ON DUAL EXTREMAL VARIATIONAL PRINCIPLES FOR GEOMETRICALLY NONLINEAR COMPOSITE PLATES

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Thin composite geometrically nonlinear plates are under consideration. The plates are linearly elastic, obeying the Kirchhoff straight normal hypothesis and the classical theory of laminated plates. The layer fibers may have different orientation angles depending on the local position of a point in the midplane. The deformation of the plates obeys the von Karman approximation. The kinematic and static variational principles for the plates are considered. The proven statement is: for everywhere positive principal in-plane (membrane) force resultants, the principles lead to a minimum and a maximum, respectively. The duality gap for the considered variational principles is absent. The sum of the total plate strain potential energy and the total complementary energy is equal to the potential of external forces. For the composite plates, a generalization of the Clapeyron theorem of the linear theory of elasticity is proven. A comparison of the behavior of geometrically linear and geometrically nonlinear composite plates (under the same loading) suggests greater generalized stiffness and lower generalized compliance of the latter plate. An example of a composite plate illustrates the variational principles discussed and their error bounds. The results of the paper are applicable to the analysis and design of composite structures, such as lower panels of large-span composite wings.

Keywords: composite plate, geometric nonlinearity, von Karman approach, curvilinear layer fiber lay-up, extremal variational principles

1. Introduction.

Composite structures, particularly plates, rods, beams, shells, complicated structures, attract considerable attention from engineers and researchers due to their widespread use in aerospace, shipbuilding, marine, and construction applications.

Since the 1990s, as composite manufacturing technologies have improved, the potential for creating structures with curved fiber lay-up (referred to in scientific and technical literature as steered fiber or variable-angle tow or VAT structures) has been intensively studied. Several recent studies in this area include [1-4], which demonstrate active theoretical and computational research, as well as experimental studies of the behavior of structures with curved fiber lay-up, including panels and aircraft wing sections. These studies have demonstrated the high efficiency of such structures. Thus, in [3], it was found through calculations and experiments that the use of curved layer fibers leads to an increase in the plate load-bearing capacity by several tens of percent compared to a design with straight layer fibers.

As is well known, variational principles form the foundation of many numerical methods for the analysis and design of structures, particularly the currently widely used finite element method [5]. Therefore, the calculation and design of plates with curved layer placement requires the development of appropriate variational principles, preferably of the extremal type.

Elongated composite structures, being more flexible than metal structures of the same dimensions, require mandatory consideration of geometric nonlinearity in the variational principles used.

We highlight the most important works devoted to variational principles for geometrically nonlinear structures and related in one way or another to the topic of this paper.

Thus, the most important studies of the time on variational principles for isotropic nonlinear (physically and geometrically) plates and shells are listed in [6]. In particular, dual variational principles for isotropic von Karman plates are described. It is indicated that the dual kinematic and static variational principles (using the displacement gradient tensor and the first Piola stress tensor) for such plates, under positive principal membrane stresses, are the minimum and maximum principles, respectively. Full proof of the latter assertion is not provided, but a method for its implementation is indicated, the method is to prove the positive definiteness of a certain 5x5 matrix via its LU decomposition into a product of matrices

In [7], variational principles for finite elastic displacements of an isotropic body are formulated in terms of the Green strain tensor and the Kirchhoff-Treftz stress tensor. The first considered principle deals with the functional of the displacement field alone. The second considered principle involves both stresses and displacements. The derivation of these principles utilizes the polar Jacobian expansion theorem, the engineering strain tensor and its conjugate stress tensor to be regarded as a function of the Piola stress tensor and the material rotation. The formulation of the complementary energy principle is discussed in terms of first-order stress functions.

In [8], a static variational principle (stationarity of the total additional energy) was proven for post-buckled thin von Karman composite plates with curved layer fibers. The results of the latter paper can also be applied to geometrically nonlinear composite plates deformed in the von Karman approximation.

This paper deals with the derivation and theoretical analysis of extremal variational principles for statically loaded geometrically nonlinear composite plates with curvilinear fiber lay-up, deformed under the condition of positive principal in-plane (membrane) force resultants, in accordance with the von Karman approximation. The variational principles considered in this paper are an extension of the variational principles of the stationary type of paper [8]. It is shown what energy differences in the behavior of structures, compared to a geometrically linear approach, result from taking geometric nonlinearity into account. The results of the study are applicable to the analysis and design of composite structures, such as the lower panel of a high-aspect-ratio wing of a passenger aircraft.

The paper consists of the Introduction and seven Sections.

Section 2 presents the main assumptions.

Section 3 discusses the kinematic variational principle and derives the conditions for the total plate potential energy functional minimization.

Section 4 examines the static variational principle and discusses the conditions of the variational functional maximization.

Section 5 is devoted to determining the dual gap for the variational principles discussed. A generalization of the Clapeyron theorem, well-known from linear elasticity theory, is also proven in the case of geometrically nonlinear composite plates.

Section 6 compares the total potential and complementary energies of the plate calculated using geometrically linear and geometrically nonlinear theory.

Section 7 presents an illustrative example of the use of the discussed variational principles, the example contains calculations of the error bounds for admissible displacement fields and internal forces.

Section 8 provides the conclusions of the paper.

2. **Assumptions**.

Thin flat composite plate of constant thickness h is considered, composed of 2K symmetrically stacked locally orthotropic layers of equal thickness (a generalization to an odd number of layers may be easily accomplished if necessary). A midplane layer of honeycomb material can also be used.

The plate is subjected to distributed dead pressure load q(x,y) (x,y are coordinates in the midplane of the plate) and in-plane (membrane) force resultants along the external contour. It is assumed that deformation occurs in accordance with the Kirchhoff hypothesis of straight normals and the von Karman approximation (see [9]).

The plate thickness is much smaller than any of its linear dimensions.

Figure 1 illustrates the Cartesian coordinate system XYZ, the midplane Γ of the plate, bounded by the smooth contour C (consisting of parts C_1 and C_2), and the normal and tangent vectors to this contour. The in-plane (membrane) force resultants $\bar{N}_{x\nu}$ and $\bar{N}_{y\nu}$, acting along the *x, y* plane on C_1, are specified, while zero displacements u, v, w along the *x, y*, *z* planes, respectively, are specified on C_2. Clamping boundary conditions along the contour of C_2 are assumed.

The fibers of the composite layers are assumed to be curved and laid in a smooth manner.

