, $\sum_{t=0}^{\infty} p^{(t)}x_t = p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}x_t \geq p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}y_t$, since since $p^{(t)} \geq 0$ for all $t\in\mathbb{N}$.

However, $p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}y_t = \sum_{t=0}^{\infty} p^{(t)}y_t$.

Thus, $\sum_{t=0}^{\infty} p^{(t)}x_t \geq \sum_{t=0}^{\infty} p^{(t)}y_t$ and so $<x_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$.

If $<y_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$ then $<y_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)$. Thus, as observed earlier in this proof, $y_0 = x_0 = x$ and $y_t \leq x_t$ for all $t\in\mathbb{N}$.

Thus, if for some T∊ℕ, $p^{(T)} > 0$ and $y_T < x_T$, then $\sum_{t=0}^{\infty} p^{(t)}x_t = \sum_{t=0}^{T-1} p^{(t)}x_t + p^{(T)}x_T + \sum_{t=T+1}^{\infty} p^{(t)}x_t > \sum_{t=0}^{T-1} p^{(t)}x_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}x_t \geq \sum_{t=0}^{T-1} p^{(t)}y_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}y_t$, since $p^{(t)} \geq 0$ and $x_t \geq y_t$ for all t∊ℕ.

However, $\sum_{t=0}^{T-1} p^{(t)}y_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}y_t = \sum_{t=0}^{\infty} p^{(t)}y_t$ and hence $\sum_{t=0}^{\infty} p^{(t)}x_t > \sum_{t=0}^{\infty} p^{(t)}y_t$.

This, contradicts our assumption that $<y_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$ and proves the proposition.

Q.E.D.

4. Optimality and associated linear programming problems: In this section we replicate for our current context the main result in section 3 of Lahiri (2025c) the latter being a discussion on infinite horizon linear optimal control problems with linear constraints.

Note 4.1: The exact mathematical interpretation of the expression (formula) $\sum_{t=0}^{\infty} p^{(t)}x_t$ is $\lim_{T\to\infty} (\sum_{t=0}^{T} p^{(t)}x_t)$. Thus, the problem we are concerned with here is in the domain of asymptotic analysis, which is very different from infinite dimensional analysis.

An **alternative version of P1**(x) is the following optimization problem:

Maximize $\sum_{t=1}^{\infty} p^{(t)}x_t$ subject to the infinite sequence $<x_t|t\in \mathbb{N}^0>$ satisfying the constraints: $(x_t, x_{t+1})\in \Omega_t$, $t\in \mathbb{N}^0$, $x_0 = x$.

We now provide one necessary condition and a somewhat stronger sufficient condition for optimality for the 1-IHLP problem in terms of linear programming problems.

Proposition 4.1: Let $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ be a 1-IHLP problem and suppose that for some x∊X, $<x_t|t\in \mathbb{N}^0>\in \mathcal{F}(x)$.

Part 1: <u>If</u> $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$ <u>then</u> for all T∈ ℕ with T ≥ 2, $<x_t|t = 0, 1, \ldots, T>$ solves the following linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all t = 0, 1, …, T-1, $y_t \geq 0$ for all t = 1, …, T-1, $y_0 = x_0 = x$ and $y_T = x_T$.

Part 2: <u>If</u> there exists T^* ∊ ℕ such that for all T∈ ℕ satisfying $T \geq T^*$, $<x_t|t = 0, 1, \ldots, T>$ solves the linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all t = 0, 1, …, T-1, $y_t \geq 0$ for all t = 1, …, T-1, and $y_0 = x_0 = x$, <u>then</u> $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$.

Proof: The proof is similar to the proof of proposition 3.1 in Lahiri (2025c). Q.E.D.

We now provide an approximation result that follows from proposition 3.1, note 4.1 and part 1 of proposition 4.1.

Proposition 4.2 (Approximation Result): Given the 1-IHLP problem <$(p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0$> and $x \in X$, there exists $T^*(\varepsilon) \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ with $T \geq T^*(\varepsilon)$, the linear programming problem [Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)} y_t$ for all $t = 0, 1, \ldots, T-1$, $y_t \geq 0$ for all $t = 1, \ldots, T-1$, and $y_0 = x_0 = x$] has a solution <$x_t^{(T)} | t = 0, 1, \ldots, T$> and $|\sum_{t=0}^{T} p^{(t)} x_t^{(T)} - V(x)| < \varepsilon$. Hence, $\lim_{T\to\infty} \sum_{t=0}^{T} p^{(t)} x_t^{(T)} = V(x)$.

Proof: Since $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$ and $\mathcal{F}(x) \neq \phi$, for all $T \in \mathbb{N}$ the linear programming problem [Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)} y_t$ for all $t = 0, 1, \ldots, T-1$, $y_t \geq 0$ for all $t = 1, \ldots, T-1$, and $y_0 = x_0 = x$] has a solution <$x_t^{(T)} | t = 0, 1, \ldots, T$>.

Towards a contradiction suppose there exists $\varepsilon > 0$ such that $|\sum_{t=0}^{T} p^{(t)} x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often.

Then, it must be the case that <u>either</u> $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ infinitely often <u>or</u> $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} \leq V(x) - \varepsilon$ infinitely often.

Let <$x_t | t \in \mathbb{N}^0$> $\in \mathcal{S}(x)$. Since <$x_t | t = 0, 1, \ldots, T$> satisfies the constraints of the linear programming problem, for all $T \in \mathbb{N}$ it must be the case that $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} \geq \sum_{t=0}^{T} p^{(t)} x_t$ for all $T \in \mathbb{N}$.

Since, $V(x) = \sum_{t=0}^{\infty} p^{(t)} x_t = \lim_{T\to\infty} \sum_{t=0}^{T} p^{(t)} x_t$, there exists $T^0 \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ with $T \geq T^0$ it is the case that $\sum_{t=0}^{T} p^{(t)} x_t + \frac{\varepsilon}{4} > V(x) > \sum_{t=0}^{T} p^{(t)} x_t - \frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} + \frac{\varepsilon}{4} \geq \sum_{t=0}^{T} p^{(t)} x_t + \frac{\varepsilon}{4} > V(x)$ for all $T \in \mathbb{N}$ with $T \geq T^0$.

Thus, $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} + \varepsilon > \sum_{t=0}^{T} p^{(t)} x_t^{(T)} + \frac{\varepsilon}{4} > V(x)$ for all $T \in \mathbb{N}$ with $T \geq T^0$.

Thus, $|\sum_{t=0}^{T} p^{(t)} x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often is incompatible with $V(x) \geq \sum_{t=0}^{T} p^{(t)} x_t^{(T)} + \varepsilon$ infinitely often.

Thus, $|\sum_{t=0}^{T} p^{(t)} x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often implies that there exists $T^1 = T^1(\varepsilon) \in \mathbb{N}$ such that $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ for all $T \in \mathbb{N}$ satisfying $T \geq T^1$.

Since, $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$, there exists $T^2 = T^2(\varepsilon) \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ satisfying $T \geq T^2$, $b\sum_{t=T}^{\infty} |p^{(t)}| < \frac{\varepsilon}{4}$.

$\sum_{t=0}^{T} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ for all T $\epsilon\mathbb{N}$ satisfying $T \geq T^1$ implies $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ for all T$\epsilon\mathbb{N}$ satisfying $T \geq \max\{T^1, T^2\}$ infinitely often.

Let T$\epsilon\mathbb{N}$ be such that $T \geq \max\{T^1, T^2\}$.

Thus, $\sum_{t=0}^{T} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$.

Let $x_{T+t}^{(T)} = c^{(T+t-1)} + a^{(T+t-1)} x_{T+t-1}^{(T)}$ for all t$\epsilon\mathbb{N}$.

Thus, $<x_t^{(T)} | t\epsilon\mathbb{N}^0> \epsilon \mathcal{F}(x)$.

Now $|\sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)}| \leq \sum_{t=T+1}^{\infty} |p^{(t)} x_t^{(T)}| \leq b\sum_{t=T+1}^{\infty} |p^{(t)}| < \frac{\varepsilon}{4}$.

