

Infinite Horizon Linear Programming With One-Dimensional State Variable

by

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Abstract

We provide a set of sufficient conditions for the optimal value of an infinite horizon linear programming problem with one-dimensional state variable to be equal to the optimal value of its implied infinite horizon dual linear programming problem. Our sufficient conditions require that in every period the linear inequality determining the constraint for the state variable for the next period is expressed in terms of a “non-constant” function of the current value of the state variable, the state variable along the optimal trajectory is always strictly positive beginning with time period one and is strictly less than its upper-bound in “at least one” time period. In addition, the transversality condition we invoke is that the product in each period, of the state variable and the co-state or dual variable for the inequality constraining the state variable in that period, converges to zero.

Keywords: infinite horizon linear programming, linear constraints, duality, infinite horizon dual linear programming problem, transversality condition

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JEL Codes: C44, C61.

1. Introduction: At the outset, let us make it clear that we are concerned with infinite “horizon” linear programming problems and not infinite linear programming problems, the latter embracing a much larger class of problems than what the former does. Hence, a counter example applicable for infinite linear programming problems need not be applicable for infinite horizon linear programming problems.

The study of infinite horizon linear programming problems originates in Hopkins (1969). The model discussed in Hopkins (1969) allows the chosen variables at any time period, to be constrained by choices made in all previous time periods. This model is discussed further in Grinold (1971). Among the earliest work on infinite horizon linear programming problems that restricts the chosen variable at any time period by just the chosen variables at the immediately previous period- and no further back in time- is the work in Evers (1973). The model in Evers (1973) assumes that the

parameters determining the constraints for the choice variables are invariant over time. Some additional work based on the model developed by Evers (1973) is available in Grinold (1977). Romeijn, Smith and Bean (1992) generalize the model in Evers (1973) and allow these parameters to vary with the time period. All works cited so far have one feature in common, i.e., they allow the state variables or chosen alternatives to be points in a finite dimensional Euclidean space, where the dimension could be greater than one.

There is a considerable body of literature on infinite horizon dynamic optimization with one dimensional state variables that is discussed in depth in Mitra (2000). The model discussed in Mitra (2000) allows for non-linear objective functions that are separable over time periods and constraint sets that allow for non-linearity. This model, referred to in Mitra (2000) as the reduced form model, has been studied with linear objective functions in Lahiri (2025a). In Lahiri (2025b), we discuss infinite horizon dynamic optimization with one dimensional state variables and linear objective function, where the constraint set in each period is determined by exactly one linear inequality. The model in Lahiri (2025b) is the framework of analysis of this paper.

The purpose of Lahiri (2025a, 2025b) was to sharpen the results about the reduced form model in Mitra (2000) in the contexts of the two former papers respectively. Thus issues concerning Euler equations and transversality conditions are discussed in Lahiri (2025b) and there is no attempt made in that paper to relate the model to linear programming.

In Lahiri (2025c), there is a model of infinite horizon optimal control with linear objective functions and with linear equations and linear inequalities constraining the evolution of a one-dimensional state variable and a one-dimensional control variable. That paper deals with the problem concerning “duality gap” that Grinold (1971) and Romeijn, Smith and Bean (1992) are concerned with. However, the model discussed in Lahiri (2025c) is conceptually different from the other works we have mentioned so far, since in Lahiri (2025c) there is a control variable as well as a state variable each evolving in its own way and both contribute to the objective function. Duality gap is about the possibility of the optimal value of an infinite horizon linear programming problem being different from the optimal value of its infinite horizon dual linear programming problem. In linear programming, primal and dual linear programming problems are defined in the context of variables in a finite dimensional

Euclidean space. Hence, we refer to the dual linear programming in the infinite horizon context as “implied infinite horizon dual”.

It needs to be noted that the model discussed in Romeijn, Smith and Bean (1992) is more general than the model discussed in Lahiri (2025b) and hence the one discussed here. The major contribution of Romeijn, Smith and Bean (1992) is a set of sufficient conditions that includes a transversality condition under which the optimal value of the infinite horizon linear programming problem is equal to the optimal value of its implied infinite horizon dual linear programming problem.

In this paper, our main result is a set of sufficient conditions that also includes a transversality condition under which the optimal value of the infinite horizon linear programming problem is equal to the optimal value of its implied infinite horizon dual linear programming problem. Our sufficient conditions require that in every period the linear inequality determining the constraint for the state variable for the next period is expressed in terms of a “non-constant” function of the current value of the state variable, the state variable along the optimal trajectory is always strictly positive beginning with time period one and is strictly less than its upper-bound in “at least one” time period. In addition the transversality condition we invoke is that the product in each period, of the state variable and the co-state or dual variable for the inequality constraining the state variable in that period, converges to zero. Our sufficient conditions are considerably simpler than the ones required in Romeijn, Smith and Bean (1992). There does not seem to be any connection between our simplicity and the fact that unlike Romeijn, Smith and Bean (1992), we focus our attention on one-dimensional state variables. In fact, our sufficient conditions here are also less restrictive than the sufficient conditions invoked in Lahiri (2025c) to address the duality gap problem.