Due to the adopted Kirchhoff hypothesis on the straight normal to the deformable surface Γ, the components *xz, yz*, and *zz* of the Green strain tensor are equal to zero.

The summation convention over repeating indices for tensors and vectors is used.

The index after the decimal point denotes differentiation with respect to the variable corresponding to that index.

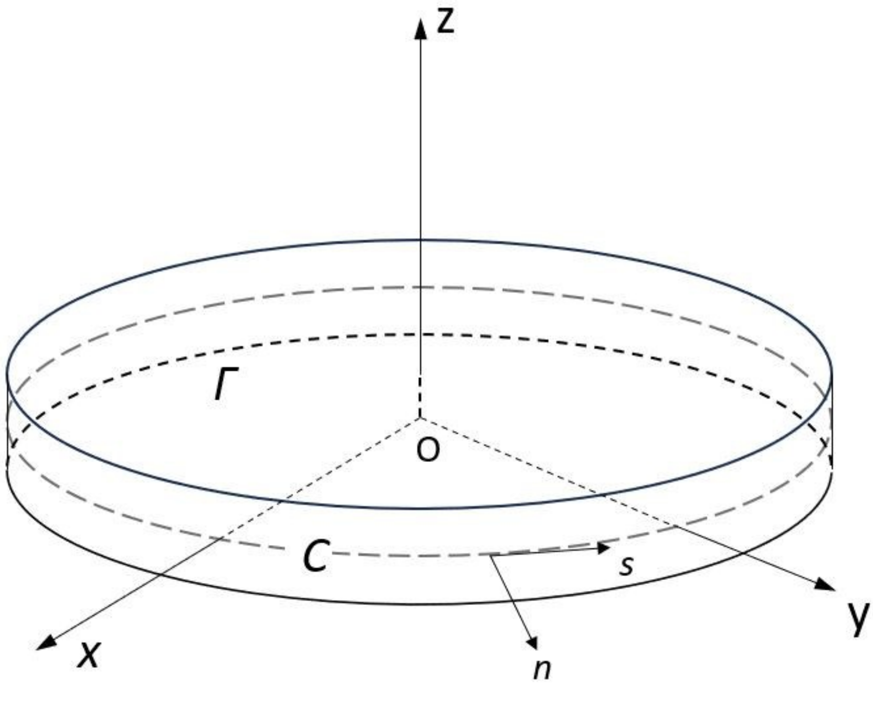

Figure 1. Thin plate.

The components of displacements $u_i, i = 1,2,3$, the components of the Green strain tensors ε_{ij} and the gradients of displacements $u_{i,j}, i,j = 1,2,3$, of points inside the plate as functions of *x, y, z* of displacements u, v, w in the midplane are written as:

$$\begin{cases} u_1 = u - z\dfrac{\partial w}{\partial x} \\ u_2 = v - z\dfrac{\partial w}{\partial y} \\ \quad u_3 = w \end{cases} \tag{2.1}$$

$$\begin{cases} \varepsilon_{xx} = u_{,x} + \frac{1}{2}\left(w_{,x}\right)^2 - zw_{,xx} \\ \varepsilon_{yy} = v_{,y} + \frac{1}{2}\left(w_{,y}\right)^2 - zw_{,yy} \\ 2\varepsilon_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y} - 2zw_{,xy} \end{cases} \tag{2.2}$$

$$u_{i,j} = \begin{cases} u_{,x} - zw_{,xx}; & u_{,y} - zw_{,xy;} & -w_{,x} \\ v_{,x} - zw_{,xy}; & v_{,y} - zw_{,yy}; & -w_{,y} \\ w_{,x}; & w_{,y}; & 0 \end{cases} \tag{2.3}$$

The following column vectors are introduced:

$$\boldsymbol{\varepsilon_0} = \left.\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{pmatrix}\right|_{z=0} \quad ; \; \boldsymbol{k} = \left.\begin{pmatrix} -\partial^2 w/\partial x^2 \\ -\partial^2 w/\partial y^2 \\ -2\,\partial^2 w/\partial x\partial y \end{pmatrix}\right|_{z=0} \tag{2.4}$$

The column vectors of the in-plane (membrane) force resultants $\boldsymbol{N}$ and the moment resultants $\boldsymbol{M}$ are also introduced [10]:

$$\boldsymbol{N} = \begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} = \begin{pmatrix} \int_{-h/2}^{h/2} \sigma_x dz \\ \int_{-h/2}^{h/2} \sigma_y dz \\ \int_{-h/2}^{h/2} \sigma_{xy} dz \end{pmatrix} \; ; \; \boldsymbol{M} = \begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} \int_{-h/2}^{h/2} \sigma_x z dz \\ \int_{-h/2}^{h/2} \sigma_y z dz \\ \int_{-h/2}^{h/2} \sigma_{xy} z dz \end{pmatrix} \tag{2.5}$$

where $\sigma_x, \sigma_y, \sigma_{xy}$ are the components of the Kirchhoff stress tensor. The components, as is known [8], are equal to the same components of the first Piola stress tensor. The integrals (2.5) of $\sigma_x, \sigma_y, \sigma_{xy}$ are expressed through the strains, curvatures, membrane stiffness and bending stiffness matrices of the composite plate $\boldsymbol{A}$ and $\boldsymbol{D}$ [10], respectively:

$$\boldsymbol{N} = \boldsymbol{A\varepsilon_0} \quad ; \quad \boldsymbol{M} = \boldsymbol{Dk} \tag{2.6}$$

3. Kinematic variational principle.

As is known [9, 11], the kinematics of deformation of the conservatively loaded plate under consideration is described by the kinematic variational principle. In accordance with this principle, the total plate potential energy *U* equal to the difference between the functionals of the total potential energy of deformation of the plate Π and the potential of external forces *W* is stationary with respect to the *x, y, z* displacements *u, v, w* of the points of the plate. We have

$$\delta U = \delta\Pi - \delta W = 0 \tag{3.1}$$

(δ is the symbol of variation), where the total plate strain potential energy reads

$$\Pi = \iint \left(\frac{1}{2}\boldsymbol{\varepsilon_0^T A \varepsilon_0} + \frac{1}{2}\boldsymbol{k^T D k}\right) d\Gamma \tag{3.2}$$

T is the transposition symbol, and the potential of external forces reads

$$W = \iint qw d\Gamma + \int dC_1 \left[\bar{N}_{xv} u + \bar{N}_{yv} v\right] \tag{3.3}$$

The integration in (3.2), (3.3) is performed over the whole undeformed mid-surface Γ.