Thus, $\frac{\varepsilon}{4} > \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} > -\frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{\infty} p^{(t)} x_t^{(T)} = \sum_{t=0}^{T} p^{(t)} x_t^{(T)} + \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon + \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} > V(x) + \varepsilon - \frac{\varepsilon}{4} = V(x) + \frac{3\varepsilon}{4} > V(x)$.

This, contradicts the definition of V(x) and proves the proposition. Q.E.D.

5. Duality Theory for 1-IHLP problem and "necessary" conditions for optimality:

Given the 1-IHLP $<(p^{(t)},c^{(t)},a^{(t)})| t\in \mathbb{N}^0>$ suppose that for all t$\epsilon\mathbb{N}^0$, $c^{(t)}$, $c^{(t)} + a^{(t)}b > 0$, so that for all (x, t)ϵX$\times$ $\mathbb{N}^0$, $c^{(t)} + a^{(t)}x > 0$. This is equivalent to the assumption that for all (x, t)ϵX$\times\mathbb{N}^0$, {0} is a proper subset of $\Omega_t(x)$.

Given x$\in$X, let $<x_t|t\in \mathbb{N}^0> \epsilon\mathcal{F}(x)$.

For T$\in\mathbb{N}$ with $T \geq 3$ consider the linear programming problem in part 1 of proposition 4.1.

Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all t = 0, 1, …, T-1, $y_t \geq 0$ for all t = 1, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$ for all t = 0, 1, …, T-1, & $y_T = x_T$.

Since $\sum_{t=0}^{T} p^{(t)} y_t = p^{(0)}x + \sum_{t=1}^{T} p^{(t)} y_t$ along with $y_0 = x_0 = x$ and $y_T = x_T$, the linear programming problem in part 1 of proposition 4.1 reduces to the following.

Maximize $\sum_{t=1}^{T-1} p^{(t)} y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t \leq c^{(t)}$ for all t = 1, …, T-2, $-a^{(T-1)} y_{T-1} \leq c^{(T-1)} - x_T$, $y_t \geq 0$ for all t = 1, …, T-1.

The dual of this linear programming problem is the following linear programming problem.

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

Hence we have the following proposition.

Proposition 5.1: Suppose <$(p^{(t)},c^{(t)},a^{(t)})$|t∈ $\mathbb{N}^0$> is a 1-IHLP satisfying $c^{(t)}$ and $c^{(t)}$ + $a^{(t)}b > 0$ for all t∈$\mathbb{N}^0$. Suppose that for some x∈X, <x_t|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x). Then, for all T∈ ℕ with T ≥ 3, there exists <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> such that:

(A) <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> solves the linear programming problem

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

(B) $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^{*(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)$.

(C) $\lim_{T\to\infty} [\alpha_0^{*(T)}(c^{(0)} + a^{(0)})x + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)]$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$.

Proof: From the discussion in this section preceding the statement of this proposition, we know if <x_t|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x), then that for all T∈ ℕ with T ≥ 3, <x_t|t = 0, …, T> solves:

Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t \leq c^{(t)}$ for all t = 1, …, T-2, $-a^{(T-1)} y_{T-1} \leq c^{(T-1)} - x_T$, $y_t \geq 0$ for all t = 1, …, T-1.

By the strong duality theorem of linear programming (see topic 2 of Lahiri (2020)) we know that <x_t|t = 0, …, T> solves the above problem <u>if and only if</u> its dual has a solution, in which case the optimal value of the maximization problem and the optimal value of its dual are equal. The dual of the linear programming maximization problem is the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

For T∈ ℕ with T ≥ 3, let <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> be such a solution.

Thus, for T∈ ℕ with T ≥ 3, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^{*(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)$.

Since, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$, it must be the case that $\lim_{T\to\infty} [\alpha_0^{*(T)}(c^{(0)} + a^{(0)})x + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)]$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$. Q.E.D.

An immediate corollary of proposition 5.1 is the following result.

Corollary of Proposition 5.1: Suppose <$(p^{(t)},c^{(t)},a^{(t)})$| t∈ $\mathbb{N}^0$> is a 1-IHLP satisfying $c^{(t)}$ and $c^{(t)} + a^{(t)}b > 0$ for all t∈$\mathbb{N}^0$. Suppose that for some x∈X, <x_t|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x). Then, for all T∈ ℕ with T ≥ 3, there exists <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> such that:

(1) $\alpha_{t-1}^{*(T)}$ - $a^{(t)}\alpha_t^{*(T)} \geq p^{(t)}$ and ($\alpha_{t-1}^{*(T)}$ - $a^{(t)}\alpha_t^{*(T)}$ - $p^{(t)})x_t = 0$ for all t = 1, …, T-1;

(2) (x_1 - $c^{(0)}$ - $a^{(0)}x)\alpha_0^{*(T)} = 0$, ($x_{t+1}$ - $a^{(t)}x_t$ - $c^{(t)})\alpha_t^{*(T)} = 0$ for all t = 1, …, T-2, (-$a^{(T-1)}x_{T-1}$ - $c^{(T-1)}$ + $x_T)\alpha_{T-1}^{*(T)} = 0$.

Proof: By proposition 5.1 we know that for all T∈ ℕ with T ≥ 3, there exists <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> such that:

(A) <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> solves the linear programming problem

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)}$ - $a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

From the complementary slackness condition (see topic 2 of Lahiri (2020)) we know that since <x_t|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x) ⊂ $\mathcal{F}$(x), <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> along with <x_t|t = 0, …, T> satisfy (1) and (2). Q.E.D.

6. Interiority condition and infinite horizon dual linear programming problem:

In this section we consider 1-IHLP problems for which, in addition to the conditions assumed in section 5 (i.e., $c^{(t)} > 0$ and $c^{(t)} + a^{(t)}b > 0$ for all t∈$\mathbb{N}^0$), it is the case that for t∈ℕ, $a^{(t)} \neq 0$.

For our main result in this section we assume that along the optimal trajectory the control variable is always strictly positive, as for instance is the case with the solution in the statement of proposition 3.2. We also assume that in some time period t^*∈ℕ, the chosen value of the control variable is less than the upper bound for the possible set of values for the control variable in period t^*. We refer to this condition on the optimal trajectory as "interiority condition".

Proposition 6.1: Suppose <($p^{(t)}$,$c^{(t)}$,$a^{(t)}$))| t∈ $\mathbb{N}^0$> is a 1-IHLP satisfying $c^{(t)}$, $c^{(t)} + a^{(t)}b > 0$, for all t∈$\mathbb{N}^0$ and $a^{(t)} \neq 0$ for all t∈ℕ. Suppose that for some x∈X, <x_t|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x), satisfies the following "**interiority condition**": For all t∈ℕ, $x_t > 0$ and there exists t^*∈ℕ such that $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$.

Then, there exists a sequence <α_t^*|$\alpha_t^* \geq 0$, t∈$\mathbb{N}^0$> satisfying $\alpha_{t^*-1}^* = 0$ and α_{t-1}^* - $a^{(t)}\alpha_t^* = p^{(t)}$ for all t∈ ℕ such that:

(1) <α_t^*| t = 0, 1, 2, …, T-1> solves the following linear programming problem for all T∈ ℕ, T ≥ max{3, t^*}:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2}\alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)}$ - $a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

(2) For all T∈ ℕ, T ≥ max{3, t*}, $\sum_{t=1}^{T-1} p^{(t)} x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

Proof: Suppose <x_t|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x).

Then from proposition 5.1 we know that for all T∈ ℕ with T ≥ 3, there exists <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)}$≥ 0 for all t = 0, 1, ..., T-1> such that: <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)}$≥ 0 for all t = 0, 1, ..., T-1> solves the linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

From the corollary of proposition 5.1 we know that this <$\alpha_t^{*(T)}$| $\alpha_t^{*(T)}$≥ 0 for all t = 0, 1, ..., T-1> satisfies the following:

(1) $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} \geq p^{(t)}$ and $(\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} - p^{(t)})x_t = 0$ for all t = 1, …, T-1;

(2) $(x_1 - c^{(0)} - a^{(0)}x)\alpha_0^{*(T)} = 0$, $(x_{t+1} - a^{(t)}x_t - c^{(t)})\alpha_t^{*(T)} = 0$ for all t = 1, …, T-2, $(-a^{(T-1)} x_{T-1} - c^{(T-1)} + x_T)\alpha_{T-1}^{*(T)} = 0$.