The requirement that the state variable is always strictly positive beginning with the first period, leads to a difference equation for the evolution of the dual variable. The requirement that the state variable is strictly less than its upper-bound in “at least one” time period implies that the corresponding value of the co-state variable is zero. This leads to a unique optimal solution of the implied infinite horizon dual linear programming problem. If the objective function is a discounted sum of the instantaneous values of the state variables and the slopes of the function determining the upper bounds of the state variable converges to a real number that is not equal to one, then from the difference equation governing the evolution of the dual variables it

follows that the transversality condition must be satisfied, i.e., the transversality condition as we have defined here is a necessary condition for optimality.

2. The Framework of Analysis: Our framework of analysis presented below is “almost” identical to the one in Lahiri (2025b).

Let \mathbb{R} denote the set of real numbers, \mathbb{R}_+ the set of non-negative real numbers and \mathbb{R}_{++} the set of strictly positive real numbers. Let \mathbb{N} denote the set of natural numbers and $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$.

Let $X = [0, b] \subset \mathbb{R}_+$, with $b \in \mathbb{R}_{++}$ denote the set of available alternatives. Time is measured in discrete periods $t \in \mathbb{N}^0$. At each time period ‘t’ an alternative is chosen, and the chosen alternative is denoted by $x_t \in X$.

Let $\langle c^{(t)} | t \in \mathbb{N}^0 \rangle$ be a sequence in \mathbb{R}_+ and let $\langle a^{(t)} | t \in \mathbb{N}^0 \rangle$ be a sequence in \mathbb{R} such that $c^{(t)} \in [0, b]$ and $c^{(t)} + a^{(t)}b \in [0, b]$ for all $t \in \mathbb{N}^0$.

Thus, $\langle c^{(t)} | t \in \mathbb{N}^0 \rangle$ is a bounded sequence in \mathbb{R}_+ .

Note 2.1: $\langle c^{(t)} | t \in \mathbb{N}^0 \rangle$ is a bounded sequence in \mathbb{R}_+ combined with $b > 0$, $-c^{(t)} \leq a^{(t)}b \leq b - c^{(t)}$ for all $t \in \mathbb{N}^0$ implies $\inf\{-\frac{c^{(t)}}{b} | t \in \mathbb{N}^0\} \leq a^{(t)} \leq \sup\{1 - \frac{c^{(t)}}{b} | t \in \mathbb{N}^0\}$, where both $\liminf\{-\frac{c^{(t)}}{b} | t \in \mathbb{N}^0\} > -\infty$ and $\limsup\{1 - \frac{c^{(t)}}{b} | t \in \mathbb{N}^0\} < +\infty$.

Thus, $\langle a^{(t)} | t \in \mathbb{N}^0 \rangle$ must also be a bounded sequence.

Since for all $x \in [0, b]$, $x = \frac{x}{b}b + (1 - \frac{x}{b})0$ with $\frac{x}{b} \in [0, 1]$ it must be the case that $c^{(t)} + a^{(t)}x \in X = [0, b]$ for all $(x, t) \in X \times \mathbb{N}^0$.

For $t \in \mathbb{N}^0$, let $\Omega_t = \{(x, y) \in X \times X | y \leq c^{(t)} + a^{(t)}x\} = \{(x, y) | 0 \leq y \leq c^{(t)} + a^{(t)}x, x \in [0, b]\}$.

Is it easy to see that for all $t \in \mathbb{N}^0$, Ω_t is a trapezium with its extreme points being $(0, 0)$, $(0, c^{(t)})$, $(b, c^{(t)} + a^{(t)}b)$, $(b, 0)$.

For $t \in \mathbb{N}^0$, Ω_t is **the two-period linearly constrained set at (time-period) t**.

For $(x, t) \in X \times \mathbb{N}^0$, let $\Omega_t(x) = \{y | 0 \leq y \leq c^{(t)} + a^{(t)}x\} = [0, c^{(t)} + a^{(t)}x]$.

Note that for all $t \in \mathbb{N}^0$, Ω_t is a non-empty, closed and bounded subset of $X \times X$ and for each $(x, t) \in X \times \mathbb{N}^0$, the set $\Omega_t(x)$ is a non-empty, closed and bounded interval in X , though the interval $\Omega_t(x)$ may be a singleton (i.e., degenerate) as for instance when $a^{(t)} = 0$.

For $(x, t) \in X \times \mathbb{N}^0$, the set $\Omega_t(x)$ is said to be **the transition set from x at (time-period) t**.

For $x \in X$, let $\mathcal{F}(x) = \{\langle x_t | t \in \mathbb{N}^0 \rangle | x_{t+1} \leq c^{(t)} + a^{(t)}x_t \text{ for all } t \in \mathbb{N}^0, x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$ as a **trajectory starting at (from) x** .

Let $\langle p^{(t)} | t \in \mathbb{N}^0 \rangle$ be a sequence in \mathbb{R} satisfying $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$.