The functional of the total plate potential energy *U* reads:

$$U=\frac{1}{2}\iint d\Gamma[\boldsymbol{\varepsilon}_{\mathbf{0}}^{T}\boldsymbol{A}\boldsymbol{\varepsilon}_{\mathbf{0}}+\boldsymbol{k}^{T}\boldsymbol{D}\boldsymbol{k}]-\iint qwd\Gamma-\int dC_{1}\left[\bar{N}_{xv}u+\bar{N}_{yv}v\right] \tag{3.4}$$

where the last integral is calculated over the contour C_1. As is known, the first variation of the functional (3.4) with respect to the variables *u, v, w* leads to the equilibrium equations with respect to the corresponding degrees of freedom. We write the variation as:

$$\delta U=\iint d\Gamma[\boldsymbol{\varepsilon}_{\mathbf{0}}^{T}\boldsymbol{A}\delta\boldsymbol{\varepsilon}_{\mathbf{0}}+\boldsymbol{k}^{T}\boldsymbol{D}\delta\boldsymbol{k}]-\iint q\delta wd\Gamma-\int dC_{1}\left[\bar{N}_{xv}\delta u+\bar{N}_{yv}\delta v\right] \tag{3.5}$$

Let us calculate the second variation from (3.4) taking into account (3.5).

$$\delta^{2}U=\frac{1}{2}\iint d\Gamma[\delta\boldsymbol{\varepsilon}_{\mathbf{0}}^{T}\boldsymbol{A}\delta\boldsymbol{\varepsilon}_{\mathbf{0}}+\boldsymbol{\varepsilon}_{\mathbf{0}}^{T}\boldsymbol{A}\delta^{2}\boldsymbol{\varepsilon}_{\mathbf{0}}+\delta\boldsymbol{k}^{T}\boldsymbol{D}\delta\boldsymbol{k}+\boldsymbol{k}^{T}\boldsymbol{D}\delta^{2}\boldsymbol{k}] \tag{3.6}$$

We consider in the fourth term on the right in (3.6) the column vector $\boldsymbol{k}$, which is linear in the second derivatives of the deflections, and its second variation. We have:

$$\delta^{2}\begin{pmatrix}-w_{,xx}\\ -w_{,yy}\\ -2w_{,xy}\end{pmatrix}=\begin{pmatrix}-\partial^{2}/\partial x^{2}\\ -\partial^{2}/\partial y^{2}\\ -2\,\partial^{2}/\partial x\partial y\end{pmatrix}\delta^{2}w \tag{3.7}$$

Since *w* is an independent kinematic variable, its second variation is identically zero. Consequently, the fourth term in (3.6) is zero.

Next, consider the second term on the right in (3.6). We have:

$$\boldsymbol{\varepsilon}_{\mathbf{0}}^{T}\boldsymbol{A}\delta^{2}\boldsymbol{\varepsilon}_{\mathbf{0}}=\boldsymbol{N}^{T}\delta^{2}\begin{pmatrix}u_{,x}+\frac{1}{2}(w_{,x})^{2}\\ v_{,y}+\frac{1}{2}(w_{,y})^{2}\\ u_{,y}+v_{,x}+w_{,x}w_{,y}\end{pmatrix}=\boldsymbol{N}^{T}\delta^{2}\begin{pmatrix}u_{,x}\\ v_{,y}\\ u_{,y}+v_{,x}\end{pmatrix}+\boldsymbol{N}^{T}\delta^{2}\begin{pmatrix}\frac{1}{2}(w_{,x})^{2}\\ \frac{1}{2}(w_{,y})^{2}\\ w_{,x}w_{,y}\end{pmatrix} \tag{3.8}$$

By reasons analogous to those outlined in the previous paragraph, the first term on the right in (3.8) is zero. Using these considerations to transform the second term in (3.8), we obtain:

$$\boldsymbol{\varepsilon}_{\mathbf{0}}^{T}\boldsymbol{A}\delta^{2}\boldsymbol{\varepsilon}_{\mathbf{0}}=\boldsymbol{N}^{T}\begin{pmatrix}(\delta w_{,x})^{2}\\ (\delta w_{,y})^{2}\\ 2\delta w_{,x}\delta w_{,y}\end{pmatrix} \tag{3.9}$$

It is easy to see that, if both principal in-plane (membrane) force resultants are positive at each point of the midplane of the plate, then in the coordinate system coinciding with the directions of these forces, the value (3.9) is positive.

The matrices $\boldsymbol{A}$ and $\boldsymbol{D}$ are positively definite [9, 10]. Consequently, the second variation of the functional of the total potential energy *U* is positive, and the functional *U* of the plate in equilibrium under conditions of everywhere positive principal in-plane (membrane) force resultants reaches a minimum.

4. Static variational principle.

Now we proceed with the consideration of the static variational principle for the von Karman composite plate.

In [8], a complementary variational principle for composite von Karman plate subjected to post-buckling is presented. The results presented in that paper are also applicable to the case considered in the present paper (the only addition is that in the present paper, the kinematic variational principle for the functional of the total potential energy considers the potential of the plate deflection under the pressure *q* of the form $\iint qwd\Gamma$).

Hereinafter, the *x* and *y* components of the external normal n_x, n_y to the undeformed boundary contour are used (the direction along the external normal to the boundary contour is *n*, and the tangential direction to this contour is *s*), as well as the relations

$$\begin{cases} \frac{\partial}{\partial x} = n_x \frac{\partial}{\partial n} - n_y \frac{\partial}{\partial s} \\ \frac{\partial}{\partial y} = n_y \frac{\partial}{\partial n} + n_x \frac{\partial}{\partial s} \end{cases} \tag{4.1}$$

