Infinite Horizon Linear Programming With One-Dimensional State Variable

by

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Abstract

We provide a set of sufficient conditions for the optimal value of an infinite horizon linear programming problem with one-dimensional state variable to be equal to the optimal value of its implied infinite horizon dual linear programming problem. Our sufficient conditions require that in every period the linear inequality determining the constraint for the state variable for the next period is expressed in terms of a "non-constant" function of the current value of the state variable, the state variable along the optimal trajectory is always strictly positive beginning with time period one and is strictly less than its upper-bound in "at least one" time period. In addition, the transversality condition we invoke is that the product in each period, of the state variable and the co-state or dual variable for the inequality constraining the state variable in that period, converges to zero. The simplicity of this characterization is entirely due to the fact that in each period, there is only one inequality constraint that the state variable is required to satisfy. We show this, by introducing a generalized infinite horizon linear programming problem with one-dimensional state variable. In this more general framework the duality gap problem can be partially resolved if and only if a transversality condition is satisfied.

Keywords: infinite horizon linear programming, linear constraints, duality, infinite horizon dual linear programming problem, transversality condition

AMS Subject Classification: 49K05, 90C05, 90C46

JEL Codes: C44, C61.

1. Introduction: At the outset, let us make it clear that we are concerned with infinite "horizon" linear programming problems <u>and not</u> infinite linear programming problems, the latter embracing a much larger class of problems than what the former does. Hence, a counter example applicable for infinite linear programming problems need not be applicable for infinite horizon linear programming problems.

The study of infinite horizon linear programming problems originates in Hopkins (1969). The model discussed in Hopkins (1969) allows the chosen variables at any time period, to be constrained by choices made in all previous time periods. This model is discussed further in Grinold (1971). Among the earliest work on infinite horizon linear programming problems that restricts the chosen variable at any time period by just the chosen variables at the immediately previous period- and no further back in time- is the work in Evers (1973). The model in Evers (1973) assumes that the parameters determining the constraints for the choice variables are invariant over time. Some additional work based on the model developed by Evers (1973) is available in Grinold (1977). Romeijn, Smith and Bean (1992) along with its sequel- Romeijn and Smith (1998)- generalize the model in Evers (1973) and allow these parameters to vary with the time period. All works cited so far have one feature in common, i.e., they allow the state variables or chosen alternatives to be points in a finite dimensional Euclidean space, where the dimension could be greater than one.

There is a considerable body of literature on infinite horizon dynamic optimization with one dimensional state variable that is discussed in depth in Mitra (2000). The model discussed in Mitra (2000) allows for non-linear objective functions that are separable over time periods and constraint sets that allow for non-linearity. This model, referred to in Mitra (2000) as the reduced form model, has been studied with linear objective functions in Lahiri (2025a). In Lahiri (2025b), we discuss infinite horizon dynamic optimization with one dimensional state variables and linear objective function, where the constraint set in each period is determined by exactly one linear inequality. The model in Lahiri (2025b) is the framework of analysis that we start out with this paper.

The purpose of Lahiri (2025a, 2025b) was to sharpen the results about the reduced form model in Mitra (2000) in the contexts of the two former papers respectively. Thus issues concerning Euler equations and transversality conditions are discussed in Lahiri (2025b) and there is no attempt made in that paper to relate the model to linear programming.

In Lahiri (2025c), there is a model of infinite horizon optimal control with linear objective functions and with linear equations and linear inequalities constraining the evolution of a one-dimensional state variable and a one-dimensional control variable. That paper deals with the problem concerning “duality gap” that Grinold (1971), Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) are concerned with.

However, the model discussed in Lahiri (2025c) is conceptually different from the other works we have mentioned so far, since in Lahiri (2025c) there is a control variable as well as a state variable, each evolving in its own way and both contribute to the objective function. There is a way to "mathematically" reconcile the optimal control model discussed in Lahiri (2025c) with the framework that Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) are concerned with as we shall show later in this paper. Such a reconciliation would require the non-trivial assumption that *in every time period, the coefficient of the control variable in the equation of motion of the state variable is "non-zero"*. This could be a way to make the model discussed in Lahiri (2025c) a one-dimensional state variable version of the multi-dimensional state variable model pursued in Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998). We will refer to this a one-dimensional state variable version of the model in Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) as a "Generalized Infinite Horizon Linear Programming Problem with One-Dimensional State Variable". Duality gap is about the possibility of the optimal value of an infinite horizon linear programming problem being different from the optimal value of its infinite horizon dual linear programming problem. In linear programming, primal and dual linear programming problems are defined in the context of finite number unknown variables. Hence, we refer to the dual linear programming in the infinite horizon context as "implied infinite horizon dual".

It needs to be noted that the model discussed in Romeijn, Smith and Bean (1992) and Romeijn and Smith (1998) is more general than the model discussed in Lahiri (2025b) and hence the one discussed here. The major contribution of Romeijn, Smith and Bean (1992) that is reproduced in Romeijn and Smith (1998), is a set of sufficient conditions that includes a transversality condition under which the optimal value of the infinite horizon linear programming problem is equal to the optimal value of its implied infinite horizon dual linear programming problem.

In this paper, our first significant result is a set of sufficient conditions that also includes a transversality condition under which the optimal value of the infinite horizon linear programming problem is equal to the optimal value of its implied infinite horizon dual linear programming problem. Our sufficient conditions require that in every period the linear inequality determining the constraint for the state variable for the next period is expressed in terms of a "non-constant" function of the current value of the state variable, the state variable along the optimal trajectory is always strictly positive beginning with time period one and is strictly less than its upper-bound in “at least one” time period. In addition the transversality condition we invoke is that the product in each period, of the state variable and the co-state or dual variable for the inequality constraining the state variable in that period, converges to zero. Our sufficient conditions are considerably simpler than the ones required in the earlier papers mentioned here, to obtain similar conclusions. There does not seem to be any connection between our simplicity and the fact that unlike the “predecessor parers”, we focus our attention on one-dimensional state variables. In fact, our sufficient conditions here are also less restrictive than the sufficient conditions invoked in Lahiri (2025c) to address the duality gap problem.

The requirement that the state variable is always strictly positive beginning with the first period, leads to a difference equation for the evolution of the dual variable. The requirement that the state variable is strictly less than its upper-bound in “at least one” time period implies that the corresponding value of the co-state variable is zero. This leads to a unique optimal solution of the implied infinite horizon dual linear programming problem. If the objective function is a discounted sum of the instantaneous values of the state variables and the slopes of the function determining the upper bounds of the state variable converges to a real number that is not equal to one, then from the difference equation governing the evolution of the dual variables it follows that the transversality condition must be satisfied.

In a final section of this paper we show that the major stumbling block towards defining the implied infinite horizon dual linear programming problem is the “multiplicity of constraints” in the more general context of a generalized infinite horizon linear programming problem with one-dimensional state variable. The possibility of multiple constraints in each period complicates the resolution of the duality gap problem in a simple manner. However, we are able to resolve the duality gap theorem to some extent, without invoking any additional assumption on the model or the optimal trajectory, other than the ones that are used in the definitions for the general framework. This result says the the optimal value of the dual of the truncated “free end-point” linear programming problem with the initial value of the state variable in the latter being the same as the initial value of the state variable in the optimal trajectory (i.e., the kind of linear programming problems in the “approximation result”) converges to the optimal value of the generalized infinite horizon linear programming problem with one-dimensional state variable, if and only if, a reasonable and simple transversality condition is satisfied. This transversality condition says that the product of the state variable and a weighted sum of the dual variables of the constraints that that the state variable is required to satisfy, converges to zero. The weights for the dual variables (unlike traditional weights) are only required to be non-zero. It is really “a matter of opinion” (like much else in science!) whether this result contributes substantially to infinite horizon linear programming.