By the "interiority condition", for all t∊ℕ, x_t > 0 and hence from (i) we get $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} = p^{(t)}$ for all t = 1, …, T-1.

By the "interiority condition" once again $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)} x_{t^*-1}$ and hence for T∊ℕ with T ≥ max{3, t*}, $\alpha_{t^*-1}^{*(T)} = 0$.

Thus, for all T∈ ℕ with T ≥ max{3, t*}, <$\alpha_t^{*(T)}$|$\alpha_t^{*(T)}$ ≥ 0 for all t = 0, …, T-1> satisfies: $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} = p^{(t)}$ for all t = 1, …, T-1 and $\alpha_{t^*-1}^{*(T)} = 0$.

Thus, for all T∈ ℕ with T ≥ max{3, t*}, <$\alpha_t^{*(T)}$|$\alpha_t^{*(T)}$ ≥ 0 for all t = 0, …, T-1> must be the unique solution of the difference equation $\alpha_{t-1} - a^{(t)}\alpha_t = p^{(t)}$, $\alpha_{t^*-1}^{*(T)} = 0$.

Let <α_t^*|α_t^* ≥ 0, t∊$\mathbb{N}^0$> be the unique solution of this system of difference equation satisfying the condition $\alpha_{t^*-1} = 0$.

Since the optimal value of the primal must be equal to the optimal value of its dual, it must be the case that for all T∈ ℕ with T ≥ max{3, t*}, <α_t^*|α_t^* ≥ 0, t = 0, 1, …, T-1> solves

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

Hence, for all T∈ ℕ with T ≥ max{3, t*} it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$. Q.E.D.

Note 6.1: We know from proposition 6.1 that $\alpha_{t^*-1}^* = 0$ and for all t∈ℕ, $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$. We have assumed for proposition 6.1 that $a^{(t)} \neq 0$ for all t∈ℕ. Thus, $\alpha_{t^*-1}^* = 0$ and $\alpha_t^* = \frac{\alpha_{t-1}^* - p^{(t)}}{a^{(t)}}$ for all t∈ℕ. If the 1-IHLP problem is a discounted 1-IHLP problem with $p^{(t)} = \delta^t$ for all t∈ℕ and δ∈(0, 1), then $\delta^t = \alpha_{t-1}^* - a^{(t)}\alpha_t^*$ for all t∈ℕ.

Thus, if $<a^{(t)}| t\in\mathbb{N}^0>$ converges to a∈ℝ with a ≠ 1, then it must be the case that $\lim_{T\to\infty} \alpha_T^* = 0$.

For x∈X, consider the alternative version of P1(x) defined in section 4:

Maximize $\sum_{t=1}^{\infty} p^{(t)}y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t \leq c^{(t)}$, $y_t \geq 0$ for all t∈ℕ.

This is the maximization problem that we are really concerned with.

Its "**implied dual linear programming (IDLP) problem**" is the following:

Minimize $\alpha_0(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t c^{(t)}$, subject to $\alpha_{t-1} - a^{(t)}\alpha_t \geq p^{(t)}$, for all t∈ℕ and $\alpha_t \geq 0$, t∈ℕ⁰.

Theorem 6.1: Suppose $<(p^{(t)},c^{(t)},a^{(t)})| t\in\mathbb{N}^0>$ is a 1-IHLP satisfying $c^{(t)}$, $c^{(t)} + a^{(t)}b > 0$, for all t∈ℕ⁰ and $a^{(t)} \neq 0$ for all t∈ℕ. Suppose that for some x∈X, $<x_t|t\in\mathbb{N}^0> \in \mathcal{S}(x)$, satisfies the following "**interiority condition**": For all t∈ℕ, $x_t > 0$ and there exists t*∈ℕ such that $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$.

Then, there exists a sequence $<\alpha_t^*|\alpha_t^* \geq 0, t\in\mathbb{N}^0>$ satisfying $\alpha_{t^*-1}^* = 0$ and $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$ for all t∈ ℕ such that:

(1) $<\alpha_t^*| t = 0, 1, 2, \ldots, T-1>$ solves the following linear programming problem for all T∈ ℕ with T ≥ max{3, t*}:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

(2) For all T∈ ℕ with T ≥ max{3, t*}, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

(3) $\sum_{t=1}^{\infty} p^{(t)}x_t = (c^{(0)} + a^{(0)}x)\alpha_0^* + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$ <u>if and only if</u> $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$.

Proof: (1) and (2) follow from proposition 6.1. Hence we need to prove (3).

From (2) of proposition 6.1 we know that for all T∈ ℕ with T ≥ max{3, t*} it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

Thus, for all T∈ ℕ with T ≥ max{3, t*} it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Since, $\langle x_t | t\in \mathbb{N}^0\rangle \in \mathcal{S}(x)$, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t$ exists and $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \sum_{t=1}^{\infty} p^{(t)}x_t$.

If $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$, then $\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T]$ exists and $\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T] = \lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t + \lim_{T\to\infty} \alpha_{T-1}^* x_T = \lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t$.

$\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T]$ exists implies $\alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ exists and hence $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ for all T ≥ max{3, t*} implies $\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T] = \alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Thus, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Since $\lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} = \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$ it follows that $\sum_{t=1}^{\infty} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$.

Conversely, if $\sum_{t=1}^{\infty} p^{(t)}x_t = (c^{(0)} + a^{(0)}x)\alpha_0^* + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)} = \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$, then $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = (c^{(0)} + a^{(0)}x)\alpha_0^* + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ for all T∈ ℕ with T ≥ max{3, t*}, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \sum_{t=1}^{\infty} p^{(t)}x_t$ and $\lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} = \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$ together imply $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$. Q.E.D.

Note 6.2: Since for all T∈ℕ, α_{T-1} is the dual variable corresponding to x_T, $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$ is a version of the well-known "**transversality condition**" of optimal control theory. Part (3) of theorem 6.1 says that under the assumptions common to both proposition 6.1 and theorem 6.1, the optimal value of the IDLP problem is equal to the optimal value of the alternative version of P1(x) defined in section 4 "*if and only if*" the **transversality condition** (in the way defined here) is satisfied by the solutions of the two infinite horizon linear programming problems.

Note 6.3: If as in note 6.1, the 1-IHLP problem is a discounted 1-IHLP problem with $p^{(t)} = \delta^t$ for all t∈ℕ and δ∈(0, 1), and $\langle a^{(t)} | t\in\mathbb{N}^0\rangle$ converges to a∈ℝ with a ≠ 1, then it must be the case that $\lim_{T\to\infty} \alpha_T^* = 0$. Thus $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$. Hence the transversality condition is satisfied for a discounted 1-IHLP problem satisfying the additional convergence criterion for $<a^{(t)}| t\in\mathbb{N}^0>$.