Let us write, in accordance with [8], the geometric boundary conditions on C_2, namely, the values of the displacements and the normal derivative of the deflection (marked with a line above) in the form:

$$u = \bar{u}\ ,\ v = \bar{v}\ ,\ w = \bar{w}\ ,\ w_{,n} = \bar{w}_{,n} \tag{4.2}$$

and the boundary condition on the contour C_1

$$\begin{cases} N_{x\nu} - \bar{N}_{x\nu} = 0\ ;\ N_{y\nu} - \bar{N}_{y\nu} = 0 \\ V_z + M_{\nu s,s} - \bar{F}_z - \bar{M}_{\nu s,s} = 0\ ;\ M_\nu - \bar{M}_\nu = 0 \end{cases} \tag{4.3}$$

where the derivative with respect to *s* in (4.1), (4.3) means the tangential derivative along the boundary contour, and on the contour the quantities are given: the *z*-force $\bar{F}_z$ and the normal bending moment $\bar{M}_{\nu s,s}$. Also, in (4.3) the notations are used [5, 8]:

$$\begin{array}{lcl} N_{x\nu} = n_x N_x + n_y N_{yx} & & M_{x\nu} = n_x M_x + n_y M_{yx} \\ N_{y\nu} = n_x N_{xy} + n_y N_y & ; & M_{y\nu} = n_x M_{xy} + n_y M_y \\ N_{z\nu} = n_x N_{xz} + n_y N_{yz} & & M_\nu = n_x M_{x\nu} + n_y M_{y\nu} \end{array}$$

$$V_z = \left(M_{x,x} + M_{xy,y}\right)n_x + \left(M_{xy,x} + M_{y,y}\right)n_y + N_{z\nu}$$
$$M_{\nu s} = -M_{x\nu}n_x + M_{y\nu}n_y \tag{4.4}$$

and N_{xz} , N_{yz} are the quantities of integrals over the plate thickness of the *xz* , *yz* components of the first Piola stress tensor.

The complementary variational functional U_c, according to [8], reads

$$U_c = -\int_\Gamma \pi_c d\Gamma + \int_{C_2}\left[N_{x\nu}\bar{u} + N_{y\nu}\bar{v} + \left(V_z + M_{\nu s,s}\right)\bar{w} - M_\nu \bar{w}_{,n}\right] dC_2 - M_{\nu s}\bar{w}|_{C_2} \tag{4.5}$$

where π_c is the complementary energy density per unit area of the undeformed plate. As usual, it is necessary to add to the functional the products of the equilibrium equations

$$\begin{gathered} N_{x,x} + N_{xy,y} = 0 \\ N_{xy,x} + N_{y,y} = 0 \\ M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + N_{xz,x} + N_{yz,y} + q = 0 \end{gathered} \tag{4.6}$$

and the Lagrange multipliers $\alpha_x, \alpha_y, \alpha_z$.

The complementary energy density per unit area of the undeformed plate π_c in (4.5) is given by the expression:

$$\pi_c = \frac{1}{2}\boldsymbol{N}^T\boldsymbol{A}^{-1}\boldsymbol{N} + \frac{1}{2}\boldsymbol{M}^T\boldsymbol{D}^{-1}\boldsymbol{M} + \frac{1}{2}N_{xz}^2\frac{N_y}{N_xN_y - N_{xy}^2} + \frac{1}{2}N_{yz}^2\frac{N_x}{N_xN_y - N_{xy}^2} - \frac{N_{xz}N_{yz}N_{xy}}{N_xN_y - N_{xy}^2} \tag{4.7}$$

or

$$\pi_c = \frac{1}{2}\boldsymbol{N}^T\boldsymbol{A}^{-1}\boldsymbol{N} + \frac{1}{2}\boldsymbol{M}^T\boldsymbol{D}^{-1}\boldsymbol{M} + \frac{1}{2}N_{xz}\gamma_1(N_{ij}) + \frac{1}{2}N_{yz}\gamma_2(N_{ij}) \tag{4.8}$$

where:

$$\gamma_1(N_{ij}) = \frac{N_{yy}N_{xz} - N_{xy}N_{yz}}{N_{xx}N_{yy} - N_{xy}^2} \tag{4.9}$$

$$\gamma_2(N_{ij}) = \frac{N_{xx}N_{yz} - N_{xy}N_{xz}}{N_{xx}N_{yy} - N_{xy}^2} \tag{4.10}$$

The first variation of the functional (4.5) with respect to the forces and moments reads:

$$\delta U_c = -\int_\Gamma \begin{bmatrix} (\delta \boldsymbol{N}^T)\boldsymbol{A}^{-1}\boldsymbol{N} + (\delta \boldsymbol{M}^T)\boldsymbol{D}^{-1}\boldsymbol{M} - \left(\frac{1}{2}\gamma_1^2 + \alpha_{x,x}\right)\delta N_{xx} - \left(\frac{1}{2}\gamma_2^2 + \alpha_{y,y}\right)\delta N_{yy} - \\ -(\gamma_1\gamma_2 + \alpha_{x,y} + \alpha_{y,x})\delta N_{xy} + (\gamma_1 - \alpha_{z,x})\delta N_{xz} + (\gamma_2 - \alpha_{z,y})\delta N_{yz} + \\ +\alpha_{z,xx}\delta M_{xx} + \alpha_{z,yy}\delta M_{yy} + 2\alpha_{z,xy}\delta M_{xy} \end{bmatrix} d\Gamma$$

$$- \int_{C_2}\left[(\alpha_x - \bar{u})\delta N_{x\nu} + (\alpha_y - \bar{v})\delta N_{y\nu} + (\alpha_z - \bar{w})\delta(V_z + M_{\nu s,s}) - (\alpha_{z,n} - \bar{w}_{,n})\delta M_\nu\right] dC_2 - (\alpha_z - \bar{w})\delta M_{\nu s}|_{C_2} = 0 \tag{4.11}$$

It is proven in the paper [8] that the variation is equal to zero in the actual state.

In general, proving the extremum of the functional (4.5) with respect to force factors requires demonstrating either the positivity or negativity of the second variation of this functional. However, in the case considered here, another proof method is possible.

As shown in [6] for an isotropic plate, when the strain potential energy density is convex (that is the case under consideration, see Section 3), the functional U_c is conjugate to the functional U, and the volume density of the strain potential energy is related to the volume density of the complementary energy by the Legendre transformation. As a result of this (see also [12]), minimizing U with respect to displacements (more precisely, with respect to the tensor of displacement gradients) is a problem dual to maximizing U_c with respect to the first Piola stress tensor. Consequently, in the problem considered in this paper, the complementary variational functional U_c is dual to the functional of the total potential energy U minimized with respect to displacements and has a maximum with respect to in-plane (membrane) force resultants and moment resultants (depending on the integrals over the plate thickness of the corresponding components of the first Piola stress tensor) in the actual stress state.