2. The Framework of Analysis: Our framework of analysis presented below is “almost” identical to the one in Lahiri (2025b).

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers and $\mathbb{R}_{++}$ the set of strictly positive real numbers. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}^0 = \mathbb{N}\cup\{0\}$.

Let $X = [0, b] \subset \mathbb{R}_+$, with $b\in\mathbb{R}_{++}$ denote the set of available alternatives. Time is measured in discrete periods $t\in\mathbb{N}^0$. At each time period ‘t’ an alternative is chosen, and the chosen alternative is denoted by $x_t \in X$.

Let $\langle c^{(t)}|t\in\mathbb{N}^0\rangle$ be a sequence in $\mathbb{R}_+$ and let $\langle a^{(t)}|t\in\mathbb{N}^0\rangle$ be a sequence in $\mathbb{R}$ such that $c^{(t)} \in [0, b]$ and $c^{(t)} + a^{(t)}b \in [0, b]$ for all $t\in\mathbb{N}^0$.

Thus, $\langle c^{(t)}|t\in\mathbb{N}^0\rangle$ is a bounded sequence in $\mathbb{R}_+$.

Note 2.1: $\langle c^{(t)}|t\in\mathbb{N}^0\rangle$ is a bounded sequence in $\mathbb{R}_+$ combined with $b > 0$, $-c^{(t)} \leq a^{(t)}b \leq b - c^{(t)}$ for all $t\in\mathbb{N}^0$ implies $\inf\{-\frac{c^{(t)}}{b}|t\in\mathbb{N}^0\} \leq a^{(t)} \leq \sup\{1-\frac{c^{(t)}}{b}|t\in\mathbb{N}^0\}$, where both $|\inf\{-\frac{c^{(t)}}{b}|t\in\mathbb{N}^0\}| < +\infty$ and $|\sup\{1-\frac{c^{(t)}}{b}|t\in\mathbb{N}^0\}| < +\infty$.

Thus, $\langle a^{(t)}|t\in\mathbb{N}^0\rangle$ must also be a bounded sequence.

Since for all $x\in[0, b]$, $x = \frac{x}{b}b + (1-\frac{x}{b})0$ with $\frac{x}{b} \in[0, 1]$ it must be the case that $c^{(t)} + a^{(t)}x \in X = [0, b]$ for all $(x, t) \in X\times\mathbb{N}^0$.

For $t\in\mathbb{N}^0$, let $\Omega_t = \{(x, y)\in X\times X|\ y \leq c^{(t)} + a^{(t)}x\} = \{(x, y)|0 \leq y \leq c^{(t)} + a^{(t)}x, x\in[0, b]\}$.

Is it easy to see that for all $t\in\mathbb{N}^0$, Ω_t is a trapezium with its extreme points being (0, 0), $(0, c^{(t)})$, $(b, c^{(t)} + a^{(t)}b)$, (b, 0).

For $t\in\mathbb{N}^0$, Ω_t is **the two-period linearly constrained set at (time-period)** t.

For $(x, t) \in X\times\mathbb{N}^0$, let $\Omega_t(x) = \{y|0 \leq y \leq c^{(t)} + a^{(t)}x\} = [0, c^{(t)} + a^{(t)}x]$.

Note that for all $t\in\mathbb{N}^0$, Ω_t is a non-empty, closed and bounded subset of $X\times X$ and for each $(x, t) \in X\times\mathbb{N}^0$, the set $\Omega_t(x)$ is a non-empty, closed and bounded interval in X, though the interval $\Omega_t(x)$ may be a singleton (i.e., degenerate) as for instance when $a^{(t)} = 0$.

For $(x, t) \in X\times\mathbb{N}^0$, the set $\Omega_t(x)$ is said to be **the transition set from** x **at (time-period)** t.

For $x\in X$, let $\mathcal{F}(x) = \{<x_t|t\in\mathbb{N}^0>|\ x_{t+1} \leq c^{(t)} + a^{(t)}x_t$ for all $t\in\mathbb{N}^0$, $x_0 = x\}$.

We will (whenever necessary) refer to an infinite sequence $<x_t|t\in\mathbb{N}^0>\in \mathcal{F}(x)$ as **a trajectory starting at (from)** x.

Let $<p^{(t)}|\ t\in\mathbb{N}^0>$ be a sequence in $\mathbb{R}$ satisfying $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$.

Note 2.2: $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$ implies $\lim_{t\to\infty} |p^{(t)}| = 0$ and for all sequences $<x_t|t\in\mathbb{N}^0>$, it must be the case that $|\sum_{t=0}^{\infty} p^{(t)}x_t| \leq \sum_{t=0}^{\infty} |p^{(t)}|x_t < +\infty$.

We shall refer to the sequence $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ as a **infinite horizon linear programming problem with one-dimensional state variable (1-IHLP problem)**.

Given a 1-IHLP problem $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ and $x\in X$ we shall be concerned with the following optimization problem denoted by **P1**(x):

Maximize $\sum_{t=0}^{\infty} p^{(t)}x_t$, subject to $<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)$.

If for some $c \in\mathbb{R}_+$ and $a\in\mathbb{R}$ the 1-IHLP $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ satisfies $c^{(t)} = c$ and $a^{(t)} = a$ for all $t\in\mathbb{N}^0$, then we may refer to the 1-IHLP problem as a **quasi time-invariant 1-IHLP problem**.

We will denote a quasi time-invariant 1-IHLP problem such as above by $(<p^{(t)}|t\in\mathbb{N}^0>, c, a)$.

Example 2.1: The discounted 1-IHLP problem: There exists $\delta\in(0, 1)$ such that for all $t\in\mathbb{N}^0$, $p^{(t)} = \delta^t$.

Note 2.3: If a "discounted 1-IHLP problem", is at the same time a quasi time-invariant 1-IHLP such that there exists a real numbers $c \in [0, b]$ and a real number a such that $c + ab \in [0, b]$ satisfying $c^{(t)} = c$ and $a^{(t)} = a$ for all $t\in\mathbb{N}^0$, then we have a model that closely resembles a one-dimensional control variable version of the infinite horizon linear programming discussed in Grinold (1977). We may refer to such a problem as a **time-invariant 1-IHLP problem** and denote it by (δ, c, a).

3. Some Preliminary results about trajectories: Given a 1-IHLP problem $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ for each $x\in X$, let $\mathcal{S}(x) = \underset{<x_t|t\in\mathbb{N}^0>\in\mathcal{F}(x)}{\operatorname{argmax}} \sum_{t=0}^{\infty} p^{(t)}x_t$. $\mathcal{S}(x)$ is the **set of solutions for P1**(x) **starting from** x **(for the 1-IHLP problem).**

The following result is proposition 4 in Lahiri (2025b).

Proposition 3.1: For all $x\in X$, $\mathcal{S}(x) \neq \phi$.