Note 6.4: The satisfaction of the "interiority condition" by $<x_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$ implies a structural restriction on the 1-IHLP $<(p^{(t)},c^{(t)},a^{(t)}))| t\in\mathbb{N}^0>$. First note that since the condition requires $x_t > 0$ for all $t\in\mathbb{N}$, for $s\in\mathbb{N}$, there exists $\varepsilon > 0$ sufficiently small such that $0 < x_s - \varepsilon < c^{(s-1)} + a^{(s-1)}x_{s-1}$. If $a^{(s)} \leq 0$, then $x_{s+1} \leq c^{(s)} + a^{(s)}x_s \leq c^{(s)} + a^{(s)}(x_s - \varepsilon)$. Thus, $<y_t|t\in\mathbb{N}^0>$ with $y_t = x_t$ for all $t\in\mathbb{N}^0\backslash\{s\}$, $y_s = x_s - \varepsilon$ satisfies $<y_t|t\in\mathbb{N}^0> \in \mathcal{F}(x)$. Since $<x_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$, it must be the case that $\sum_{t=1}^{\infty} p^{(t)}x_t \geq \sum_{t=1}^{\infty} p^{(t)}y_t = \sum_{t=1}^{\infty} p^{(t)}x_t - \varepsilon p^{(s)}$, i.e., $\varepsilon p^{(s)} \geq 0$. Since $\varepsilon > 0$, $\varepsilon p^{(s)} \geq 0$ implies $p^{(s)} \geq 0$. Thus, for all $s\in\mathbb{N}$, $a^{(s)} \leq 0$ implies $p^{(s)} \geq 0$. Further, the "interiority condition" requires $0 < x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$. Thus, for $\varepsilon > 0$ sufficiently small, $0 < x_{t^*} + \varepsilon < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$. If $a^{(t^*)} \geq 0$, then for $\varepsilon > 0$, $x_{t^*+1} \leq c^{(t^*)} + a^{(t^*)}(x_{t^*} + \varepsilon)$ and thus $<y_t|t\in\mathbb{N}^0>$ with $y_t = x_t$ for all $t\in\mathbb{N}^0\backslash\{t^*\}$, $y_{t^*} = x_{t^*} + \varepsilon$ satisfies $<y_t|t\in\mathbb{N}^0> \in \mathcal{F}(x)$. Since $<x_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$, it must be the case that $\sum_{t=1}^{\infty} p^{(t)}x_t \geq \sum_{t=1}^{\infty} p^{(t)}y_t = \sum_{t=1}^{\infty} p^{(t)}x_t + \varepsilon p^{(t^*)}$, i.e., $\varepsilon p^{(t^*)} \leq 0$. Since $\varepsilon > 0$, $\varepsilon p^{(t^*)} \leq 0$ implies $p^{(t^*)} \leq 0$. Thus, $a^{(t^*)} \geq 0$ implies $p^{(t^*)} \leq 0$.

7. Discrete-Time Infinite Horizon Linear Programming Problem (with One-Dimensional Control Variable): With a little bit of algebra along with a non-trivial assumption applied to the model of linear optimal control with linear constraints in Lahiri (2025c), we can obtain a generalization of a 1-IHLP problem. The additional assumption required is that in every period the coefficient of the control variable in the equation determining the evolution of the control variable is strictly positive. Suppose that for all $t\in\mathbb{N}^0$, there is a positive integer m_t, such that $A^{(t)}$, $C^{(t)}$ are m_t dimensional real-valued column vectors whose i^{th} entry for $i\in I(t) = \{1, \ldots, m(t)\}$are $A_i^{(t)}$ and $C_i^{(t)}$ respectively and satisfy the following condition: For all $x\in X$

$0\in\bigcap_{i\in I(t)}(\{y\in\mathbb{R}_+|y - A_i^{(t)}x \leq C_i^{(t)}\})\subset X$.

Thus, for all $x\in X$ and $i\in I(t)$ it must be the case that $\{y\in\mathbb{R}_+|y - A_i^{(t)}x \leq C_i^{(t)}\}$ is a non-empty closed interval in X whose left-hand end point is 0.

For $t\in\mathbb{N}^0$, the real valued function on [0, b] defined by $x\mapsto\min\{C_i^{(t)} + A_i^{(t)}x| i\in I(t)\}$ is a continuous, piece-wise affine and concave function. The assumption for all $x\in X$, $0\in\bigcap_{i\in I(t)}(\{y\in\mathbb{R}_+|y - A_i^{(t)}x \leq C_i^{(t)}\})\subset X$ implies that the graph of this function is contained in $X\times X$.

For what follows we assume:

$\max\{\sum_{i\in I(t)} |A_i^{(t)}| | t\in\mathbb{N}^0\} < +\infty$, $\max\{\sum_{i\in I(t)} |C_i^{(t)}| | t\in\mathbb{N}^0\} < +\infty$ and $\max\{\sum_{i\in I(t)} |D_i^{(t)}| | t\in\mathbb{N}^0\} < +\infty$.

The above assumption allows us to choose b > 0 (sufficiently large) so that the requirement $\{y\in\mathbb{R}_+ | y - A_i^{(t)}x \leq C_i^{(t)}\}$ is a closed interval in X whose left-hand end point is 0, is satisfied.

For $t\in\mathbb{N}^0$, let $\Omega_t = \{(x, y)\in X\times X | y - A_i^{(t)}x \leq C_i^{(t)}$ for <u>all</u> $i\in I(t)\}$ and for $(x, t) \in X\times \mathbb{N}^0$, let $\Omega_t(x) = \{y\in X | (x, y)\in\Omega_t\}$.

For $x\in X$, let $\mathcal{F}(x) = \{<x_t|t\in\mathbb{N}^0> | x_{t+1} \in \Omega_t(x_t)$ for all $t\in\mathbb{N}^0$, $x_0 = x\}$

As before, let $<p^{(t)}| t\in\mathbb{N}^0>$ be a sequence in $\mathbb{R}$ satisfying $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$.

We shall refer to the sequence $<(p^{(t)}, A^{(t)}, C^{(t)}) | t\in\mathbb{N}^0>$ as a **discrete-time infinite horizon linear programming (DT-IHLP) problem**.

Thus, a DT-IHLP problem $<(p^{(t)}, A^{(t)}, C^{(t)}) | t\in\mathbb{N}^0>$ with I(t) = (1} for all $t\in\mathbb{N}^0$ is a 1-IHLP problem.

Given a DT-IHLP problem $<(p^{(t)}, A^{(t)}, C^{(t)}) | t\in\mathbb{N}^0>$ and $x\in X$, we shall be concerned with the following optimization problem denoted by **G-P1**(x):

Maximize $\sum_{t=0}^{\infty} p^{(t)}x_t$, subject to $<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)$.

Let $\mathcal{S}(x) = \underset{<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)}{\operatorname{argmax}} \sum_{t=0}^{\infty} p^{(t)}x_t$.

An alternative version of G-P1(x) is the following:

Maximize $\sum_{t=1}^{\infty} p^{(t)}x_t$, subject to $<x_t|t\in\mathbb{N}^0> \in\mathcal{F}(x)$.

If for $x\in X$, $<x_t|t\in\mathbb{N}^0>\in\mathcal{S}(x)$, $x_t \in\{0, \min\{C_i^{(t-1)} + A_i^{(t-1)}x_{t-1}\}$ for all $t\in\mathbb{N}$ satisfying $p^{(t)} \neq 0$, then we may, in keeping with the terminology in optimal control theory (as for instance in pages 202–208 in Kamien and Schwartz (1991)), refer to such a trajectory as a **bang-bang solution starting at** x.

For instance, if $A_i^{(t)} > 0$ for all $i\in I(t)$ and $t\in\mathbb{N}^0$ and $p^{(t)} > 0$ for all $t\in\mathbb{N}$, then $\mathcal{S}(x) = \{<x_t|t\in\mathbb{N}^0>\}$ such that $x_t = \min\{C_i^{(t-1)} + A_i^{(t-1)}x_{t-1}\}$ for all $t\in\mathbb{N}$.

On the other hand, if $p^{(t)} < 0$ for all $t\in\mathbb{N}$, then $\mathcal{S}(x) = \{<x_t|t\in\mathbb{N}^0>\}$ such that $x_t = 0$ for all $t\in\mathbb{N}$.

Both are "extreme" examples of a bang-bang solution starting at x.

The following is an example that incorporates "periodicity" in the pattern (or behavior) of bang-bang solution.

Example 7.1: Let <($p^{(t)}$, $A^{(t)}$, $C^{(t)}$)|t$\epsilon\mathbb{N}^0$> be a DT-IHLP such that for all t$\epsilon\mathbb{N}$, I(t-1) = {1} and:

(1) If t is odd, then $C_1^{(t-1)} > 0$, $A_1^{(t-1)} \geq 0$, $C_1^{(t-1)} + A_1^{(t-1)}b \leq b$ and $p^{(t)} > 0$.