Note 2.2: $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$ implies $\lim_{t \rightarrow \infty} |p^{(t)}| = 0$ and for all sequences $\langle x_t | t \in \mathbb{N}^0 \rangle$, it must be the case that $|\sum_{t=0}^{\infty} p^{(t)} x_t| \leq \sum_{t=0}^{\infty} |p^{(t)}| |x_t| < +\infty$.

We shall refer to the sequence $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ as a **one variable infinite horizon linear programming (1-IHLP) problem** and represent it as $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$.

Given a 1-IHLP problem $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ we shall be concerned with the following optimization problem denoted by **P1(x)**:

Given $x \in X$: Maximize $\sum_{t=0}^{\infty} p^{(t)} x_t$, subject to $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$.

If for some $c \in \mathbb{R}_+$ and $a \in \mathbb{R}$ the 1-IHLP $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ satisfies $c^{(t)} = c$ and $a^{(t)} = a$ for all $t \in \mathbb{N}^0$, then we may refer to the 1-IHLP problem as a **quasi time-invariant 1-IHLP problem**.

We will denote a quasi time-invariant 1-IHLP problem such as above by $\langle (p^{(t)} | t \in \mathbb{N}^0), c, a \rangle$.

Example 2.1: The discounted 1-IHLP problem: There exists $\delta \in (0, 1)$ such that for all $t \in \mathbb{N}^0$, $p^{(t)} = \delta^t$.

Note 2.3: If a “discounted 1-IHLP problem”, is at the same time a quasi time-invariant 1-IHLP such that there exists a real numbers $c \in [0, b]$ and a real number a such that $c + ab \in [0, b]$ satisfying $c^{(t)} = c$ and $a^{(t)} = a$ for all $t \in \mathbb{N}^0$, then we have a model that closely resembles a one-dimensional control variable version of the infinite horizon linear programming discussed in Grinold (1977). We may refer to such a problem as a **time-invariant 1-IHLP problem** and denote it by (δ, c, a) .

3. Some Preliminary results about trajectories: Given a 1-IHLP problem $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ for each $x \in X$, let $\mathcal{S}(x) = \operatorname{argmax}_{\langle x_t | t \in \mathbb{N}^0 \rangle} \sum_{t=0}^{\infty} p^{(t)} x_t$. $\mathcal{S}(x)$ is the **set of solutions for**

P1(x) starting from x (for the 1-IHLP problem).

The following result is proposition 4 in Lahiri (2025b).

Proposition 3.1: For all $x \in X$, $\mathcal{S}(x) \neq \emptyset$.

An immediate consequence of proposition 1, is that there exists a function $V: X \rightarrow \mathbb{R}$ such that for all $x \in X$: $V(x) = \sum_{t=0}^{\infty} p^{(t)} x_t$, where $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$.

In note 2.3 we have defined a time-invariant 1-IHLP problem, It is well known that for a time invariant 1-IHLP problem (δ, c, a) for all $x \in X$: $V(x) = x + \delta \max_{y \in [0, c+ax]} V(y)$.

This equation is known as Bellman's fundamental equation of dynamic programming for the 1-IHLP problem.

For more on the fundamental equation of dynamic programming one may refer to Mitra (2000).

The following proposition provides a "solution procedure" for a large class of 1-IHLP problems.

Proposition 3.2: Suppose $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ is a 1-IHLP problem satisfying $p^{(t)} \geq 0$ for all $t \in \mathbb{N}$ and $a^{(t)} \geq 0$ for all $t \in \mathbb{N}^0$. Let $x \in X$ and let $\langle x_t | t \in \mathbb{N}^0 \rangle$ be the sequence with $x_0 = x$, and $x_{t+1} = c^{(t)} + a^{(t)}x_t$ for all $t \in \mathbb{N}^0$. Then, $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$.

Further, if $\langle y_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$, then for all $T \in \{t \in \mathbb{N} | p^{(t)} > 0\}$ it must be the case that $y_T = x_T$.

Proof: Clearly $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$. Let $\langle y_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$. Clearly $y_1 \in [0, c^{(0)} + a^{(0)}x] = [0, x_1]$ and hence $y_1 \leq x_1$. Suppose that for some $T \in \mathbb{N}$ it is the case that $y_T \leq x_T$. Since, $a^{(T)} \geq 0$, $y_{T+1} \in [0, c^{(T)} + a^{(T)}y_T] \subset [0, c^{(T)} + a^{(T)}x_T] = [0, x_{T+1}]$, i.e., $y_{T+1} \leq x_{T+1}$. Thus, $y_T \leq x_T$ implies $y_{T+t} \leq x_{T+t}$ for all $t \in \mathbb{N}^0$.

Since $y_1 \leq x_1$, it must be the case that $y_t \leq x_t$ for all $t \in \mathbb{N}$.

Thus, $\sum_{t=0}^{\infty} p^{(t)}x_t = p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}x_t \geq p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}y_t$, since $p^{(t)} \geq 0$ for all $t \in \mathbb{N}$.

However, $p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}y_t = \sum_{t=0}^{\infty} p^{(t)}y_t$.