An immediate consequence of proposition 3.1, is that there exists a function V: X $\rightarrow$ $\mathbb{R}$ such that for all xϵX: V(x) = $\sum_{t=0}^{\infty} p^{(t)} x_t$, where $<x_t|t\epsilon\mathbb{N}^0>\epsilon\ \mathcal{S}(x)$.

In note 2.3 we have defined a time-invariant 1-IHLP problem, It is well known that for a time invariant 1-IHLP problem (δ, c, a) for all xϵX: $V(x) = x + \delta \max_{y\epsilon[0,\, c+ax]} V(y)$.

This equation is known as Bellman's fundamental equation of dynamic programming for the 1-IHLP problem.

For more on the fundamental equation of dynamic programming one may refer to Mitra (2000).

The following proposition provides a "solution procedure" for a large class of 1-IHLP problems.

Proposition 3.2: Suppose $<(p^{(t)}, c^{(t)}, a^{(t)})|t\epsilon\mathbb{N}^0>$ is a 1-IHLP problem satisfying $p^{(t)} \geq 0$ for all t$\epsilon\mathbb{N}$ and $a^{(t)} \geq 0$ for all t$\epsilon\mathbb{N}^0$. Let xϵX and let $<x_t|t\epsilon\mathbb{N}^0>$ be the sequence with x_0 = x, and $x_{t+1} = c^{(t)} + a^{(t)}x_t$ for all t$\epsilon\mathbb{N}^0$. Then, $<x_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{S}(x)$.

Further, if $<y_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{S}(x)$, then for all T$\epsilon$$\{t\epsilon\mathbb{N}|\ p^{(t)} > 0\}$ it must be the case that $y_T = x_T$.

Proof: Clearly $<x_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{F}(x)$. Let $<y_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{F}(x)$. Clearly $y_1\epsilon\ [0, c^{(0)} + a^{(0)}x] = [0, x_1]$ and hence $y_1 \leq x_1$. Suppose that for some T$\epsilon\mathbb{N}$ it is the case that $y_T \leq x_T$. Since, $a^{(T)} \geq 0$, $y_{T+1}\ \epsilon\ [0, c^{(T)} + a^{(T)}y_T] \subset [0, c^{(T)} + a^{(T)}x_T] = [0, x_{T+1}]$, i.e., $y_{T+1} \leq x_{T+1}$. Thus, $y_T \leq x_T$ implies $y_{T+t} \leq x_{T+t}$ for all t$\epsilon\mathbb{N}^0$.

Since $y_1 \leq x_1$, it must be the case that $y_t \leq x_t$ for all t$\epsilon\mathbb{N}$.

Thus, $\sum_{t=0}^{\infty} p^{(t)}x_t = p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}x_t \ \geq p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}y_t$, since since $p^{(t)} \geq 0$ for all t$\epsilon\mathbb{N}$.

However, $p^{(0)}x + \sum_{t=1}^{\infty} p^{(t)}y_t = \sum_{t=0}^{\infty} p^{(t)}y_t$.

Thus, $\sum_{t=0}^{\infty} p^{(t)}x_t \geq \sum_{t=0}^{\infty} p^{(t)}y_t$ and so $<x_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{S}(x)$.

If $<y_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{S}(x)$ then $<y_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{F}(x)$. Thus, as observed earlier in this proof, $y_0 = x_0$ = x and $y_t \leq x_t$ for all t$\epsilon\mathbb{N}$.

Thus, if for some T$\epsilon\mathbb{N}$, $p^{(T)} > 0$ and $y_T < x_T$, then $\sum_{t=0}^{\infty} p^{(t)}x_t = \sum_{t=0}^{T-1} p^{(t)}x_t + p^{(T)}x_T + \sum_{t=T+1}^{\infty} p^{(t)}x_t > \sum_{t=0}^{T-1} p^{(t)}x_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}x_t \geq \sum_{t=0}^{T-1} p^{(t)}y_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}y_t$, since $p^{(t)} \geq 0$ and $x_t \geq y_t$ for all t$\epsilon\mathbb{N}$.

However, $\sum_{t=0}^{T-1} p^{(t)}y_t + p^{(T)}y_T + \sum_{t=T+1}^{\infty} p^{(t)}y_t = \sum_{t=0}^{\infty} p^{(t)}y_t$ and hence $\sum_{t=0}^{\infty} p^{(t)}x_t > \sum_{t=0}^{\infty} p^{(t)}y_t$.

This, contradicts our assumption that $<y_t|t\epsilon\mathbb{N}^0>\epsilon\mathcal{S}(x)$ and proves the proposition.

Q.E.D.

4. Optimality and associated linear programming problems: In this section we replicate for our current context the main result in section 3 of Lahiri (2025c) the latter being a discussion on infinite horizon linear optimal control problems with linear constraints.

Note 4.1: The exact mathematical interpretation of the expression (formula) $\sum_{t=0}^{\infty} p^{(t)} x_t$ is $\lim_{T\to\infty} (\sum_{t=0}^{T} p^{(t)} x_t)$. Thus, the problem we are concerned with here is in the domain of asymptotic analysis, which is very different from infinite dimensional analysis.

An **alternative version of P1**(x) is the following optimization problem:

Maximize $\sum_{t=1}^{\infty} p^{(t)} x_t$ subject to the infinite sequence $<x_t|t\in \mathbb{N}^0>$ satisfying the constraints: $(x_t, x_{t+1})\in \Omega_t$, $t\in \mathbb{N}^0$, $x_0 = x$.

We now provide one necessary condition and a somewhat stronger sufficient condition for optimality for the 1-IHLP problem in terms of linear programming problems.

Proposition 4.1: Let $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ be a 1-IHLP problem and suppose that for some $x\in X$, $<x_t|t\in \mathbb{N}^0>\in \mathcal{F}(x)$.

Part 1: <u>If</u> $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$ <u>then</u> for all $T\in \mathbb{N}$, $<x_t|t = 0, 1, \ldots, T>$ solves the following linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)} y_t$ for all $t = 0, 1, \ldots, T-1$, $y_0 = x_0 = x$ and $y_T = x_T$.

Part 2: <u>If</u> there exists $T^* \in \mathbb{N}$ such that for all $T\in \mathbb{N}$ satisfying $T \geq T^*$, $<x_t|t = 0, 1, \ldots, T>$ solves the linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)} y_t$ for all $t = 0, 1, \ldots, T-1$ and $y_0 = x_0 = x$, <u>then</u> $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$.

Proof: The proof is similar to the proof of proposition 3.1 in Lahiri (2025c). Q.E.D.

We now provide an approximation result that follows from proposition 3.1, note 4.1 and part 1 of proposition 4.1.

Proposition 4.2 (Approximation Result): Given the 1-IHLP problem $<(p^{(t)}, c^{(t)}, a^{(t)})|t\in\mathbb{N}^0>$ and $x\in X$, there exists $T^*(\varepsilon)\in\mathbb{N}$ such that for all $T\in\mathbb{N}$ with $T \geq T^*(\varepsilon)$, the linear programming problem [Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)} y_t$ for all $t = 0, 1, \ldots, T-1$ and $y_0 = x_0 = x$] has a solution $<x_t^{(T)}| t = 0, 1, \ldots, T>$ and $|\sum_{t=0}^{T} p^{(t)} x_t^{(T)} - V(x)| < \varepsilon$.