Thus, the total complementary energy of the plate (entering with a minus sign into the complementary variational functional U_c), under the assumptions made and the equilibrium equations satisfied, has a minimum with respect to in-plane (membrane) force resultants and moment resultants in the actual stress state.

5. On duality gap.

We consider the sum of the total strain potential energy Π and the complementary energy Π_c of the plate. As is known [5, 6], for the volume densities of these energies p and p_c the following relation is valid:

$$p + p_c = \sigma_{ij}^p u_{j,i} \tag{5.1}$$

where σ_{ij}^p is the first Piola stress tensor, and $u_{j,i}$ is the displacement gradient tensor. We consider how the sum $\Pi + \Pi_c$ is related to the potential of external forces W given by the relation (3.3).

As stated above, the complementary energy density per unit area of the plate π_c is given by the formula (4.7).

Further in the Section we use the notations: $N_x^p, N_y^p, N_{xy}^p, N_{xz}^p, N_{yz}^p$ are the corresponding in-plane (membrane) force resultants calculated through the components of the first Piola stress tensor $\sigma_x^p, \sigma_y^p, \sigma_{xy}^p$ by integration over the plate thickness [8]. The in-plane (membrane) force resultants N_x, N_y, N_{xy} and N_x^p, N_y^p, N_{xy}^p (as well as the corresponding membrane stresses) are pairwise equal to each other and calculated according to formulas (2.5), using either the Kirchhoff stress tensor in the first case, or the first Piola stress tensor in the second case. The moment resultants M_x^p, M_y^p, M_{xy}^p and the column vector composed of them are calculated according to (2.5), where the components of the first Piola stress tensor are used instead of the Kirchhoff stress tensor components. Obviously, the moment resultants are equal to M_x, M_y, M_{xy}, respectively.

It is easy to see that, in case of geometrically nonlinear deformation and positive values of the membrane force resultants, the complementary energy density of unit area (4.7) is greater than the strain potential energy density.

We calculate the integral over thickness of the quantity $\sigma_{ij}^p u_{j,i}$ for *i,j=1,2,3*. We have:

$$\int_{-h/2}^{h/2} dz\,(\sigma_{ij}^p u_{j,i}) = \int_{-h/2}^{h/2} dz\{\sigma_x^p(u_{,x} - zw_{,xx}) + \sigma_y^p(v_{,y} - zw_{,yy}) + \sigma_{xy}^p(u_{,y}+v_{,x} - 2zw_{,xx}) + \sigma_{xz}^p w_{,x} + \sigma_{yz}^p w_{,y}\} \tag{5.3}$$

The components of the first Piola stress tensor absent in (5.3) are equal to zero, as it is seen from the general formula [5]:

$$\sigma_{ij}^p = \sigma_{ik}(\delta_{jk} + u_{j,k}) \tag{5.4}$$

where δ_{jk} is the Kroneker symbol.

After simple transformation of (5.3) with account of (2.5) we obtain:

$$\int_{-h/2}^{h/2} dz\,(\sigma_{ij}^p u_{j,i}) = \frac{\partial}{\partial x}\left[N_x^p u - M_x^p w_{,x} + M_{x,x}^p w + N_{xy}^p v - M_{xy}^p w_{,y} + M_{xy,y}^p w + N_{xz}^p w\right] + \frac{\partial}{\partial y}\left[N_y^p v - M_y^p w_{,y} + M_{y,y}^p w + N_{xy}^p u + M_{xy,x}^p w - M_{xy}^p w_{,x} + N_{yz}^p w\right] - uN_{x,x}^p - wM_{x,xx}^p - vN_{y,y}^p - wM_{y,yy}^p - uN_{xy,y}^p - vN_{xy,x}^p - 2wM_{xy,xy}^p - wN_{xz,x}^p - wN_{yz,y}^p \tag{5.5}$$

Regrouping the terms in (5.5), we have:

$$\int_{-h/2}^{h/2} dz\,(\sigma_{ij}^p u_{j,i}) = \frac{\partial}{\partial x}\left[N_x^p u - M_x^p w_{,x} + M_{x,x}^p w + N_{xy}^p v - M_{xy}^p w_{,y} + M_{xy,y}^p w + N_{xz}^p w\right] + \frac{\partial}{\partial y}\left[N_y^p v - M_y^p w_{,y} + M_{y,y}^p w + N_{xy}^p u + M_{xy,x}^p w - M_{xy}^p w_{,x} + N_{yz}^p w\right] - u(N_{x,x}^p + N_{xy,y}^p) - v(N_{xy,x}^p + N_{y,y}^p) - w\left[M_{x,xx}^p + 2M_{xy,xy}^p + M_{y,yy}^p + N_{xz,x}^p + N_{yz,y}^p + q\right] + qw \tag{5.6}$$

Analyzing (5.6) we see that the expression contains the equilibrium equations for *x, y, z* (4.6), multiplied by *u, v, w*, respectively.

Next, we integrate (5.6) over the entire area of the plate and use the Ostrogradsky-Gauss theorem (the divergence theorem) to transform this integral. We omit the terms with the equilibrium equations, as they are equal to zero. We have:

$$\iint d\Gamma \int_{-h/2}^{h/2} dz\,(\sigma_{ij}^p u_{j,i}) = \oint\{n_x(N_x^p u + N_{xy}^p v) + n_x(-M_x^p w_{,x} + M_{x,x}^p w - M_{xy}^p w_{,y} + M_{xy,y}^p w + N_{xz}^p w) + n_y(N_{xy}^p u + N_y^p v) + n_y(-M_y^p w_{,y} + M_{y,y}^p w + M_{xy,x}^p w - M_{xy}^p w_{,x} + N_{yz}^p w)\} + \iint d\Gamma[qw] \tag{5.7}$$

Accounting (4.4):

$$\iint d\Gamma \int_{-h/2}^{h/2} dz\,(\sigma_{ij}^p u_{j,i}) = \oint dC\{N_{x\nu} u + N_{y\nu} v + n_x(-M_x^p w_{,x} + M_{x,x}^p w - M_{xy}^p w_{,y} + M_{xy,y}^p w + N_{xz}^p w) + n_y(-M_y^p w_{,y} + M_{y,y}^p w + M_{xy,x}^p w - M_{xy}^p w_{,x} + N_{yz}^p w)\} + \iint d\Gamma[qw] \tag{5.8}$$

As is known from the formulation of the equilibrium problem (Section 2 and [5]), the quantities $N_{x\nu}$ and $N_{y\nu}$ are given on C_1, and the displacements u, v, w are equal to zero on C_2.