(2) If t is even, then $C_1^{(t-1)} = 0$, $A_1^{(t-1)} = 0$.

From (1) it follows that if t is an odd natural number then $C_1^{(t-1)} \leq b$, so that $C_1^{(t-1)} + A_1^{(t-1)}x \epsilon X$ for all xϵX.

From (2) it follows that for all xϵX and <x_t|t$\epsilon\mathbb{N}^0$>$\epsilon\mathcal{F}$(x), $x_t = 0$ for all t $\epsilon\mathbb{N}$ if t is even.

Hence, for all xϵX and <x_t|t$\epsilon\mathbb{N}^0$> $\epsilon\mathcal{F}$(x), $\sum_{t=0}^{\infty} p^{(t)}x_t = p^{(0)}x + \sum_{t=1}^{\infty} p^{(2t-1)}x_{2t-1}$.

Thus, for all xϵX, if <x_t|t$\epsilon\mathbb{N}^0$> $\epsilon\mathcal{S}$(x), $x_1 = C_1^{(0)} + A_1^{(0)}x > 0$, since $C_1^{(0)} > 0$ and $A_1^{(0)} \geq 0$.

For t $\epsilon\mathbb{N}$, $x_{2t+1} = C_1^{(2t)} + A_1^{(2t)}x_{2t} = C_1^{(2t)} > 0$, since $x_{2t} = 0$.

Thus, <x_t|t$\epsilon\mathbb{N}^0$> $\epsilon\mathcal{S}$(x) if and only if x_0 = x, $x_t = C_1^{(t-1)} + A_1^{(t-1)}x_{t-1}$ if t is an odd natural number and $x_t = 0$ if t is an even natural number.

The possibility of "bang-bang solution" illustrates the wide scope of DT-IHLP problems (in fact, in the context of our examples above 1-IHLP problems).

As in section 3 we have the following result.

Proposition 7.1: For all xϵX, $\mathcal{S}$(x) ≠ ϕ.

An immediate consequence of proposition 7.1, is that there exists a function V: X → $\mathbb{R}$ such that for all xϵX: V(x) = $\sum_{t=0}^{\infty} p^{(t)}x_t$, where <$x_t$|t$\epsilon\mathbb{N}^0$>$\epsilon$ $\mathcal{S}$(x).

As in section 4, the following result can be obtained from corresponding results in Lahiri (2025c).

Proposition 7.2: Let <($p^{(t)}$, $A^{(t)}$, $C^{(t)}$)|t$\epsilon\mathbb{N}^0$> be a DT-IHLP problem and suppose that for some xϵX, <x_t|t∈ $\mathbb{N}^0$>∈$\mathcal{F}$(x).

Part 1: <u>If</u> <x_t|t∈ $\mathbb{N}^0$>∈$\mathcal{S}$(x) <u>then</u> for all T∈ $\mathbb{N}$ with T ≥2, <x_t|t = 0, 1, …, T> solves the following linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all iϵI(t) and t = 0, 1, …, T-1, $y_t \geq 0$ for all t = 1, …, T-1, $y_0 = x_0$ = x and $y_T = x_T$.

Part 2: <u>If</u> there exists T^* ϵ $\mathbb{N}$ such that for all T∈ $\mathbb{N}$ satisfying T ≥ T^*, <x_t|t = 0, 1, …, T> solves the linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all iϵI(t) and t = 0, 1, …, T-1, $y_t \geq 0$ for all t = 1, …, T-1, and $y_0 = x_0$ = x, <u>then</u> <x_t|t∈ $\mathbb{N}^0$>∈$\mathcal{S}$(x).

The approximation result in section 4 continues to hold for a DT-IHLP problem and hence we have the following.

Proposition 7.3: Given the DT-IHLP problem $<(p^{(t)}, A^{(t)}, C^{(t)})|t\in\mathbb{N}^0>$ and $x\in X$, there exists $T^*(\varepsilon)\in\mathbb{N}$ such that for all $T\in\mathbb{N}$ with $T \geq T^*(\varepsilon)$, the linear programming problem [Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all $i\in I(t)$ and $t = 0, 1, \ldots, T-1$, $y_t \geq 0$ for all $t = 1, \ldots, T-1$, and $y_0 = x_0 = x$] has a solution $<x_t^{(T)}| t = 0, 1, \ldots, T>$ and $|\sum_{t=0}^{T} p^{(t)}x_t^{(T)} - V(x)| < \varepsilon$.

For $x\in X$, let $<x_t|t\in \mathbb{N}^0> \in\mathcal{F}(x)$.

For $T\in \mathbb{N}$ with $T \geq 3$ the linear programming problem in part 1 of proposition 7.2 is equivalent to the following linear programming problem:

Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $y_1 \leq C_i^{(0)} + A_i^{(0)}x$ for all $i\in I(0)$, $y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all $i\in I(t)$ and $t = 1, \ldots, T-2$, $-A_i^{(T-1)}y_{T-1} \leq C_i^{(T-1)} - x_T$ for all $i\in I(T-1)$, $y_t \geq 0$ for all $t = 1, \ldots, T-1$.

The dual of this linear programming problem is the following linear programming minimization problem.

Minimize $\sum_{i\in I(0)} \beta_i^{(T,0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2} (\sum_{i\in I(t)} \beta_i^{(T,t)} C_i^{(t)}) + \sum_{i\in I(T-1)} \beta_i^{(T,T-1)}(C_i^{(T-1)} - x_T)$ subject to $\sum_{i\in I(t-1)} \beta_i^{(T,t-1)} - \sum_{i\in I(t)} \beta_i^{(T,t)} A_i^{(t)} \geq p^{(t)}$ for all $t = 1, \ldots, T-1$, $\beta_i^{(T,t)} \geq 0$ for all $i\in I(t)$ and $t = 0, 1, \ldots, T-1$.

Thus, even if we assume that $x_t > 0$ for all $t\in\mathbb{N}^0$, the system of difference equations that emerged almost immediately from the application of duality theorem and complementary slackness conditions in the proof of proposition 6.1, fail to do so once we move into the more general framework of DT-IHLP problems.

However, we can obtain a somewhat weaker version of theorem 6.1 in this considerably more general context and with less restrictive conditions on both the problem as well as the optimal solution, if we pursue the line of reasoning for the approximation result (proposition 7.3 in this section).

Note that for $T\in \mathbb{N}$ with $T \geq 3$, the linear programming problem [Maximize $\sum_{t=1}^{T} p^{(t)}y_t$, subject to $y_1 \leq C_i^{(0)} + A_i^{(0)}x$ for all $i\in I(0)$, $y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all $i\in I(t)$ and $t = 1, \ldots, T-1$, $y_t \geq 0$ for all $t = 1, \ldots, T$] is the sequence of "free end-point" linear programming programming problems that was used for the approximation result (i.e. proposition 7.3 above).

Proposition 7.3 says that for all T∈ ℕ, this linear programming has a solution $<x_t^{(T)}| t = 0, 1, \ldots, T>$ and $\lim_{T\to\infty} \sum_{t=1}^{T} p^{(t)} x_t^{(T)} = V(x) - p^{(0)}x$.

The dual of this linear programming problem is the linear programming problem [Minimize $\sum_{i\in I(0)} \beta_i^{(T,0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-1} (\sum_{i\in I(t)} \beta_i^{(T,t)} C_i^{(t)})$ subject to $\sum_{i\in I(t-1)} \beta_i^{(T,t-1)} - \sum_{i\in I(t)} \beta_i^{(T,t)} A_i^{(t)} \geq p^{(t)}$ for all t = 1, …, T-1, $\sum_{i\in I(t-1)} \beta_i^{(T,T-1)} \geq p^{(T)}$, $\beta_i^{(T,t)} \geq 0$ for all i∈I(t) and t = 0, 1, …, T-1].