Thus, $\sum_{t=0}^{\infty} p^{(t)}x_t \geq \sum_{t=0}^{\infty} p^{(t)}y_t$ and so $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$.

If $\langle y_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$ then $\langle y_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$. Thus, as observed earlier in this proof, $y_0 = x_0 = x$ and $y_t \leq x_t$ for all $t \in \mathbb{N}$.

Thus, if for some $T \in \mathbb{N}$, $p^{(T)} > 0$ and $y_T < x_T$, then $\sum_{t=0}^{\infty} p^{(t)}x_t = \sum_{t=0}^{T-1} p^{(t)}x_t + p^{(T)}x_T + \sum_{t=T+1}^{\infty} p^{(t)}x_t > \sum_{t=0}^{T-1} p^{(t)}x_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}x_t \geq \sum_{t=0}^{T-1} p^{(t)}y_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}y_t$, since $p^{(t)} \geq 0$ and $x_t \geq y_t$ for all $t \in \mathbb{N}$.

However, $\sum_{t=0}^{T-1} p^{(t)}y_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}y_t = \sum_{t=0}^{\infty} p^{(t)}y_t$ and hence $\sum_{t=0}^{\infty} p^{(t)}x_t > \sum_{t=0}^{\infty} p^{(t)}y_t$.

This, contradicts our assumption that $\langle y_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$ and proves the proposition.

Q.E.D.

4. Optimality and associated linear programming problems: In this section we replicate for our current context the main result in section 3 of Lahiri (2025c) the latter being a discussion on infinite horizon linear optimal control problems with linear constraints.

Note 4.1: The exact mathematical interpretation of the expression (formula)

$\sum_{t=0}^{\infty} p^{(t)}x_t$ is $\lim_{T \rightarrow \infty} (\sum_{t=0}^T p^{(t)}x_t)$. Thus, the problem we are concerned with here is in the domain of asymptotic analysis, which is very different from infinite dimensional analysis.

An **alternative version of P1(x)** is the following optimization problem:

Given $x \in X$, Maximize $\sum_{t=1}^{\infty} p^{(t)}x_t$ subject to the infinite sequence $\langle x_t | t \in \mathbb{N}^0 \rangle$

satisfying the constraints: $(x_t, x_{t+1}) \in \Omega_t$, $t \in \mathbb{N}^0$, $x_0 = x$.

We now provide one necessary condition and a somewhat stronger sufficient condition for optimality for the 1-IHLP problem in terms of linear programming problems.

Proposition 4.1: Let $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ be a 1-IHLP problem and suppose that for some $x \in X$, $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$.

Part 1: If $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$ then for all $T \in \mathbb{N}$, $\langle x_t | t = 0, 1, \dots, T \rangle$ solves the following linear programming problem: Maximize $\sum_{t=0}^T p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all $t = 0, 1, \dots, T-1$, $y_0 = x_0 = x$ and $y_T = x_T$.

Part 2: If there exists $T^* \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ satisfying $T \geq T^*$, $\langle x_t | t = 0, 1, \dots, T \rangle$ solves the linear programming problem: Maximize $\sum_{t=0}^T p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all $t = 0, 1, \dots, T-1$ and $y_0 = x_0 = x$, then $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$.

Proof: The proof is similar to the proof of proposition 3.1 in Lahiri (2025c). Q.E.D.

We now provide an approximation result that follows from proposition 3.1, note 4.1 and part 1 of proposition 4.1.

Proposition 4.2 (Approximation Result): Given the 1-IHLP problem $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ and $x \in X$, there exists $T^*(\epsilon) \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ with $T \geq T^*$, the linear programming problem [Maximize $\sum_{t=0}^T p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all $t = 0, 1, \dots, T-1$ and $y_0 = x_0 = x$] has a solution $\langle x_t^{(T)} | t = 0, 1, \dots, T \rangle$ and $|\sum_{t=0}^T p^{(t)}x_t^{(T)} - V(x)| < \epsilon$.

Proof: Since $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$ and $\mathcal{F}(x) \neq \emptyset$, for all $T \in \mathbb{N}$ the linear programming problem [Maximize $\sum_{t=0}^T p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all $t = 0, 1, \dots, T-1$ and $y_0 = x_0 = x$] has a solution $\langle x_t^{(T)} | t = 0, 1, \dots, T \rangle$.

Towards a contradiction suppose there exists $\varepsilon > 0$ such that $|\sum_{t=0}^T p^{(t)} x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often.

Then, it must be the case that either $\sum_{t=0}^T p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ infinitely often or $\sum_{t=0}^T p^{(t)} x_t^{(T)} \leq V(x) - \varepsilon$ infinitely often.

Let $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$. Since $\langle x_t | t = 0, 1, \dots, T \rangle$ satisfies the constraints of the linear programming problem for all $T \in \mathbb{N}$, it must be the case that $\sum_{t=0}^T p^{(t)} x_t^{(T)} \geq \sum_{t=0}^T p^{(t)} x_t$ for all $T \in \mathbb{N}$.