Proof: Since $\sum_{t=0}^{\infty}|p^{(t)}| < +\infty$ and $\mathcal{F}(x) \neq \phi$, for all T$\epsilon\mathbb{N}$ the linear programming problem [Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all t = 0, 1, …, T-1 and $y_0 = x_0 = x$] has a solution $<x_t^{(T)}| \ t = 0, 1, \ldots, T>$.

Towards a contradiction suppose there exists $\varepsilon > 0$ such that $|\sum_{t=0}^{T} p^{(t)}x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often.

Then, it must be the case that <u>either</u> $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} \geq V(x) + \varepsilon$ infinitely often <u>or</u> $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} \leq V(x) - \varepsilon$ infinitely often.

Let $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$. Since $<x_t|t = 0, 1, \ldots, T>$ satisfies the constraints of the linear programming problem for all T$\epsilon\mathbb{N}$, it must be the case that $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} \geq \sum_{t=0}^{T} p^{(t)}x_t$ for all T$\epsilon\mathbb{N}$.

Since, $V(x) = \sum_{t=0}^{\infty} p^{(t)}x_t = \lim_{T\to\infty}\sum_{t=0}^{T} p^{(t)}x_t$, there exists $T^0\epsilon\mathbb{N}$ such that for all T$\epsilon\mathbb{N}$ with $T \geq T^0$ it is the case that $\sum_{t=0}^{T} p^{(t)}x_t + \frac{\varepsilon}{4} > V(x) > \sum_{t=0}^{T} p^{(t)}x_t - \frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} + \frac{\varepsilon}{4} \geq \sum_{t=0}^{T} p^{(t)}x_t + \frac{\varepsilon}{4} > V(x)$ for all T$\epsilon\mathbb{N}$ with $T \geq T^0$.

Thus, $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} + \varepsilon > \sum_{t=0}^{T} p^{(t)}x_t^{(T)} + \frac{\varepsilon}{4} > V(x)$ for all T$\epsilon\mathbb{N}$ with $T \geq T^0$.

Thus, $|\sum_{t=0}^{T} p^{(t)}x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often is incompatible with $V(x) \geq \sum_{t=0}^{T} p^{(t)}x_t^{(T)} + \varepsilon$ infinitely often.

Thus, $|\sum_{t=0}^{T} p^{(t)}x_t^{(T)} - V(x)| \geq \varepsilon$ infinitely often implies that there exists $T^1 = T^1(\varepsilon)\epsilon\mathbb{N}$ such that $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} \geq V(x) + \varepsilon$ for all T $\epsilon\mathbb{N}$ satisfying $T \geq T^1$.

Since, $\sum_{t=0}^{\infty}|p^{(t)}| < +\infty$, there exists $T^2 = T^2(\varepsilon)\epsilon\mathbb{N}$ such that for all T$\epsilon\mathbb{N}$ satisfying $T \geq T^2$, $b\sum_{t=T}^{\infty}|p^{(t)}| < \frac{\varepsilon}{4}$.

$\sum_{t=0}^{T} p^{(t)}x_t^{(T)} \geq V(x) + \varepsilon$ for all T $\epsilon\mathbb{N}$ satisfying $T \geq T^1$ implies $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} \geq V(x) + \varepsilon$ for all T$\epsilon\mathbb{N}$ satisfying $T \geq \max\{T^1, T^2\}$ infinitely often.

Let T$\epsilon\mathbb{N}$ be such that $T \geq \max\{T^1, T^2\}$.

Thus, $\sum_{t=0}^{T} p^{(t)}x_t^{(T)} \geq V(x) + \varepsilon$.

Let $x_{T+t}^{(T)} = c^{(T+t-1)} + a^{(T+t-1)}x_{T+t-1}^{(T)}$ for all t$\epsilon\mathbb{N}$.

Thus, $<x_t^{(T)}| \ t\epsilon\mathbb{N}^0>\epsilon\mathcal{F}(x)$.

Now $|\sum_{t=T+1}^{\infty} p^{(t)}x_t^{(T)}| \leq \sum_{t=T+1}^{\infty}|p^{(t)}x_t^{(T)}| \leq b\sum_{t=T+1}^{\infty}|p^{(t)}| < \frac{\varepsilon}{4}$.

Thus, $\frac{\varepsilon}{4} > \sum_{t=T+1}^{\infty} p^{(t)}x_t^{(T)} > -\frac{\varepsilon}{4}$.

Thus, $\sum_{t=0}^{\infty} p^{(t)} x_t^{(T)} = \sum_{t=0}^{T} p^{(t)} x_t^{(T)} + \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} \geq V(x) + \varepsilon + \sum_{t=T+1}^{\infty} p^{(t)} x_t^{(T)} > V(x) + \varepsilon - \frac{\varepsilon}{4} = V(x) + \frac{3\varepsilon}{4} > V(x)$.

This, contradicts the definition of V(x) and proves the proposition. Q.E.D.

5. Duality Theory for 1-IHLP problem and "necessary" conditions for optimality:

Given the 1-IHLP $<(p^{(t)},c^{(t)},a^{(t)})| t\in \mathbb{N}^0>$ suppose that for all $t\in\mathbb{N}^0$, $c^{(t)}$, $c^{(t)} + a^{(t)}b > 0$, so that for all $(x, t)\in X\times \mathbb{N}^0$, $c^{(t)} + a^{(t)}x > 0$. This is equivalent to the assumption that for all $(x, t)\in X\times\mathbb{N}^0$, $\{0\}$ is a proper subset of $\Omega_t(x)$.

Given $x\in X$, let $<x_t|t\in \mathbb{N}^0> \in\mathcal{F}(x)$.

For $T\in \mathbb{N}$ with $T \geq 3$ consider the linear programming problem in part 1 of proposition 4.1.

Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $y_{t+1} \leq c^{(t)} + a^{(t)}y_t$ for all t = 0, 1, …, T-1, $y_0 = x_0 = x$, $y_t \geq 0$ for all t = 0, 1, …, T, & $y_T = x_T$.

Since $\sum_{t=0}^{T} p^{(t)}y_t = p^{(0)}x + \sum_{t=1}^{T} p^{(t)}y_t$ along with $y_0 = x_0 = x$ and $y_T = x_T$, the linear programming problem in part 1 of proposition 4.1 reduces to the following.

Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t\leq c^{(t)}$ for all t = 1, …, T-2, $-a^{(T-1)} y_{T-1} \leq c^{(T-1)} - x_T$, $y_t \geq 0$ for all t = 1, …, T-1.

The dual of this linear programming problem is the following linear programming problem.

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)}c^{(t)}+ \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)}\geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)}\geq 0$ for all t = 0, …, T-1.

Hence we have the following proposition.

Proposition 5.1: Suppose $<(p^{(t)},c^{(t)},a^{(t)})|t\in \mathbb{N}^0>$ is a 1-IHLP satisfying $c^{(t)}$ and $c^{(t)} + a^{(t)}b > 0$ for all $t\in\mathbb{N}^0$. Suppose that for some $x\in X$, $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$. Then, for all $T\in \mathbb{N}$ with $T \geq 3$, there exists $<\alpha_t^{*(T)}| \alpha_t^{*(T)}\geq 0$ for all t = 0, 1, ..., T-1> such that:

(A) $<\alpha_t^{*(T)}| \alpha_t^{*(T)}\geq 0$ for all t = 0, 1, ..., T-1> solves the linear programming problem Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)}c^{(t)}+ \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)}\geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)}\geq 0$ for all t = 0, …, T-1.