Theorem 7.1: Let $<(p^{(t)}, A^{(t)}, C^{(t)})|t\in\mathbb{N}^0>$ be a DT-IHLP problem and suppose that for some x∈X, $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$. Then for all T∈ℕ with T ≥ 3, for each t = 0, 1, 2, …, T-1, and for each i∈I(t) there exists $\beta_i^{*(T,t)} \geq 0$ satisfying the following conditions:

(G-1) The array $<\beta_i^{*(T,t)}|i\in I(t), t = 0, \ldots, T-1>$ solves:

Minimize $\sum_{i\in I(0)} \beta_i^{(T,0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-1} (\sum_{i\in I(t)} \beta_i^{(T,t)} C_i^{(t)})$ subject to $\sum_{i\in I(t-1)} \beta_i^{(T,t-1)} - \sum_{i\in I(t)} \beta_i^{(T,t)} A_i^{(t)} \geq p^{(t)}$ for all t = 1, …, T-1, $\sum_{i\in I(t-1)} \beta_i^{(T,T-1)} \geq p^{(T)}$, $\beta_i^{(T,t)} \geq 0$ for all i∈I(t) and t = 0, 1, …, T-1.

(G-2) $\sum_{i\in I(0)} \beta_i^{*(T,0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-1} (\sum_{i\in I(t)} \beta_i^{*(T,t)} C_i^{(t)}) = \sum_{t=1}^{T-1} p^{(t)}x_t$.

(G-3) $V(x) - p^{(0)}x = \sum_{t=1}^{\infty} p^{(t)}x_t = \lim_{T\to\infty} [\sum_{i\in I(0)} \beta_i^{*(T,0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-1} (\sum_{i\in I(t)} \beta_i^{*(T,t)} C_i^{(t)})]$.

Note 7.1: What theorem 7.1 says is, without any additional assumption (such as for all (x, t)∈X×ℕ⁰, {0} is a proper subset of $\Omega_t(x)$ as required in sections 5 and 6, or "interiority condition" as in section 6), the optimal value of the dual of the truncated "free end-point" linear programming problem with the initial value of the control variable in the latter being the same as the initial value of the control variable in the optimal trajectory(i.e., dual of the the linear programming problems in the "approximation result" proposition 7.3), converges to the optimal value of the DT-IHLP problem.

8. Conclusion: Part 1 of proposition 7.2 imlies the following statement: If $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$ then for all T∈ ℕ with T ≥ 3, $<x_t|t = 1, \ldots, T-1>$ solves the following linear programming problem: Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $y_1 \leq C_i^{(0)} + A_i^{(0)}x$ for all i∈I(0), $y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all i∈I(t) and t = 1, …, T-2, $-A_i^{(T-1)}y_{T-1} \leq C_i^{(T-1)} - x_T$ for all i∈I(T-1), $y_t \geq 0$ for all t = 1, …, T-1.

By the complementary slackness conditions of linear programming we know that for T∈ ℕ with T ≥ 3, <x_t|t = 1, …, T-1> solves the linear programming problem above <u>if and only if</u> there exists <$\beta_i^{*(T,t)}$|i∈I(t), t = 0, …, T-1> that along with <x_t|t = 1, …, T-1> satisfy the following conditions:

(1) $x_1 \leq C_i^{(0)} + A_i^{(0)}x$, $x_1 \geq 0$ and $(x_1 - C_i^{(0)} - A_i^{(0)}x)\beta_i^{*(T,0)} = 0$ for all i∈I(0)

(2) $x_{t+1} - A_i^{(t)}x_t \leq C_i^{(t)}$, $(x_{t+1} - A_i^{(t)}x_t - C_i^{(t)})\beta_i^{*(T,t)} = 0$ for all i∈I(t) and $x_{t+1} \geq 0$, t = 1, …, T-2.

(3) $-A_i^{(T-1)}x_{T-1} \leq C_i^{(T-1)} - x_T$ and $(-A_i^{(T-1)}x_{T-1} - C_i^{(T-1)} + x_T)\beta_i^{*(T,T-1)} = 0$ for all i∈I(T-1)

(4) $\sum_{i\in I(t-1)}\beta_i^{*(T,t-1)} - \sum_{i\in I(t)}\beta_i^{*(T,t)}A_i^{(t)} \geq p^{(t)}$, $(\sum_{i\in I(t-1)}\beta_i^{*(T,t-1)} - \sum_{i\in I(t)}\beta_i^{*(T,t)}A_i^{(t)} - p^{(t)})x_t = 0$ for all t = 1, …, T-1, $\beta_i^{*(T,t)} \geq 0$ for all i∈I(t) and t = 0, 1, …, T-1.

These four conditions are equivalent to the statement <$\beta_i^{*(T,t)}$|i∈I(t), t = 0, …, T-1> solves the dual of the linear programming problem mentioned above, i.e.,

Minimize $\sum_{i\in I(0)}\beta_i^{(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i\in I(t)}\beta_i^{(t)}C_i^{(t)}) + \sum_{i\in I(T-1)}\beta_i^{(T-1)}(C_i^{(T-1)} - x_T)$ subject to $\sum_{i\in I(t-1)}\beta_i^{(t-1)} - \sum_{i\in I(t)}\beta_i^{(t)}A_i^{(t)} \geq p^{(t)}$ for all t = 1, …, T-1, $\beta_i^{(t)} \geq 0$ for all i∈I(t) and t = 0, 1, …, T-1.

However, the four conditions in this note are also satisfied by <$\beta_i^{*(T+s,t)}$|i∈I(t), t = 0, …, T-1> for all s∈ℕ.

Thus, for all s∈ℕ, <$\beta_i^{*(T+s,t)}$|i∈I(t), t = 0, …, T-1> solves the minimization problem.

Hence, $\sum_{t=1}^{T-1}p^{(t)}x_t = \sum_{i\in I(0)}\beta_i^{*(T+s,0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i\in I(t)}\beta_i^{*(T+s,t)}C_i^{(t)}) + \sum_{i\in I(T-1)}\beta_i^{*(T+s,T-1)}(C_i^{(T-1)} - x_T)$ for all s∈ℕ.

Thus, we arrive at the following theorem.

Theorem 8.1: Let <($p^{(t)}$, $A^{(t)}$, $C^{(t)}$)|t∈$ℕ^0$> be a DT-IHLP problem and suppose that for some x∈X, <x_t|t∈ $ℕ^0$>∈ $\mathcal{S}$(x). Then for all T∈ ℕ with T ≥ 3, there exists an array <$\beta_i^{*(T,t)}$|i∈I(t), t = 0, …, T-1> such that:

(A) For all T∈ ℕ with T ≥ 3 and s∈$ℕ^0$, <$\beta_i^{*(T+s,t)}$|i∈I(t), t = 0, …, T-1> along with <x_t|t = 1, …, T-1> satisfy the following conditions:

(1) $x_1 \leq C_i^{(0)} + A_i^{(0)}x$, $x_1 \geq 0$ and $(x_1 - C_i^{(0)} - A_i^{(0)}x)\beta_i^{*(T+s,0)} = 0$ for all i∈I(0)

(2) $x_{t+1} - A_i^{(t)}x_t \leq C_i^{(t)}$, $(x_{t+1} - A_i^{(t)}x_t - C_i^{(t)})\beta_i^{*(T+s,t)} = 0$ for all i∈I(t) and $x_{t+1} \geq 0$, t = 1, …, T-2.

(3) $-A_i^{(T-1)}x_{T-1} \leq C_i^{(T-1)} - x_T$ and $(-A_i^{(T-1)}x_{T-1} - C_i^{(T-1)} + x_T)\beta_i^{*(T+s,T-1)} = 0$ for all $i \in I(T-1)$

(4) $\sum_{i \in I(t-1)} \beta_i^{*(T+s,t-1)} - \sum_{i \in I(t)} \beta_i^{*(T+s,t)} A_i^{(t)} \geq p^{(t)}$, $(\sum_{i \in I(t-1)} \beta_i^{*(T+s,t-1)} - \sum_{i \in I(t)} \beta_i^{*(T+s,t)} A_i^{(t)} - p^{(t)})x_t = 0$ for all t = 1, …, T-1, $\beta_i^{*(T+s,t)} \geq 0$ for all $i \in I(t)$ and t = 0, 1, …, T-1.