Since, $V(x) = \sum_{t=0}^{\infty} p^{(t)} x_t = \lim_{T \rightarrow \infty} \sum_{t=0}^T p^{(t)} x_t$, there exists $T^0 \in \mathbb{N}$ such that for all $T \in \mathbb{N}$

with $T \geq T^0$ it is the case that $\sum_{t=0}^T p^{(t)} x_t + \frac{\varepsilon}{4} > V(x) > \sum_{t=0}^T p^{(t)} x_t - \frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^T p^{(t)} x_t^{(T)} + \frac{\varepsilon}{4} \geq \sum_{t=0}^T p^{(t)} x_t + \frac{\varepsilon}{4} > V(x)$ for all $T \in \mathbb{N}$ with $T \geq T^0$.

Thus, $\sum_{t=0}^T p^{(t)} x_t^{(T)} + \varepsilon > \sum_{t=0}^T p^{(t)} x_t^{(T)} + \frac{\varepsilon}{4} > V(x)$ for all $T \in \mathbb{N}$ with $T \geq T^0$.

Thus, $|\sum_{t=0}^T p^{(t)} x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often is incompatible with $V(x) \geq \sum_{t=0}^T p^{(t)} x_t^{(T)} + \varepsilon$ infinitely often.

Thus, $|\sum_{t=0}^T p^{(t)} x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often implies that there exists $T^1 \in \mathbb{N}$ such that $\sum_{t=0}^T p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ for all $T \in \mathbb{N}$ satisfying $T \geq T^1$.

Since, $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$, there exists $T^2 \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ satisfying $T \geq T^2$,

$$b \sum_{t=T}^{\infty} |p^{(t)}| < \frac{\varepsilon}{4}.$$

$\sum_{t=0}^T p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ for all $T \in \mathbb{N}$ satisfying $T \geq T^1$ implies $\sum_{t=0}^T p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$ for all $T \in \mathbb{N}$ satisfying $T \geq \max\{T^1, T^2\}$ infinitely often.

Let $T \in \mathbb{N}$ be such that $T \geq \max\{T^1, T^2\}$.

Thus, $\sum_{t=0}^T p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon$.

Let $x_{T+t}^{(T)} = c^{(T+t-1)} + a^{(T+t-1)} x_{T+t-1}^{(T)}$ for all $t \in \mathbb{N}$.

Thus, $\langle x_t^{(T)} | t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$.

Now $|\sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)}| \leq \sum_{t=T+1}^{\infty} |p^{(t)} x_t^{(T)}| \leq b \sum_{t=T+1}^{\infty} |p^{(t)}| < \frac{\varepsilon}{4}$.

Thus, $\frac{\varepsilon}{4} > \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} > -\frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{\infty} p^{(t)} x_t^{(T)} = \sum_{t=0}^T p^{(t)} x_t^{(T)} + \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon + \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} > V(x) + \varepsilon - \frac{\varepsilon}{4} = V(x) + \frac{3\varepsilon}{4} > V(x)$.

This, contradicts the definition of $V(x)$ and proves the proposition. Q.E.D.

5. Duality Theory for 1-IHLP problem and “necessary” conditions for optimality:

Given the 1-IHLP $\langle p^{(t)}, c^{(t)}, a^{(t)} \mid t \in \mathbb{N}^0 \rangle$ suppose that for all $t \in \mathbb{N}^0$, $c^{(t)}, c^{(t)} + a^{(t)}b > 0$, so that for all $(x, t) \in X \times \mathbb{N}^0$, $c^{(t)} + a^{(t)}x > 0$.

Given $x \in X$, let $\langle x_t \mid t \in \mathbb{N}^0 \rangle \in \mathcal{F}(x)$.

For $T \in \mathbb{N}$ with $T \geq 3$ consider the linear programming problem in part 1 of proposition 4.1.

Maximize $\sum_{t=0}^T p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all $t = 0, 1, \dots, T-1$, $y_0 = x_0 = x$, $y_t \geq 0$ for all $t = 0, 1, \dots, T$, & $y_T = x_T$.

Since $\sum_{t=0}^T p^{(t)}y_t = p^{(0)}x + \sum_{t=1}^T p^{(t)}y_t$ along with $y_0 = x_0 = x$ and $y_T = x_T$, the linear programming problem in part 1 of proposition 4.1 reduces to the following.

Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t \leq c^{(t)}$ for all $t = 1, \dots, T-2$, $-a^{(T-1)}y_{T-1} \leq c^{(T-1)} - x_T$, $y_t \geq 0$ for all $t = 1, \dots, T-1$.

The dual of this linear programming problem is the following linear programming problem.

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \dots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \dots, T-1$.

Hence we have the following proposition.

Proposition 5.1: Suppose $\langle p^{(t)}, c^{(t)}, a^{(t)} \mid t \in \mathbb{N}^0 \rangle$ is a 1-IHLP satisfying $c^{(t)}$ and $c^{(t)} + a^{(t)}b > 0$ for all $t \in \mathbb{N}^0$. Suppose that for some $x \in X$, $\langle x_t \mid t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$. Then, for all $T \in \mathbb{N}$ with $T \geq 3$, there exists $\langle \alpha_t^{*(T)} \mid \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ such that:

(A) $\langle \alpha_t^{*(T)} \mid \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ solves the linear programming problem

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \dots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \dots, T-1$.