(B) $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^{*(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{*(T)}c^{(t)}+ \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)$.

(C) $\lim_{T\to\infty} [\alpha_0^{*(T)}(c^{(0)} + a^{(0)})x + \sum_{t=1}^{T-2} \alpha_t^{*(T)}c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)]$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$.

Proof: From the discussion in this section preceding the statement of this proposition, we know if $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$, then that for all $T\in \mathbb{N}$ with $T \geq 3$, $<x_t|t = 0, \ldots, T>$ solves:

Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t \leq c^{(t)}$ for all t = 1, …, T-2, $-a^{(T-1)} y_{T-1} \leq c^{(T-1)} - x_T$, $y_t \geq 0$ for all t = 1, …, T-1.

By the strong duality theorem of linear programming (see topic 2 of Lahiri (2020)) we know that $<x_t|t = 0, \ldots, T>$ solves the above problem if and only if its dual has a solution, in which case the optimal value of the maximization problem and the optimal value of its dual are equal. The dual of the linear programming maximization problem is the following linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

For $T\in \mathbb{N}$ with $T \geq 3$, let $<\alpha_t^{*(T)}|\ \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> be such a solution.

Thus, for $T\in \mathbb{N}$ with $T \geq 3$, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^{*(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)$.

Since, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$, it must be the case that $\lim_{T\to\infty} [\alpha_0^{*(T)}(c^{(0)} + a^{(0)})x + \sum_{t=1}^{T-2} \alpha_t^{*(T)} c^{(t)} + \alpha_{T-1}^{*(T)}(c^{(T-1)} - x_T)]$ exists and is equal to $\sum_{t=1}^{\infty} p^{(t)}x_t$. Q.E.D.

An immediate corollary of proposition 5.1 is the following result.

Corollary of Proposition 5.1: Suppose $<(p^{(t)},c^{(t)},a^{(t)})|\ t\in \mathbb{N}^0>$ is a 1-IHLP satisfying $c^{(t)}$ and $c^{(t)} + a^{(t)}b > 0$ for all $t\in\mathbb{N}^0$. Suppose that for some $x\in X$, $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$. Then, for all $T\in \mathbb{N}$ with $T \geq 3$, there exists $<\alpha_t^{*(T)}|\ \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> such that:

(1) $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} \geq p^{(t)}$ and $(\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} - p^{(t)})x_t = 0$ for all t = 1, …, T-1;

(2) $(x_1 - c^{(0)} - a^{(0)}x)\alpha_0^{*(T)} = 0$, $(x_{t+1} - a^{(t)}x_t - c^{(t)})\alpha_t^{*(T)} = 0$ for all t = 1, …, T-2, $(-a^{(T-1)} x_{T-1} - c^{(T-1)} + x_T)\alpha_{T-1}^{*(T)} = 0$.

Proof: By proposition 5.1 we know that for all $T\in \mathbb{N}$ with $T \geq 3$, there exists $<\alpha_t^{*(T)}|\ \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> such that:

(A) $<\alpha_t^{*(T)}|\ \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> solves the linear programming problem

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

From the complementary slackness condition (see topic 2 of Lahiri (2020)) we know that since $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x) \subset \mathcal{F}(x)$, $<\alpha_t^{*(T)}|\ \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> along with $<x_t|t = 0, \ldots, T>$ satisfy (1) and (2). Q.E.D.

6. Interiority condition and infinite horizon dual linear programming problem:

In this section we consider 1-IHLP problems for which, in addition to the conditions assumed in section 5 (i.e., $c^{(t)} > 0$ and $c^{(t)} + a^{(t)}b > 0$ for all $t\epsilon\mathbb{N}^0$), it is the case that for $t\epsilon\mathbb{N}$, $a^{(t)} \neq 0$.

For our main result in this section we assume that along the optimal trajectory the state variable is always strictly positive, as for instance is the case with the solution in the statement of proposition 3.2. We also assume that in some time period $t^*\epsilon\mathbb{N}$, the chosen value of the state variable is less than the upper bound for the possible set of values for the state variable in period t^*. We refer to this condition on the optimal trajectory as "interiority condition".

Proposition 6.1: Suppose $<(p^{(t)},c^{(t)},a^{(t)})|\ t\in \mathbb{N}^0>$ is a 1-IHLP satisfying $c^{(t)}$, $c^{(t)} + a^{(t)}b >$ 0, for all $t\epsilon\mathbb{N}^0$ and $a^{(t)} \neq 0$ for all $t\epsilon\mathbb{N}$. Suppose that for some $x\epsilon X$, $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$, satisfies the following "**interiority condition**": For all $t\epsilon\mathbb{N}$, $x_t > 0$ and there exists $t^*\epsilon\mathbb{N}$ such that $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$.

Then, there exists a sequence $<\alpha_t^*|\alpha_t^* \geq 0, t\epsilon\mathbb{N}^0>$ satisfying $\alpha_{t^*-1}^* = 0$ and $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$ for all $t\in \mathbb{N}$ such that:

(1) $<\alpha_t^*|\ t = 0, 1, 2, \ldots, T-1>$ solves the following linear programming problem for all $T\in \mathbb{N}$, $T \geq \max\{3, t^*\}$:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

(2) For all $T\in \mathbb{N}$, $T \geq \max\{3, t^*\}$, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

Proof: Suppose $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$.

Then from proposition 5.1 we know that for all $T\in \mathbb{N}$ with $T \geq 3$, there exists $<\alpha_t^{*(T)}|\ \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> such that: $<\alpha_t^{*(T)}|\ \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> solves the linear programming problem:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

From the corollary of proposition 5.1 we know that this $<\alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all t = 0, 1, ..., T-1> satisfies the following:

(1) $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} \geq p^{(t)}$ and $(\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} - p^{(t)})x_t = 0$ for all t = 1, …, T-1;

(2) $(x_1 - c^{(0)} - a^{(0)}x)\alpha_0^{*(T)} = 0$, $(x_{t+1} - a^{(t)}x_t - c^{(t)})\alpha_t^{*(T)} = 0$ for all t = 1, …, T-2, $(-a^{(T-1)}x_{T-1} - c^{(T-1)} + x_T)\alpha_{T-1}^{*(T)} = 0$.

By the "interiority condition", for all t∈ℕ, $x_t > 0$ and hence from (i) we get $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} = p^{(t)}$ for all t = 1, …, T-1.

By the "interiority condition" once again $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$ and hence for T∈ℕ with T ≥ max{3, t*}, $\alpha_{t^*-1}^{*(T)} = 0$.

Thus, for all T∈ ℕ with T ≥ max{3, t*}, $<\alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all t = 0, …, T-1> satisfies: $\alpha_{t-1}^{*(T)} - a^{(t)}\alpha_t^{*(T)} = p^{(t)}$ for all t = 1, …, T-1 and $\alpha_{t^*-1}^{*(T)} = 0$.

Thus, for all T∈ ℕ with T ≥ max{3, t*}, $<\alpha_t^{*(T)} | \alpha_t^{*(T)} \geq 0$ for all t = 0, …, T-1> must be the unique solution of the difference equation $\alpha_{t-1} - a^{(t)}\alpha_t = p^{(t)}$, $\alpha_{t^*-1}^{*(T)} = 0$.