(B) (A) is equivalent to the following statement: For all $T \in \mathbb{N}$ with $T \geq 3$ and $s \in \mathbb{N}^0$, $<\beta_i^{*(T+s,t)} | i \in I(t)$, t = 0, …, T-1> solves the linear programming problem:

Minimize $\sum_{i \in I(0)} \beta_i^{(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i \in I(t)} \beta_i^{(t)} C_i^{(t)}) + \sum_{i \in I(T-1)} \beta_i^{(T-1)}(C_i^{(T-1)} - x_T)$ subject to $\sum_{i \in I(t-1)} \beta_i^{(t-1)} - \sum_{i \in I(t)} \beta_i^{(t)} A_i^{(t)} \geq p^{(t)}$ for all t = 1, …, T-1, $\beta_i^{(t)} \geq 0$ for all $i \in I(t)$ and t = 0, 1, …, T-1.

Further, $\sum_{i \in I(0)} \beta_i^{*(T+s,\, 0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i \in I(t)} \beta_i^{*(T+s,t)} C_i^{(t)}) + \sum_{i \in I(T-1)} \beta_i^{*(T+s,T-1)}(C_i^{(T-1)} - x_T) = \sum_{t=1}^{T-1} p^{(t)}x_t$.

(C) If the set of non-negative numbers $\{\beta_i^{*(T,t)} | i \in I(t)$, t = 0, …, T-1 for $T \in \mathbb{N}$, $T \geq 3\}$ is bounded above then there exists an array of non-negative real numbers $<\beta_i^{*(t)} | i \in I(t)$, $t \in \mathbb{N}^0>$ such that for all $T \in \mathbb{N}$ with $T \geq 3$, $<\beta_i^{*(t)} | i \in I(t)$, t = 0, 1, …, T-1> solve the linear programming problem in (B).

(D) If in addition to the assumptions in (C), we assume that $<\beta_i^{*(t)} | i \in I(t)$, $t \in \mathbb{N}^0>$ along with $<x_t | t \in \mathbb{N}^0>$ satisfy the "**transversality condition**" $\lim_{T \to \infty} x_T(\sum_{i \in I(T-1)} \beta_i^{*(T-1)}) = 0$, then $<\beta_i^{*(t)} | i \in I(t)$, $t \in \mathbb{N}^0>$ solves the "**implied infinite horizon dual linear programming problem**": Minimize $\sum_{i \in I(0)} \beta_i^{(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{\infty}(\sum_{i \in I(t)} \beta_i^{t)} C_i^{(t)})$ subject to $\sum_{i \in I(t-1)} \beta_i^{(t-1)} - \sum_{i \in I(t)} \beta_i^{(t)} A_i^{(t)} \geq p^{(t)}$ for all $t \in \mathbb{N}$, $\beta_i^{(t)} \geq 0$ for all $i \in I(t)$ and $t \in \mathbb{N}^0$.

Further, $V(x) - p^{(0)}x = \sum_{t=1}^{\infty} p^{(t)}x_t = \sum_{i \in I(0)} \beta_i^{*(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{\infty}(\sum_{i \in I(t)} \beta_i^{*(t)} C_i^{(t)})$.

Proof: (A) and (B) follow from the discussion preceding the statement of this theorem.

Let us prove (C).

Suppose $\{<\beta_i^{*(T,t)} | i \in I(t)$, t = 0, …, T-1 for $T \in \mathbb{N}$, $T \geq 3\}$ is bounded above. For each t = 0, …, T-1 for $T \in \mathbb{N}$, $T \geq 3$, let the array $<\beta_i^{*(T,t)} | i \in I(t)>$ be denoted by $\beta^{*(T,t)}$. Thus for

$T\epsilon\mathbb{N}$, $T \geq 3$ and $t = 0, \ldots, T-1$, $\beta^{*(T,t)}\epsilon\mathbb{R}_{+}^{I(t)}$ and for each $t\epsilon\mathbb{N}^{0}$, $<\beta^{*(T,t)}|\ T\epsilon\mathbb{N}, T \geq \max\{3, t+1\}>$ is a bounded sequence in $\mathbb{R}_{+}^{I(t)}$.

Let $t = 0$ and consider the sequence $<\beta^{*(T,0)}|\ T\epsilon\mathbb{N}, T \geq 3>$. Since it is a bounded infinite sequence in $\mathbb{R}_{+}^{I(0)}$by the Bolzano-Weirstass's theorem, there exists a convergent subsequence $<\beta^{*(R^{(0)}(n),0)}|\ n\epsilon\mathbb{N}, R^{0}(n) \geq 3>$ converging $\beta^{*(0)}\epsilon\mathbb{R}_{+}^{I(0)}$.

Consider $<\beta^{*(R^{(0)}(n),1)}|\ n\epsilon\mathbb{N}, R^{0}(n) \geq 3>$ which is a bounded infinite sequence in $\mathbb{R}_{+}^{I(1)}$.

By the Bolzano-Weirstass's theorem, this sequence has a convergent subsequence $<\beta^{*(R^{(1)}(n),1)}|\ n\epsilon\mathbb{N}, R^{1}(n) \geq 3>$ converging $\beta^{*(1)}\epsilon\mathbb{R}_{+}^{I(1)}$.

Further, $<\beta^{*(R^{(1)}(n),0)}|\ n\epsilon\mathbb{N}, R^{1}(n) \geq 3>$ is a subsequence of $<\beta^{*(R^{(0)}(n),0)}|\ n\epsilon\mathbb{N}, R^{0}(n) \geq 3>$ that converges to $\beta^{*(0)}\epsilon\mathbb{R}_{+}^{I(0)}$. Hence, $<\beta^{*(R^{(1)}(n),0)}|\ n\epsilon\mathbb{N}, R^{1}(n) \geq 3>$ converges to $\beta^{*(0)}$.

Consider $<\beta^{*(R^{(1)}(n),2)}|\ n\epsilon\mathbb{N}, R^{1}(n) \geq 3>$ which is a bounded infinite sequence in $\mathbb{R}_{+}^{I(2)}$.

By the Bolzano-Weirstass's theorem, this sequence has convergent subsequence $<\beta^{*(R^{(2)}(n),2)}|\ n\epsilon\mathbb{N}, R^{2}(n) \geq 3>$ converging to $\beta^{*(2)}\epsilon\mathbb{R}_{+}^{I(1)}$.

Further, $<\beta^{*(R^{(2)}(n),0)}|\ n\epsilon\mathbb{N}, R^{2}(n) \geq 3>$ is a subsequence of $<\beta^{*(R^{(1)}(n),0)}|\ n\epsilon\mathbb{N}, R^{1}(n) \geq 3>$ and $<\beta^{*(R^{(2)}(n),1)}|\ n\epsilon\mathbb{N}, R^{2}(n) \geq 3>$ is a subsequences of $<\beta^{*(R^{(1)}(n),1)}|\ n\epsilon\mathbb{N}, R^{1}(n)$.

Hence, the first subsequence converges to $\beta^{*(0)}$ and the second subsequence converges to $\beta^{*(1)}$.

Suppose that for $t\epsilon\mathbb{N}$, with $t \geq 2$, there are bounded subsequences $<\beta^{*(R^{(t)}(n),s)}|\ n\epsilon\mathbb{N}, R^{t}(n) \geq t + 1>$ in $\mathbb{R}_{+}^{I(s)}$, $s = 0, 1, 2, \ldots, t$, such that $<\beta^{*(R^{(t)}(n),s)}|\ T\epsilon\mathbb{N}, R^{t}(n) \geq t + 1>$ converges to $\beta^{*(s)}\epsilon\ \mathbb{R}_{+}^{I(s)}$, $s = 0, 1, 2, \ldots, t$.