(B) $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^{*(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)$.

(C) $\lim_{T \rightarrow \infty} [\alpha_0^{*(T)}(c^{(0)} + a^{(0)})x + \sum_{t=1}^{T-2} \alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)]$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$.

Proof: From the discussion in this section preceding the statement of this proposition, we know if $\langle x_t \mid t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$, then that for all $T \in \mathbb{N}$ with $T \geq 3$, $\langle x_t \mid t = 0, \dots, T \rangle$ solves:

Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t \leq c^{(t)}$ for all $t = 1, \dots, T-2$, $-a^{(T-1)}y_{T-1} \leq c^{(T-1)} - x_T$, $y_t \geq 0$ for all $t = 1, \dots, T-1$.

By the strong duality theorem of linear programming (see topic 2 of Lahiri (2020)) we know that $\langle x_t | t = 0, \dots, T \rangle$ solves the above problem if and only if its dual has a solution, in which case the optimal value of the maximization problem and the optimal value of its dual are equal. The dual of the linear programming maximization problem is the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \dots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \dots, T-1$.

For $T \in \mathbb{N}$ with $T \geq 3$, let $\langle \alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ be such a solution.

Thus, for $T \in \mathbb{N}$ with $T \geq 3$, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^{*(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T) - x_T$.

Since, $\lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} p^{(t)}x_t$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$, it must be the case that

$\lim_{T \rightarrow \infty} [\alpha_0^{*(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)]$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$. Q.E.D.

An immediate corollary of proposition 5.1 is the following result.

Corollary of Proposition 5.1: Suppose $\langle (p^{(t)}, c^{(t)}, a^{(t)}) | t \in \mathbb{N}^0 \rangle$ is a 1-IHLP satisfying $c^{(t)}$ and $c^{(t)} + a^{(t)}b > 0$ for all $t \in \mathbb{N}^0$. Suppose that for some $x \in X$, $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$. Then, for all $T \in \mathbb{N}$ with $T \geq 3$, there exists $\langle \alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ such that:

- (1) $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} \geq p^{(t)}$ and $(\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} - p^{(t)})x_t = 0$ for all $t = 1, \dots, T-1$;
- (2) $(x_1 - c^{(0)} - a^{(0)}x)\alpha_0^{*(T)} = 0$, $(x_{t+1} - a^{(t)}x_t - c^{(t)})\alpha_t^{*(T)} = 0$ for all $t = 1, \dots, T-2$, $(-a^{(T-1)}x_{T-1} - c^{(T-1)} + x_T)\alpha_{T-1}^{*(T)} = 0$.

Proof: By proposition 5.1 we know that for all $T \in \mathbb{N}$ with $T \geq 3$, there exists $\langle \alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ such that:

(A) $\langle \alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ solves the linear programming problem Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \dots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \dots, T-1$.

From the complementary slackness condition (see topic 2 of Lahiri (2020)) we know that since $\langle x_t | t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x) \subset \mathcal{F}(x)$, $\langle \alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ along with $\langle x_t | t = 0, \dots, T \rangle$ satisfy (1) and (2). Q.E.D.

6. Interiority condition and infinite horizon dual linear programming problem:

In this section we consider 1-IHLP problems for which, in addition to the conditions assumed in section 5 (i.e., $c^{(t)} > 0$ and $c^{(t)} + a^{(t)}b > 0$ for all $t \in \mathbb{N}^0$), it is the case that for $t \in \mathbb{N}$, $a^{(t)} \neq 0$.

For our main result in this section we assume that along the optimal trajectory the state variable is always strictly positive, as for instance is the case with the solution in the statement of proposition 3.2. We also assume that in some time period $t^* \in \mathbb{N}$, the chosen value of the state variable is less than the upper bound for the possible set of values for the state variable in period t^* . We refer to this condition on the optimal trajectory as “interiority condition”.

Proposition 6.1: Suppose $\langle p^{(t)}, c^{(t)}, a^{(t)} \mid t \in \mathbb{N}^0 \rangle$ is a 1-IHLP satisfying $c^{(t)}, c^{(t)} + a^{(t)}b > 0$, for all $t \in \mathbb{N}^0$ and $a^{(t)} \neq 0$ for all $t \in \mathbb{N}$. Suppose that for some $x \in X$, $\langle x_t \mid t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$, satisfies the following “**interiority condition**”: For all $t \in \mathbb{N}$, $x_t > 0$ and there exists $t^* \in \mathbb{N}$ such that $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$.

Then, there exists a sequence $\langle \alpha_t^* \mid \alpha_t^* \geq 0, t \in \mathbb{N}^0 \rangle$ satisfying $\alpha_{t^*-1}^* = 0$ and $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$ for all $t \in \mathbb{N}$ such that:

(1) $\langle \alpha_t^* \mid t = 0, 1, 2, \dots, T-1 \rangle$ solves the following linear programming problem for all $T \in \mathbb{N}$, $T \geq \max\{3, t^*\}$:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \dots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \dots, T-1$.