Let $<\alpha_t^* | \alpha_t^* \geq 0$, t∈ℕ⁰> be the unique solution of this system of difference equation satisfying the condition $\alpha_{t^*-1} = 0$.

Since the optimal value of the primal must be equal to the optimal value of its dual, it must be the case that for all T∈ ℕ with T ≥ max{3, t*}, $<\alpha_t^* | \alpha_t^* \geq 0$, t = 0, 1, …, T-1> solves

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all t = 1, …, T-1, $\alpha_t^{(T)} \geq 0$ for all t = 0, …, T-1.

Hence, for all T∈ ℕ with T ≥ max{3, t*} it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$. Q.E.D.

Note 6.1: We know from proposition 6.1 that $\alpha_{t^*-1}^* = 0$ and for all t∈ℕ, $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$. We have assumed for proposition 6.1 that $a^{(t)} \neq 0$ for all t∈ℕ. Thus, $\alpha_{t^*-1}^* = 0$ and $\alpha_t^* = \frac{\alpha_{t-1}^* - p^{(t)}}{a^{(t)}}$ for all t∈ℕ. If the 1-IHLP problem is a discounted 1-IHLP problem with $p^{(t)} = \delta^t$ for all t∈ℕ and δ∈(0, 1), then $\delta^t = \alpha_{t-1}^* - a^{(t)}\alpha_t^*$ for all t∈ℕ.

Thus, if $<a^{(t)}|$ t∈ℕ⁰> converges to a∈ℝ with a ≠ 1, then it must be the case that $\lim_{T\to\infty} \alpha_T^* = 0$.

For x∈X, consider the alternative version of P1(x) defined in section 4:

Maximize $\sum_{t=1}^{\infty} p^{(t)}y_t$, subject to $y_1 \leq c^{(0)} + a^{(0)}x$, $y_{t+1} - a^{(t)}y_t \leq c^{(t)}$, $y_t \geq 0$ for all $t\epsilon\mathbb{N}$.

This is the maximization problem that we are really concerned with.

Its "**implied dual linear programming (IDLP) problem**" is the following:

Minimize $\alpha_0(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t c^{(t)}$, subject to $\alpha_{t-1} - a^{(t)}\alpha_t \geq p^{(t)}$, for all $t\epsilon\mathbb{N}$ and $\alpha_t \geq 0$, $t\epsilon\mathbb{N}^0$.

Theorem 6.1: Suppose $<(p^{(t)},c^{(t)},a^{(t)})| t\in \mathbb{N}^0>$ is a 1-IHLP satisfying $c^{(t)}$, $c^{(t)} + a^{(t)}b > 0$, for all $t\epsilon\mathbb{N}^0$ and $a^{(t)} \neq 0$ for all $t\epsilon\mathbb{N}$. Suppose that for some $x\epsilon X$, $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$, satisfies the following "**interiority condition**": For all $t\epsilon\mathbb{N}$, $x_t > 0$ and there exists $t^*\epsilon\mathbb{N}$ such that $x_{t^*} < c^{(t^*-1)} + a^{(t^*-1)}x_{t^*-1}$.

Then, there exists a sequence $<\alpha_t^*|\alpha_t^* \geq 0, t\epsilon\mathbb{N}^0>$ satisfying $\alpha_{t^*-1}^* = 0$ and $\alpha_{t-1}^* - a^{(t)}\alpha_t^* = p^{(t)}$ for all $t\in \mathbb{N}$ such that:

(1) $<\alpha_t^*| t = 0, 1, 2, \ldots, T-1>$ solves the following linear programming problem for all $T\in \mathbb{N}$ with $T \geq \max\{3, t^*\}$:

Minimize $\alpha_0^{(T)}(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^{(T)} c^{(t)} + \alpha_{T-1}^{(T)}(c^{(T-1)} - x_T)$, subject to $\alpha_{t-1}^{(T)} - a^{(t)}\alpha_t^{(T)} \geq p^{(t)}$ for all $t = 1, \ldots, T-1$, $\alpha_t^{(T)} \geq 0$ for all $t = 0, \ldots, T-1$.

(2) For all $T\in \mathbb{N}$ with $T \geq \max\{3, t^*\}$, $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

(3) $\sum_{t=1}^{\infty} p^{(t)}x_t = (c^{(0)} + a^{(0)}x)\alpha_0^* + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$ <u>if and only if</u> $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$.

Proof: (1) and (2) follow from proposition 6.1. Hence we need to prove (3).

From (2) of proposition 6.1 we know that for all $T\in \mathbb{N}$ with $T \geq \max\{3, t^*\}$ it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-2} \alpha_t^* c^{(t)} + \alpha_{T-1}^*(c^{(T-1)} - x_T) = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} - \alpha_{T-1}^* x_T$.

Thus, for all $T\in \mathbb{N}$ with $T \geq \max\{3, t^*\}$ it must be the case that $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Since, $<x_t|t\in \mathbb{N}^0>\in \mathcal{S}(x)$, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t$ exists and $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \sum_{t=1}^{\infty} p^{(t)}x_t$.

If $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$, then $\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T]$ exists and $\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T] = \lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t + \lim_{T\to\infty} \alpha_{T-1}^* x_T = \lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t$.

$\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T]$ exists implies $\alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ exists and hence $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ for all $T \geq \max\{3, t^*\}$ implies $\lim_{T\to\infty}[\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T] = \alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Thus, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$.

Since $\lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} = \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$ it follows that $\sum_{t=1}^{\infty} p^{(t)}x_t = \alpha_0^*(c^{(0)} + a^{(0)}x) + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$.

Conversely, if $\sum_{t=1}^{\infty} p^{(t)}x_t = (c^{(0)} + a^{(0)}x)\alpha_0^* + \sum_{t=1}^{\infty} \alpha_t^* c^{(t)} = \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$, then $\sum_{t=1}^{T-1} p^{(t)}x_t + \alpha_{T-1}^* x_T = (c^{(0)} + a^{(0)}x)\alpha_0^* + \sum_{t=1}^{T-1} \alpha_t^* c^{(t)}$ for all $T \in \mathbb{N}$ with $T \geq \max\{3, t^*\}$, $\lim_{T\to\infty} \sum_{t=1}^{T-1} p^{(t)}x_t = \sum_{t=1}^{\infty} p^{(t)}x_t$ and $\lim_{T\to\infty} \sum_{t=1}^{T-1} \alpha_t^* c^{(t)} = \sum_{t=1}^{\infty} \alpha_t^* c^{(t)}$ together imply $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$. Q.E.D.

Note 6.2: Since for all $T \in \mathbb{N}$, α_{T-1} is the dual (co-state) variable corresponding to x_T, $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$ is a version of the well-known "**transversality condition**" of optimal control theory. Part (3) of theorem 6.1 says that under the assumptions common to both proposition 6.1 and theorem 6.1, the optimal value of the IDLP problem is equal to the optimal value of the alternative version of P1(x) defined in section 4 "*if and only if*" the **transversality condition** (in the way defined here) is satisfied by the solutions of the two infinite horizon linear programming problems.