Consider the subsequence $<\beta^{*(R^{(t)}(n),t+1)}|T\epsilon\mathbb{N}, R^{t}(n) \geq t + 1>$ in $\mathbb{R}_{+}^{I(t+1)}$. Since it is a bounded sequence, by the Bolzano-Weirstass's theorem it has a convergent subsequence $<\beta^{*(R^{(t+1)}(n),t+1)}|\ n\epsilon\mathbb{N}, R^{t+1}(n) \geq t + 2>$ in $\mathbb{R}_{+}^{I(t+1)}$ converging to $\beta^{*(t+1)}\epsilon\ \mathbb{R}_{+}^{I(t)}$.

Since for $s = 0, 1, 2, ..., t$, $<\beta^{*(R^{(t+1)}(n),s)}|\ T\epsilon\mathbb{N}, R^{t+1}(n) \geq t + 2>$ is a subsequnce of $<\beta^{*(R^{(t)}(n),s)}|\ n\epsilon\mathbb{N}, R^{t}(n) \geq t + 1>$, it follows that $<\beta^{*(R^{(t+1)}(n),s)}|\ T\epsilon\mathbb{N}, R^{t}(n) \geq t + 2>$ converges to $\beta^{*(s)}$.

Hence, by a standard induction argument it follows that for all $t \in \mathbb{N}^0$ and s = 0, 1, 2, …, t, there exists a subsequence $<\beta^{*(R^{(t)}(n),s)} | n \in \mathbb{N}>$ of $<\beta^{*(T,s)} | T \in \mathbb{N}>$ such that $<\beta^{*(R^{(t)}(n),s)} | n \in \mathbb{N}>$ converges to $\beta^{*(s)} \in \mathbb{R}_{+}^{I(s)}$.

Note that for all $T \in \mathbb{N}$ with $T \geq 3$, $\beta^{*(T-1)}$ gets defined by the subsequence $<\beta^{*(R^{(T-1)}(n),T-1)} | n \in \mathbb{N}, R^{(T-1)}(n) \geq T>$ of the sequence $<\beta^{*(T-1+n,T-1)} | n \in \mathbb{N}>$.

We know that, for all $T \in \mathbb{N}$ with $T \geq 3$ and $n \in \mathbb{N}$, $<\beta_i^{*(R^{(T-1)}(n),t)} | i \in I(t), t = 0, ..., T-1>$ satisfying $R^{(T-1)}(n) \geq T$ for all $n \in \mathbb{N}$ solves the linear programming problem: Minimize $\sum_{i \in I(0)} \beta_i^{(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i \in I(t)} \beta_i^{(t)} C_i^{(t)}) + \sum_{i \in I(T-1)} \beta_i^{(T-1)}(C_i^{(T-1)} - x_T)$ subject to $\sum_{i \in I(t-1)} \beta_i^{(t-1)} - \sum_{i \in I(t)} \beta_i^{(t)} A_i^{(t)} \geq p^{(t)}$ for all t = 1, …, T-1, $\beta_i^{(t)} \geq 0$ for all $i \in I(t)$ and t = 0, 1, …, T-1.

Further, $\sum_{t=1}^{T-1} p^{(t)} x_t = \sum_{i \in I(0)} \beta_i^{*(R^{(T-1)}(n),0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i \in I(t)} \beta_i^{*(R^{(T-1)}(n),t)} C_i^{(t)}) + \sum_{i \in I(T-1)} \beta_i^{*(R^{(T-1)}(n),T-1)}(C_i^{(T-1)} - x_T)$ for all $n \in \mathbb{N}$.

Since for all t = 0, 1, 2, …, T-1, $\lim_{n \to \infty} \beta^{*(R^{(T-1)}(n),t)} = \beta^{*(t)}$, it follows that $<\beta_i^{*(t)} | i \in I(t), t = 0, ..., T-1>$ satisfies all the constraints of the linear programming minimization problem in (B) and $\sum_{t=1}^{T-1} p^{(t)} x_t = \sum_{i \in I(0)} \beta_i^{*(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i \in I(t)} \beta_i^{*(t)} C_i^{(t)}) + \sum_{i \in I(T-1)} \beta_i^{*(T-1)}(C_i^{(T-1)} - x_T)$.

Thus, for all $T \in \mathbb{N}$ with $T \geq 3$, $<\beta_i^{*(t)} | i \in I(t), t = 0, ..., T-1>$ solves the linear programming problem in (B). This proves (C).

To prove (D), first note that since for all $T \in \mathbb{N}$, $T \geq 3$, $<\beta_i^{*(t)} | i \in I(t), t = 0, ..., T-1>$ satisfies all the constraints of the linear programming minimization problem in (B), it must be the case that for all $T \in \mathbb{N}$ with $T \geq 3$, $\sum_{t=1}^{T-1} p^{(t)} x_t = \sum_{i \in I(0)} \beta_i^{*(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i \in I(t)} \beta_i^{*(t)} C_i^{(t)}) + \sum_{i \in I(T-1)} \beta_i^{*(T-1)}(C_i^{(T-1)} - x_T)$, and $<\beta_i^{*(t)} | i \in I(t), t \in \mathbb{N}^0>$ *satisfies all the constraints of the implied infinite horizon dual linear programming problem.*

We know that $\lim_{T \to \infty} \sum_{t=1}^{T-1} p^{(t)} x_t = \sum_{t=1}^{\infty} p^{(t)} x_t = V(x) - p^{(0)}x$.

Thus, $\lim_{T \to \infty} [\sum_{i \in I(0)} \beta_i^{*(0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2}(\sum_{i \in I(t)} \beta_i^{*(t)} C_i^{(t)}) + \sum_{i \in I(T-1)} \beta_i^{*(T-1)}(C_i^{(T-1)} - x_T)]$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)} x_t = V(x) - p^{(0)}x$.

However, $\lim_{T\to\infty} [\sum_{i\in I(0)} \beta_i^{*(0)}(\mathrm{C}_\mathrm{i}^{(0)} + \mathrm{A}_\mathrm{i}^{(0)}\mathrm{x}) + \sum_{t=1}^{T-2}(\sum_{i\in I(t)} \beta_i^{*(t)}\mathrm{C}_\mathrm{i}^{(t)}) +$

$\sum_{i\in I(T-1)} \beta_i^{*(T-1)}(\mathrm{C}_\mathrm{i}^{(T-1)} - \mathrm{x}_\mathrm{T})] = \lim_{T\to\infty} [\sum_{i\in I(0)} \beta_i^{*(0)}(\mathrm{C}_\mathrm{i}^{(0)} + \mathrm{A}_\mathrm{i}^{(0)}\mathrm{x}) +$

$\sum_{t=1}^{T-1}(\sum_{i\in I(t)} \beta_i^{*(t)}\mathrm{C}_\mathrm{i}^{(t)}) - \mathrm{x}_\mathrm{T}(\sum_{i\in I(T-1)} \beta_i^{*(T-1)})]$.

Thus, if $\lim_{T\to\infty} \mathrm{x}_\mathrm{T}(\sum_{i\in I(T-1)} \beta_i^{*(T-1)}) = 0$, then $\mathrm{V(x)} - \mathrm{p}^{(0)}\mathrm{x} = \sum_{t=1}^{\infty} \mathrm{p}^{(t)}\mathrm{x}_\mathrm{t} =$

$\lim_{T\to\infty} [\sum_{i\in I(0)} \beta_i^{*(0)}(\mathrm{C}_\mathrm{i}^{(0)} + \mathrm{A}_\mathrm{i}^{(0)}\mathrm{x}) + \sum_{t=1}^{T-1}(\sum_{i\in I(t)} \beta_i^{*(t)}\mathrm{C}_\mathrm{i}^{(t)})]$.

Further, since for all $T \in \mathbb{N}$ with $T \geq 3$, $<\beta_i^{*(t)} | i \in I(t), t = 0, ..., T-1>$ solves the linear programming problem in (B), under the assumption $\lim_{T\to\infty} \mathrm{x}_\mathrm{T}(\sum_{i\in I(T-1)} \beta_i^{*(T-1)}) = 0$, $<\beta_i^{*(t)} | i \in I(t), t \in \mathbb{N}^0>$ solves the infinite horizon linear programming minimization problem in (D). Q.E.D.

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