The curly brackets on the right-hand side of (5.8) contain in fact the boundary conditions (see [5, 14]).

As a result, we obtain a relation for the sum of the total strain potential energy and the total complementary energy of the composite von Karman plate:

$$\Pi + \Pi_c = W \quad (5.9)$$

From the relation, the sum $\Pi + \Pi_c$ is equal to W when the plate is deformed in the von Karman approximation. As is well known, even at small deflections, this sum is also equal to the potential of external forces W.

Thus, the dual gap for the variational principles considered is absent.

It should be noted that when deriving the relation (5.9), the condition of positivity of the principal in-plane (membrane) force resultants was not used.

As was stated above in this Section, for positive principal values of the in-plane (membrane) force resultants

$$\Pi \leq \Pi_c \quad (5.10)$$

Consequently, the relations (5.9), (5.10) prove a generalization of the Clapeyron theorem, known from the linear theory of elasticity, in the form

$$\Pi \leq \frac{1}{2} W \quad (5.11)$$

that is, the total potential energy of a composite plate deformed in the von Karman approximation and subject to everywhere positive principal in-plane (membrane) force resultants does not exceed one half of the potential of the external forces. As is well known, in accordance with the classical Clapeyron theorem of linear elasticity theory, the relation (5.11) must contain an equality symbol.

6. Comparison of the behavior of plates analyzed using geometrically linear and geometrically nonlinear theories.

We consider the behavior of a composite plate under conditions described by the von Karman approximation. The principal values of the in-plane (membrane) force resultants will be assumed to be positive. The plate can be analyzed in the geometrically linear approximation (with displacement vectors $\boldsymbol{u}_{lin}$) and, under the same loads, in the geometrically nonlinear von Karman approximation (with displacement vectors $\boldsymbol{u}_{nl}$).

For the geometrically linear approximation (see [9]), the principle of minimum total potential energy of the plate is valid. In accordance with this principle, considering $\boldsymbol{u}_{nl}$ as a kinematically admissible displacement field, the following inequality holds for the total potential energy of the geometrically linear plate U^{lin}:

$$U^{lin}(\boldsymbol{u}_{lin}) \leq U^{lin}(\boldsymbol{u}_{nl}) \quad (6.1)$$

Further, taking into account what was said above about the positivity of the principal values of the in-plane (membrane) force resultants, we can write for the total potential energy of a geometrically linear U^{lin} and geometrically nonlinear U^{nl} plate

$$U^{lin}(\boldsymbol{u}_{nl}) \leq U^{nl}(\boldsymbol{u}_{nl}) \quad (6.2)$$

Consequently, combining (6.1) and (6.2), we obtain:

$$U^{lin}(\boldsymbol{u}_{lin}) \leq U^{nl}(\boldsymbol{u}_{nl}) \quad (6.3)$$

that is, under the same boundary conditions and loading, *the total potential energy of a geometrically nonlinear plate is not less than the total potential energy of this plate analyzed in the geometrically linear approximation.*

We now use the static variational principle discussed above. According to this principle, the distribution of forces and moments for a geometrically linear plate can be considered as statically admissible for a geometrically nonlinear plate. Then, for the total complementary energy of the geometrically linear Π_c^{lin} and nonlinear Π_c^{nl} plates, the following inequality holds:

$$\Pi_c^{nl} \leq \Pi_c^{lin} \tag{6.4}$$

This means that, under the same boundary conditions and loading, *the total complementary energy of a geometrically nonlinear plate does not exceed the total additional energy of this plate analyzed in the geometrically linear approximation.*

If we consider the total potential energy of a plate as a measure of its generalized stiffness and the total complementary energy as a measure of its generalized compliance (similar to [13]), then the inequalities (6.3) and (6.4), respectively, allow us to conclude that the plate under consideration in the geometrically nonlinear von Karman approximation has greater generalized stiffness and lower generalized compliance, compared to the plate analyzed using geometrically linear theory. An approach like that described in this section was used in [13], devoted to the analysis and optimization of geometrically nonlinear trusses.

7. Example.

As an illustrating example we consider a square plate with dimensions $b*b$ (see Figure 2). A plate of thickness h is made of specially orthotropic material [10] with identical Young modulus E along the orthotropy axes (oriented along the *x, y* axes), the shear modulus G in the X-Y plane, and the Poisson ratio μ equal to 0.25. The ratio E/G is taken to be 14. The above characteristics correspond, for example, to a composite material made of woven carbon fiber-epoxy resin prepreg.

The plate is loaded with a constant pressure load q, independent of the *x, y* coordinates.

The plate is hinged on all sides; the displacements *u, v, w* along the *x, y, z* coordinate axes are zero, respectively.

The *b/h* ratio is taken to be 200, which corresponds to a plate 2 mm thick and 400x400 mm in size.

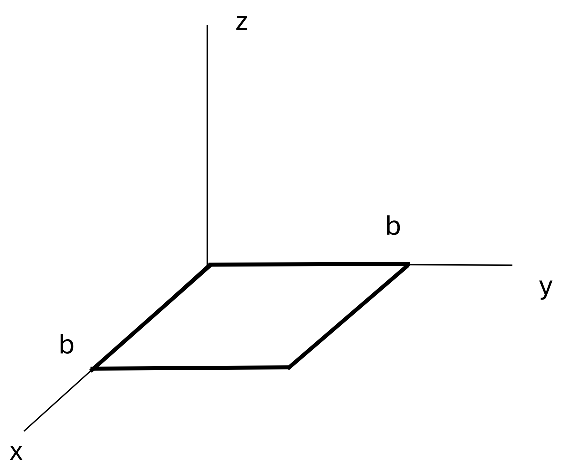

Figure 2. Plane orthotropic plate.