Note 6.3: If as in note 6.1, the 1-IHLP problem is a discounted 1-IHLP problem with $p^{(t)} = \delta^t$ for all $t \in \mathbb{N}$ and $\delta \in (0, 1)$, and $\langle a^{(t)} | t \in \mathbb{N}^0 \rangle$ converges to $a \in \mathbb{R}$ with $a \neq 1$, then it must be the case that $\lim_{T\to\infty} \alpha_T^* = 0$. Thus $\lim_{T\to\infty} \alpha_{T-1}^* x_T = 0$. Hence the transversality condition is satisfied for a discounted 1-IHLP problem satisfying the additional convergence criterion for $\langle a^{(t)} | t \in \mathbb{N}^0 \rangle$.

7. Generalized Infinite Horizon Linear Programming Problem with One-Dimensional State Variable: With a little bit of algebra along with a non-trivial assumption applied to the model of linear optimal control with linear constraints in Lahiri (2025c), we can obtain a generalization of a 1-IHLP problem. The additional assumption required is that in every period the coefficient of the control variable in the equation determining the evolution of the state variable is non-zero.

Suppose that for all $t \in \mathbb{N}^0$, there is a positive integer m_t, such that $A^{(t)}$, $C^{(t)}$, $D^{(t)}$ are m_t dimensional real-valued column vectors whose i^{th} entry for $i \in I(t) = \{1, \ldots, m(t)\}$ are

$A_i^{(t)}$, $C_i^{(t)}$ and $D_i^{(t)}$ respectively and satisfy the following condition: For all $i \in I(t)$, $D_i^{(t)} \neq 0$ and for all $x \in X$ and $i \in I(t)$, $0 \in \{y \in \mathbb{R} | D_i^{(t)} y - A_i^{(t)} x \leq C_i^{(t)}\} \subset X$.

Thus, for all $x \in X$ and $i \in I(t)$ it must be the case that $\{y \in \mathbb{R} | D_i^{(t)} y - A_i^{(t)} x \leq C_i^{(t)}\}$ is a closed interval in X whose left-hand end point is 0.

For $t \in \mathbb{N}^0$, let $\Omega_t = \{(x, y) \in X \times X | D_i^{(t)} y - A_i^{(t)} x \leq C_i^{(t)}$ for all $i \in I(t)\}$ and for $(x, t) \in X \times \mathbb{N}^0$, let $\Omega_t(x) = \{y \in X | (x, y) \in \Omega_t\}$.

For $x \in X$, let $\mathcal{F}(x) = \{<x_t | t \in \mathbb{N}^0> | \; x_{t+1} \in \Omega_t(x_t)$ for all $t \in \mathbb{N}^0$, $x_0 = x\}$

As before, let $<p^{(t)} | \; t \in \mathbb{N}^0>$ be a sequence in $\mathbb{R}$ satisfying $\sum_{t=0}^{\infty} |p^{(t)}| < +\infty$.

We shall refer to the sequence $<(p^{(t)}, A^{(t)}, C^{(t)}, D^{(t)}) | t \in \mathbb{N}^0>$ as a **generalized infinite horizon linear programming problem with one-dimensional state variable (G-1-IHLP problem**).

Given a G-1-IHLP problem $<(p^{(t)}, A^{(t)}, C^{(t)}, D^{(t)}) | t \in \mathbb{N}^0>$ and $x \in X$, we shall be concerned with the following optimization problem denoted by **G-P1**(x):

Maximize $\sum_{t=0}^{\infty} p^{(t)} x_t$, subject to $<x_t | t \in \mathbb{N}^0> \in \mathcal{F}(x)$.

Let $\mathcal{S}(x) = \underset{<x_t | t \in \mathbb{N}^0> \in \mathcal{F}(x)}{\operatorname{argmax}} \sum_{t=0}^{\infty} p^{(t)} x_t$.

An alternative version of G-P1(x) is the following:

Maximize $\sum_{t=1}^{\infty} p^{(t)} x_t$, subject to $<x_t | t \in \mathbb{N}^0> \in \mathcal{F}(x)$.

As in section 3 we have the following result.

Proposition 7.1: For all $x \in X$, $\mathcal{S}(x) \neq \phi$.

An immediate consequence of proposition 7.1, is that there exists a function V: X $\rightarrow$ $\mathbb{R}$ such that for all $x \in X$: $V(x) = \sum_{t=0}^{\infty} p^{(t)} x_t$, where $<x_t | t \in \mathbb{N}^0> \in \mathcal{S}(x)$.

As in section 4, the following result can be obtained from corresponding results in Lahiri (2025c).

Proposition 7.2: Let $<(p^{(t)}, A^{(t)}, C^{(t)}, D^{(t)}) | t \in \mathbb{N}^0>$ be a G-1-IHLP problem and suppose that for some $x \in X$, $<x_t | t \in \mathbb{N}^0> \in \mathcal{F}(x)$.

Part 1: If $<x_t | t \in \mathbb{N}^0> \in \mathcal{S}(x)$ then for all $T \in \mathbb{N}$, $<x_t | t = 0, 1, \ldots, T>$ solves the following linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $D_i^{(t)} y_{t+1} - A_i^{(t)} y_t \leq C_i^{(t)}$ for all $i \in I(t)$ and $t = 0, 1, \ldots, T-1$, $y_0 = x_0 = x$ and $y_T = x_T$.

Part 2: If there exists $T^* \in \mathbb{N}$ such that for all $T \in \mathbb{N}$ satisfying $T \geq T^*$, $<x_t | t = 0, 1, \ldots, T>$ solves the linear programming problem: Maximize $\sum_{t=0}^{T} p^{(t)} y_t$, subject to $D_i^{(t)} y_{t+1} - A_i^{(t)} y_t \leq C_i^{(t)}$ for all $i \in I(t)$ and $t = 0, 1, \ldots, T-1$ and $y_0 = x_0 = x$, then $<x_t | t \in \mathbb{N}^0> \in \mathcal{S}(x)$.

The approximation result in section 4 continues to hold for a G-1-IHLP problem and hence we have the following.

Proposition 7.3: Given the G-1-IHLP problem $<(p^{(t)}, A^{(t)}, C^{(t)}, D^{(t)})|t\in\mathbb{N}^0>$ and $x\in X$, there exists $T^*(\varepsilon)\in\mathbb{N}$ such that for all $T\in\mathbb{N}$ with $T \geq T^*(\varepsilon)$, the linear programming problem [Maximize $\sum_{t=0}^{T} p^{(t)}y_t$, subject to $D_i^{(t)}y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all $i\in I(t)$ and $t = 0, 1, \ldots, T-1$ and $y_0 = x_0 = x$] has a solution $<x_t^{(T)}| t = 0, 1, \ldots, T>$ and $|\sum_{t=0}^{T} p^{(t)}x_t^{(T)} - V(x)| < \varepsilon$.

For $x\in X$, let $<x_t|t\in \mathbb{N}^0> \in\mathcal{F}(x)$.

For $T\in \mathbb{N}$ with $T \geq 3$ the linear programming problem in part 1 of proposition 7.2 is equivalent to the following linear programming problem:

Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $D_i^{(0)}y_1 \leq C_i^{(0)} + A_i^{(0)}x$ for all $i\in I(0)$, $D_i^{(t)}y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all $i\in I(t)$ and $t = 1, \ldots, T-2$, $-A_i^{(T-1)}y_{T-1} \leq C_i^{(T-1)} - D_i^{(T-1)}x_T$ for all $i\in I(T-1)$, $y_t \geq 0$ for all $t = 1, \ldots, T-1$.