(2) For all $T \in \mathbb{N}$, $T \geq \max\{3, t^*\}$, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^*c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) - \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^*c^{(t)} - \alpha_{T-1}^*x_T$.

Proof: Suppose $\langle x_t \mid t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$.

Then from proposition 5.1 we know that for all $T \in \mathbb{N}$ with $T \geq 3$, there exists $\langle \alpha_t^{*(T)} \mid \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ such that: $\langle \alpha_t^{*(T)} \mid \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ solves the linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)}c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \dots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \dots, T-1$.

From the corollary of proposition 5.1 we know that this $\langle \alpha_t^{*(T)} \mid \alpha_t^{*(T)} \geq 0$ for all $t = 0, 1, \dots, T-1 \rangle$ satisfies the following:

(1) $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} \geq p^{(t)}$ and $(\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} - p^{(t)})x_t = 0$ for all $t = 1, \dots, T-1$;

$$(2) (x_1 - c^{(0)} - a^{(0)}x)\alpha_0^{*(T)} = 0, (x_{t+1} - a^{(t)}x_t - c^{(t)})\alpha_t^{*(T)} = 0 \text{ for all } t = 1, \dots, T-2, (-a^{(T-1)}x_{T-1} - c^{(T-1)} + x_T)\alpha_{T-1}^{*(T)} = 0.$$

By the ‘‘interiority condition’’, for all $t \in \mathbb{N}$, $x_t > 0$ and hence from (i) we get $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} = p^{(t)}$ for all $t = 1, \dots, T-1$.

By the ‘‘interiority condition’’ once again $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$ and hence for $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$, $\alpha_{t^*-1}^{*(T)} = 0$.

Thus, for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$, $\langle \alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0 \text{ for all } t = 0, \dots, T-1 \rangle$ satisfies: $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} = p^{(t)}$ for all $t = 1, \dots, T-1$ and $\alpha_{t^*-1}^{*(T)} = 0$.

Thus, for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$, $\langle \alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0 \text{ for all } t = 0, \dots, T-1 \rangle$ must be the unique solution of the difference equation $\alpha_{t-1} - a^{(t)}\alpha_t = p^{(t)}$, $\alpha_{t^*-1}^{*(T)} = 0$.

Let $\langle \alpha_t^* | \alpha_t^* \geq 0, t \in \mathbb{N}^0 \rangle$ be the unique solution of this system of difference equation satisfying the condition $\alpha_{t^*-1}^* = 0$.

Since the optimal value of the primal must be equal to the optimal value of its dual, it must be the case that for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$, $\langle \alpha_t^* | \alpha_t^* \geq 0, t = 0, 1, \dots, T-1 \rangle$ solves

$$\text{Minimize } \alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T), \text{ subject to } \alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)} \text{ for all } t = 1, \dots, T-1, \alpha_t^{(T)} \geq 0 \text{ for all } t = 0, \dots, T-1.$$

Hence, for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$ it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$. Q.E.D.

Note 6.1: We know from proposition 6.1 that $\alpha_{t^*-1}^* = 0$ and for all $t \in \mathbb{N}$, $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$. We have assumed for proposition 6.1 that $a^{(t)} \neq 0$ for all $t \in \mathbb{N}$. Thus, $\alpha_{t^*-1}^* = 0$ and $\alpha_t^* = \frac{\alpha_{t-1}^* - p^{(t)}}{a^{(t)}}$ for all $t \in \mathbb{N}$. If the 1-IHLP problem is a discounted 1-IHLP problem with $p^{(t)} = \delta^t$ for all $t \in \mathbb{N}$ and $\delta \in (0, 1)$, then $\delta^t = \alpha_{t-1}^* - a^{(t)}\alpha_t^*$ for all $t \in \mathbb{N}$.

Thus, if $\langle a^{(t)} | t \in \mathbb{N}^0 \rangle$ converges to $a \in \mathbb{R}$ with $a \neq 1$, then it must be the case that $\lim_{T \rightarrow \infty} \alpha_T^* = 0$.

For $x \in X$, consider the alternative version of P1(x) defined in section 4:

$$\text{Maximize } \sum_{t=1}^{\infty} p^{(t)}y_t, \text{ subject to } y_1 \leq c^{(0)} + a^{(0)}x, y_{t+1} - a^{(t)}y_t \leq c^{(t)}, y_t \geq 0 \text{ for all } t \in \mathbb{N}.$$

This is the maximization problem that we are really concerned with.

Its ‘‘**implied dual linear programming (IDL) problem**’’ is the following:

Minimize $\alpha_0(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t c^{(t)}$, subject to $\alpha_{t-1} - a^{(t)}\alpha_t \geq p^{(t)}$, for all $t \in \mathbb{N}$ and $\alpha_t \geq 0$, $t \in \mathbb{N}^0$.