Kinematically admissible displacements of the plate midplane are taken as:

$$\begin{cases} u = d_1\left[\left(\frac{2x}{b}-1\right)-\left(\frac{2x}{b}-1\right)^3\right]\left(1-\left(\frac{2y}{b}-1\right)^2\right) \\ v = d_2\left(1-\left(\frac{2x}{b}-1\right)^2\right)\left[\left(\frac{2y}{b}-1\right)-\left(\frac{2y}{b}-1\right)^3\right] \\ w = d_3\left(1-\left(\frac{2x}{b}-1\right)^2\right)\left(1-\left(\frac{2y}{b}-1\right)^2\right) \end{cases} \tag{7.1}$$

where d_1, d_2, d_3 are unknown factors to be determined from the above-described kinematic variational principle.

The total plate potential energy U is written as (keeping in mind the boundary conditions and the notations (2.4)):

$$U = \frac{1}{2}\iint d\Gamma[\boldsymbol{\varepsilon}_0^T \boldsymbol{A}\boldsymbol{\varepsilon}_0 + \boldsymbol{k}^T\boldsymbol{D}\boldsymbol{k}] - \iint qwd\Gamma \tag{7.2}$$

The elements of the matrices $\boldsymbol{A}$ and $\boldsymbol{D}$ are given by the formulas (the elements 16 and 26 are obviously equal to zero):

$$A_{11} = A_{22} = \frac{Eh}{1-\mu^2} \ ; \quad A_{12} = A_{21} = \frac{\mu Eh}{1-\mu^2} \ ; \quad A_{66} = Gh \quad ; \quad D_{ij} = \frac{1}{3}h^2 A_{ij}, i,j = 1,2,6 \tag{7.3}$$

The derivatives of U with respect to d_1, d_2, d_3 must be zero due to the kinematic variational principle. This leads to a nonlinear system of third-order algebraic equations, which, in turn, leads to a cubic equation for $\frac{d_3}{h}$. The quadratic term in the latter equation is absent. Since the discriminant of the equation is positive, the real-value solution of the cubic equation is unique [15] and is calculated using the well-known Cardano formula. The quantities d_1, d_2 are then determined from the above system of equations.

To evaluate the behavior of the structure using the static variational principle described above in the relevant section, we use the following specified functions for statically admissible internal force resultants and moment resultants:

$$\begin{cases} M_x = \frac{1}{16}q_1 E\left(1-\left(\frac{2x}{b}-1\right)^2\right)\left(1+q_2\left(\frac{2y}{b}-1\right)^2\right)b^2 \\ M_y = \frac{1}{16}q_1 E\left(1-\left(\frac{2y}{b}-1\right)^2\right)\left(1+q_2\left(\frac{2x}{b}-1\right)^2\right)b^2 \\ M_{xy} = \frac{1}{48}q_1 E q_2\left[\left(\frac{2x}{b}-1\right)^3\left(\frac{2y}{b}-1\right)-\left(\frac{2y}{b}-1\right)^3\left(\frac{2x}{b}-1\right)\right]b^2 \\ N_x = N_y = q_3, \qquad N_{xy} = 0 \\ N_{xz} = (q-Eq_1)\left[-\frac{1}{2}\left(\frac{2x}{b}-1\right)+q_4\left(\left(\frac{2x}{b}-1\right)\left(\frac{2y}{b}-1\right)^2-\frac{1}{3}\left(\frac{2x}{b}-1\right)^3\right)\right]\frac{b}{2} \\ N_{yz} = (q-Eq_1)\left[-\frac{1}{2}\left(\frac{2y}{b}-1\right)+q_4\left(\left(\frac{2y}{b}-1\right)\left(\frac{2x}{b}-1\right)^2-\frac{1}{3}\left(\frac{2y}{b}-1\right)^3\right)\right]\frac{b}{2} \end{cases} \tag{7.4}$$

As can be easily verified, the relations (7.4) satisfy the *z*-equilibrium equation (the third equation in (4.6)) and the hinge support conditions (the corresponding bending moment resultants at the edges are equal to zero).

The quantities q_1, q_2, q_3, q_4 must be determined from the static variational principle presented in Section 4.

The complementary variational functional U_c (4.5) is written (taking into account the boundary conditions) as:

$$U_c = -\iint dxdy\left[\frac{1}{2}\boldsymbol{N}^T\boldsymbol{A}^{-1}\boldsymbol{N} + \frac{1}{2}\boldsymbol{M}^T\boldsymbol{D}^{-1}\boldsymbol{M} + \frac{1}{2}\frac{N_{xz}^2}{N_x} + \frac{1}{2}\frac{N_{yz}^2}{N_y}\right] \tag{7.5}$$

Differentiating (7.5) with respect to q_1, q_2, q_3, q_4 and equating these derivatives to zero, we obtain a nonlinear algebraic system of four equations. From this system follows a cubic equation for $\frac{q_1}{E}$, in which the quadratic term is absent, and the discriminant [15] is positive. The unique real-value solution of this equation is given by the well-known Cardano formula. The quantities q_2, q_3, q_4 are also determined from the indicated system of equations.

Figure 3 shows the dimensionless (divided by Eh^3) results of calculating the total potential energy of the plate U and the complementary variational functional U_c for various (divided by $E \cdot 10^{-6}$) loads q. The presented results illustrate the obtained dual variational principles and possible errors when using kinematically admissible displacements and statically admissible force resultants and moment resultants. For the specified displacements, the maximum deflections, i.e., the deflections at the center of the plate (with an accuracy of *0.01*h*), are: *3.79*h* with an abscissa of 1.0 and *7.88*h* with an abscissa of 10.0.

From the presented data, it is clear that U_c is always lower than U.