The dual of this linear programming problem is the following linear programming minimization problem.

Minimize $\sum_{i\in I(0)} \beta_i^{(T,0)}(C_i^{(0)} + A_i^{(0)}x) + \sum_{t=1}^{T-2} (\sum_{i\in I(t)} \beta_i^{(T,t)} C_i^{(t)}) + \sum_{i\in I(T-1)} \beta_i^{(T,T-1)}(C_i^{(T-1)} - D_i^{(T-1)}x_T)$ subject to $\sum_{i\in I(t-1)} \beta_i^{(T,t-1)} D_i^{(t-1)} - \sum_{i\in I(t)} \beta_i^{(T,t)} A_i^{(t)} \geq p^{(t)}$ for all $t = 1, \ldots, T-1$, $\beta_i^{(T,t)} \geq 0$ for all $i\in I(t)$ and $t = 0, 1, \ldots, T-1$.

Thus, even if we assume that $x_t > 0$ for all $t\in\mathbb{N}^0$, the system of difference equations that emerged almost immediately from the application of duality theorem and complementary slackness conditions in the proof of proposition 6.1, fail to do so once we move into the more general framework of a G-1-IHLP.

However, we can obtain a somewhat weaker version of theorem 6.1 in this considerably more general context and with less restrictive conditions on both the problem as well as the optimal solution, if we pursue the line of reasoning in section 6.

Note that for $T\in \mathbb{N}$ with $T \geq 3$, the linear programming problem [Maximize $\sum_{t=1}^{T-1} p^{(t)}y_t$, subject to $D_i^{(0)}y_1 \leq C_i^{(0)} + A_i^{(0)}x$ for all $i\in I(0)$, $D_i^{(t)}y_{t+1} - A_i^{(t)}y_t \leq C_i^{(t)}$ for all $i\in I(t)$ and $t = 1, \ldots, T-2$,, $y_t \geq 0$ for all $t = 1, \ldots, T-1$] is the sequence of "free end-point" linear programming programming problems that was used for the approximation result i.e. proposition 7.3.

Its dual is the linear programming problem [Minimize $\sum_{i\in I(0)} \beta_i^{(T,0)}(\mathrm{C}_\mathrm{i}^{(0)} + \mathrm{A}_\mathrm{i}^{(0)}\mathrm{x}) + \sum_{t=1}^{T-2} (\sum_{i\in I(t)} \beta_i^{(T,t)}\mathrm{C}_\mathrm{i}^{(\mathrm{t})})$ subject to $\sum_{i\in I(t-1)} \beta_i^{(T,t-1)}\mathrm{D}_\mathrm{i}^{(\mathrm{t}-1)} - \sum_{i\in I(t)} \beta_i^{(T,t)}\mathrm{A}_\mathrm{i}^{(\mathrm{t})} \geq \mathrm{p}^{(\mathrm{t})}$ for all t = 1, …, T-2, $\beta_i^{(T,t)} \geq 0$ for all i϶I(t) and t = 0, 1, …, T-2].

Theorem 7.1: Let <($p^{(t)}$, $A^{(t)}$, $C^{(t)}$, $D^{(t)}$)|t϶$\mathbb{N}^0$> be a G-1-IHLP problem and suppose that for some x϶X, <x_t|t∈ $\mathbb{N}^0$>∈ $\mathcal{S}$(x). Then for all T϶ℕ with T ≥ 3, for each t = 0, 1, 2, …, T-1, and for each i϶I(t) there exists $\beta_i^{*(T,t)} \geq 0$ satisfying the following conditions:

(G-1) The array <$\beta_i^{*(T,t)}$|i϶I(t), t = 0, …, T-1> solves:

Minimize $\sum_{i\in I(0)} \beta_i^{(T,0)}(\mathrm{C}_\mathrm{i}^{(0)} + \mathrm{A}_\mathrm{i}^{(0)}\mathrm{x}) + \sum_{t=1}^{T-2} (\sum_{i\in I(t)} \beta_i^{(T,t)}\mathrm{C}_\mathrm{i}^{(\mathrm{t})}) + \sum_{i\in I(T-1)} \beta_i^{(T,T-1)}(\mathrm{C}_\mathrm{i}^{(\mathrm{T}-1)} - \mathrm{D}_\mathrm{i}^{(\mathrm{T}-1)}\mathrm{x}_\mathrm{T})$ subject to $\sum_{i\in I(t-1)} \beta_i^{(T,t-1)}\mathrm{D}_\mathrm{i}^{(\mathrm{t}-1)} - \sum_{i\in I(t)} \beta_i^{(T,t)}\mathrm{A}_\mathrm{i}^{(\mathrm{t})} \geq \mathrm{p}^{(\mathrm{t})}$ for all t = 1, …, T-1, $\beta_i^{(T,t)} \geq 0$ for all i϶I(t) and t = 0, 1, …, T-1.

(G-2) $\sum_{i\in I(0)} \beta_i^{*(T,0)}(\mathrm{C}_\mathrm{i}^{(0)} + \mathrm{A}_\mathrm{i}^{(0)}\mathrm{x}) + \sum_{t=1}^{T-2} (\sum_{i\in I(t)} \beta_i^{*(T,t)}\mathrm{C}_\mathrm{i}^{(\mathrm{t})}) + \sum_{i\in I(T-1)} \beta_i^{*(T,T-1)}(\mathrm{C}_\mathrm{i}^{(\mathrm{T}-1)} - \mathrm{D}_\mathrm{i}^{(\mathrm{T}-1)}\mathrm{x}_\mathrm{T}) = \sum_{t=1}^{T-1} \mathrm{p}^{(\mathrm{t})}\mathrm{x}_\mathrm{t}$.

(G-3) $\sum_{t=1}^{\infty} \mathrm{p}^{(\mathrm{t})}\mathrm{x}_\mathrm{t} = \lim_{T\to\infty} [\sum_{i\in I(0)} \beta_i^{*(T,0)}(\mathrm{C}_\mathrm{i}^{(0)} + \mathrm{A}_\mathrm{i}^{(0)}\mathrm{x}) + \sum_{t=1}^{T-1} (\sum_{i\in I(t)} \beta_i^{*(T,t)}\mathrm{C}_\mathrm{i}^{(\mathrm{t})})]$ <u>if and only if</u> $\lim_{T\to\infty} (\sum_{i\in I(T-1)} \beta_i^{*(T,T-1)}\mathrm{D}_\mathrm{i}^{(\mathrm{T}-1)})\mathrm{x}_\mathrm{T} = 0$ (i.e., <u>a</u> transversality condition is satisfied).

What the theorem 7.1 says is, without any additional assumption (such as for all (x, t)϶X×$\mathbb{N}^0$, {0} is a proper subset of Ω_t(x) as required in sections 5 and 6, or "interiority condition" as in section 6), the optimal value of the dual of the truncated "free end-point" linear programming problem with the initial value of the state variable in the latter being the same as the initial value of the state variable in the optimal trajectory(i.e., the linear programming problems in the "approximation result" proposition 7.3), converges to the optimal value of the G-1-IHLP, if and only if a reasonable and simple transversality condition is satisfied. This transversality condition says that the product of the state variable and a weighted sum of the dual variables of the constraints that it is required to satisfy, converges to zero. The weights for the dual variables (unlike traditional weights) are only required to be non-zero.

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