Theorem 6.1: Suppose $\langle p^{(t)}, c^{(t)}, a^{(t)} \mid t \in \mathbb{N}^0 \rangle$ is a 1-IHLP satisfying $c^{(t)}, c^{(t)} + a^{(t)}b > 0$, for all $t \in \mathbb{N}^0$ and $a^{(t)} \neq 0$ for all $t \in \mathbb{N}$. Suppose that for some $x \in X$, $\langle x_t \mid t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$, satisfies the following “**interiority condition**”: For all $t \in \mathbb{N}$, $x_t > 0$ and there exists $t^* \in \mathbb{N}$ such that $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$.

Then, there exists a sequence $\langle \alpha_t^* \mid \alpha_t^* \geq 0, t \in \mathbb{N}^0 \rangle$ satisfying $\alpha_{t^*-1}^* = 0$ and $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$ for all $t \in \mathbb{N}$ such that:

(1) $\langle \alpha_t^* \mid t = 0, 1, 2, \dots, T-1 \rangle$ solves the following linear programming problem for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \dots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \dots, T-1$.

(2) For all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

(3) $\sum_{t=1}^{\infty} p^{(t)}x_t = (c^{(0)} + a^{(0)}x)\alpha_0^* + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$ if and only if $\lim_{T \rightarrow \infty} \alpha_{T-1}^* x_T = 0$.

Proof: (1) and (2) follow from proposition 6.1. Hence we need to prove (3).

From (2) of proposition 6.1 we know that for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$ it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

Thus, for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$ it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Since, $\langle x_t \mid t \in \mathbb{N}^0 \rangle \in \mathcal{S}(x)$, $\lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} p^{(t)}x_t$ exists and $\lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \sum_{t=1}^{\infty} p^{(t)}x_t$.

If $\lim_{T \rightarrow \infty} \alpha_{T-1}^* x_T = 0$, then $\lim_{T \rightarrow \infty} [\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T]$ exists and $\lim_{T \rightarrow \infty} [\sum_{t=1}^{T-1} p^{(t)}x_t +$

$\alpha_{T-1}^* x_T] = \lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} p^{(t)}x_t + \lim_{T \rightarrow \infty} \alpha_{T-1}^* x_T = \lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} p^{(t)}x_t$.

$\lim_{T \rightarrow \infty} [\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T]$ exists implies $\alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ exists and

hence $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ for all $T \geq \max\{3, t^*\}$

implies $\lim_{T \rightarrow \infty} [\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T] = \alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Thus, $\lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Since $\lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} a_t^* c^{(t)} = \sum_{t=1}^{\infty} a_t^* c^{(t)}$ it follows that $\sum_{t=1}^{\infty} p^{(t)} x_t = \alpha_0^* (c^{(0)} + a^{(0)}x) + \sum_{t=1}^{\infty} a_t^* c^{(t)}$.

Conversely, if $\sum_{t=1}^{\infty} p^{(t)} x_t = (c^{(0)} + a^{(0)}x) \alpha_0^* + \sum_{t=1}^{\infty} a_t^* c^{(t)} = \sum_{t=1}^{\infty} a_t^* c^{(t)}$, then

$\sum_{t=1}^{T-1} p^{(t)} x_t + a_{T-1}^* x_T = (c^{(0)} + a^{(0)}x) \alpha_0^* + \sum_{t=1}^{T-1} a_t^* c^{(t)}$ for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$,

$\lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} p^{(t)} x_t = \sum_{t=1}^{\infty} p^{(t)} x_t$ and $\lim_{T \rightarrow \infty} \sum_{t=1}^{T-1} a_t^* c^{(t)} = \sum_{t=1}^{\infty} a_t^* c^{(t)}$ together imply

$\lim_{T \rightarrow \infty} a_{T-1}^* x_T = 0$. Q.E.D.

Note 6.2: Since for all $T \in \mathbb{N}$, α_{T-1} is the dual (co-state) variable corresponding to x_T ,

$\lim_{T \rightarrow \infty} \alpha_{T-1}^* x_T = 0$ is a version of the well-known “**transversality condition**” of optimal

control theory. Part (3) of theorem 6.1 says that under the assumptions common to both proposition 6.1 and theorem 6.1, the optimal value of the IDLP problem is equal to the optimal value of the alternative version of $P1(x)$ defined in section 4 “*if and only if*” the **transversality condition** (in the way defined here) is satisfied by the solutions of the two infinite horizon linear programming problems.

Note 6.3: If as in note 6.1, the 1-IHLP problem is a discounted 1-IHLP problem with

$p^{(t)} = \delta^t$ for all $t \in \mathbb{N}$ and $\delta \in (0, 1)$, and $\langle a^{(0)} | t \in \mathbb{N}^0 \rangle$ converges to $a \in \mathbb{R}$ with $a \neq 1$, then it

must be the case that $\lim_{T \rightarrow \infty} \alpha_T^* = 0$. Thus $\lim_{T \rightarrow \infty} \alpha_{T-1}^* x_T = 0$. Hence the transversality

condition is satisfied for a discounted 1-IHLP problem satisfying the additional convergence criterion for $\langle a^{(0)} | t \in \mathbb{N}^0 \rangle$.

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