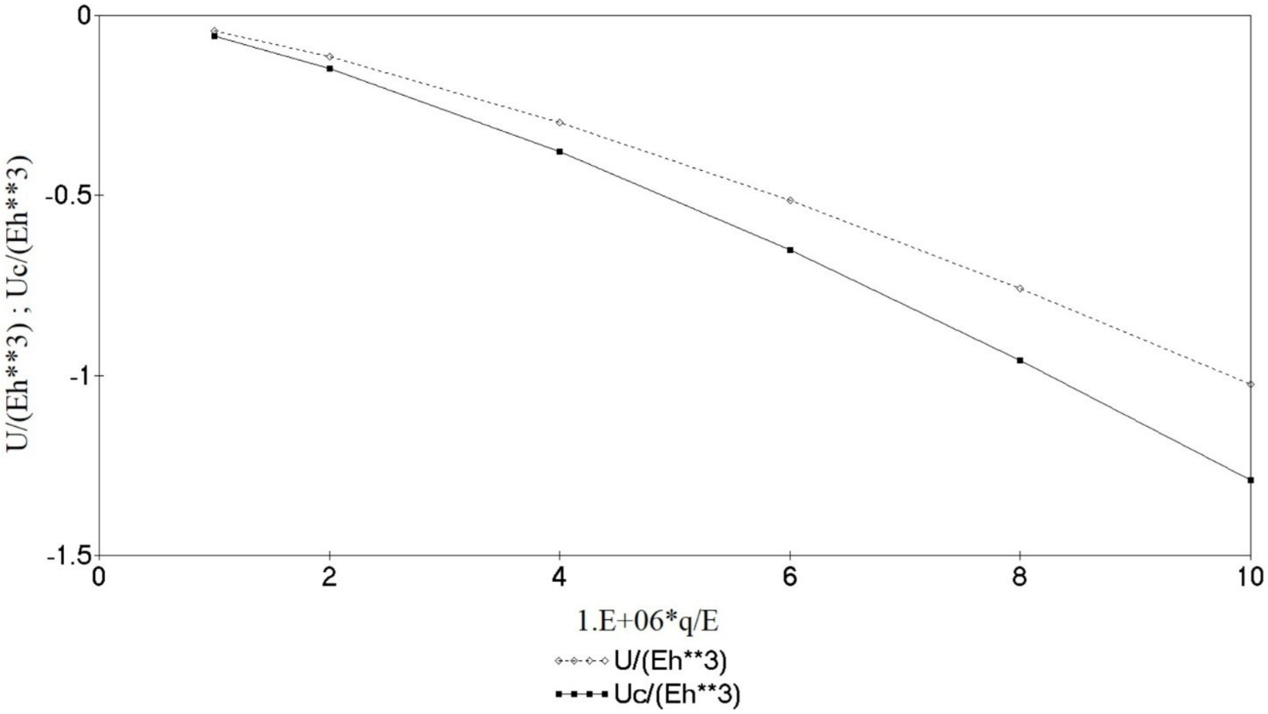

Figure 3. Comparison of the results for the dual variational principle.

8. Conclusion.

It is proven that, in equilibrium, a composite plate deformed according to the von Karman approximation has its total potential energy, with the principal in-plane (membrane) force resultants being positive everywhere, reaching a minimum.

It is proven that, in equilibrium, a composite plate deformed according to the von Karman approximation has its complementary variational functional, with the principal in-plane (membrane) force resultant being positive everywhere, reaching a maximum.

It is shown that, for the dual kinematic and static variational principles (describing the deformation of a von Karman composite plate), the dual gap is zero, and the sum of the total strain potential energy and the complementary energy of the plate is equal to the potential of external forces.

A generalization of the Clapeyron theorem, well-known from linear elasticity theory, is proven for the case of composite plates deformed according to the von Karman approximation.

A comparison of the behavior of plates with identical layer arrangements and identical loads, analyzed in geometrically linear and geometrically nonlinear formulations, suggests greater generalized stiffness and lower generalized compliance of the plate for the geometrically nonlinear von Karman approximation in the case of positive principal in-plane (membrane) force resultants.

An illustrative example of a composite plate is presented. The example demonstrates the use of the discussed variational principles and the calculations of the error bounds for some kinematically admissible displacement fields and statically admissible force resultants/ moment resultants.

The results of this work are applicable to the analysis and design of composite structures, such as the lower panel of a high-aspect-ratio composite wing of a passenger airplane.

References

1. *Zucco G., Oliveri V., Rouhi M., Telford R., Clancy G., McHalr C., O'Higgins R., Young T.M.,Weaver P.M.* Static test of a variable stiffness thermoplastic composite wingbox under shear, bending and torsion. The Aeronautical Journal. 2020; 124(1275): 635-666. doi:10.1017/aer.2019.161
2. *Pagani A., Racionero Sánchez-Majano A., Petrolo M.* Influence of Structural Theories on Optimal Fiber Distributions in Tow-Steered Composites Considering Local Strain and Stress. Aerotec. Missili Spaz. (2025). https://doi.org/10.1007/s42496-025-00275-3
3. *Zhang Y., Kong B., Gu J., Chen P., Wang B*. Experimental investigation on the buckling and post-buckling behavior of variable stiffness laminates, Thin-Walled Structures, Volume 184, 2023, 110450, ISSN 0263-8231, https://doi.org/10.1016/j.tws.2022.110450.
4. *Nozawa S., Serhat G.* Topology and fiber path optimization of composite structures: A critical review, Materials & Design, Volume 251, 2025, 113699, ISSN 0264-1275, https://doi.org/10.1016/j.matdes.2025.113699.
5. Washizu, K. Variational Methods in Elasticity & Plasticity. 3rd edition. Oxford-New York, Pergamon Press 1982.
6. *Stumpf H.* The derivation of dual extremum and complementary stationary principles in geometrically non-linear shell theory. Ing Archiv, 1979, 8, 221-237.
7. *Fraeys de Veubeke B.* A new variational principle for finite elastic displacements. Intern. Journ. Eng. Sci., vol. 10, 1972, pp. 745-763.
8. *Selyugin S.V.* Complementary Energy Theorem for Thin Composite Plates in Postbuckling. Mechanics of Solids, 2024, Vol. 59, No. 3, pp. 1290–1300.
9. *Reddy J.N.* Mechanics of laminated composite plates and shells. Theory and analysis. 2nd edition. Taylor and Francis, Inc. 2003. 831 pp. doi: https://doi.org/10.1201/b12409
10. *Gibson R.F.* Principles of composite material mechanics. 4th edition. Taylor and Francis, Inc. 2016. 425 pp. doi: https://doi.org/10.1201/b19626
11. *Vasiliev V.V.* Mechanics of Composite Structures (1st ed.). CRC Press. 1993. https://doi.org/10.1201/9780203747858

12. *Gelfand I., Fomin S.* Calculus of variations. Prentice Hall, 1963.
13. *Selyugin S.V.* On optimal geometrically non-linear trusses. Struct Multidisc Optim 28, 1–12 (2004) DOI 10.1007/s00158-004-0462-4
14. *Selyugin S.V.* Analysis and design of composite plates and panels (in Russian). MAI Publ., Moscow, 155 pp.
15. *Korn G.A., Korn T.M*. Mathematical Handbook for Scientists and Engineers. 2nd Enlarged and Revised Edition, McGraw-Hill, New York, 1968, 943 pp.