

Common Pay-off Matrix Games and Anti-Diagonal Symmetric Coordination

Games

By

Somdeb Lahiri

Email: somdeb.lahiri@gmail.com

ORCID: <https://orcid.org/0000-0002-5247-3497>

(Formerly with) PD Energy University, Gandhinagar (EU-G), India.

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Abstract

We prove the existence of equilibrium of common pay-off matrix games as an optimal solution of a bi-linear programming problem. If in a strategy profile the randomization of the column player is completely mixed then the strategy profile is an equilibrium if and only if it is the saddle-point of the pay-off matrix. We show if the pay-off matrix is skew symmetric then the randomization of the column player that minimizes the player's maximum expected pay-off is a symmetric saddle point of the pay-off matrix and if this randomization is completely mixed then it is a symmetric equilibrium of the common pay-off matrix game. For symmetric matrices we provide a necessary and sufficient condition for the existence of a randomization which is a symmetric saddle point as well as a symmetric equilibrium of the common pay-off matrix game. We prove the existence of equilibria for anti-diagonal symmetric coordination games and in addition provide formulas for calculating several symmetric equilibria for such games.

Keywords: common pay-off matrix games, equilibrium, saddle-point, symmetric equilibrium, symmetric saddle-point, symmetric matrix, skew-symmetric matrix, anti-diagonal symmetric coordination games

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JEL Classification: C61, C72

1. Introduction: A considerable amount of game theory in normal form is based on a single pay-off matrix. Two-person zero sum games (see the first four chapters of Ferguson (2000)), symmetric bi-matrix games (see Lahiri (2025) for an introductory survey on bi-matrix games) are well known classes of two-person games based on a single pay-off matrix. Games in normal form that are determined by a single matrix (single matrix games), find extensive application in evolutionary game theory which is surveyed adequately for our purpose in Lahkar (2012).

In this paper we first consider the two player version of common pay-off games discussed in Takashi (2009), where both players get equal pay-offs at each pair of pure strategies. We refer to such games as “common pay-off matrix games”. The origins of such games may be traced back to Marschak (1955), where such decision making problems are referred to as “team decision problems”. Such games are a type of “co-ordination” games. There is an easy proof of the existence of equilibrium of common pay-off matrix games. Solutions of a very simple bi-linear programming problem are shown to be equilibria of such games. However, not all equilibrium points need necessarily be recovered as an optimal solution of this bi-linear programming problem, which we show by using example 1.2 in Sorger (2015). The matrix of a common pay-off matrix game allows us to define the concept of “saddle-point of a matrix”, which is better known as “equilibrium of a two-person zero-sum game” defined by the matrix. If in a strategy profile, the randomization chosen by the column player is “completely mixed” (i.e., assigns positive probability to all pure strategies), then the strategy profile is an equilibrium if and only if it is a saddle-point of the pay-off matrix. We discuss the relationship between equilibrium of common pay-off games and saddle-point of the pay-off matrix of such games. If the pay-off matrix of a common pay-off matrix game is symmetric (which implies that the matrix is square), then a symmetric saddle-point is a symmetric equilibrium. An example of this is example 1.2 in Sorger (2015) which also shows that the converse need not be true. For skew-symmetric matrices (which implies that the matrix is square), the randomization for the column player that minimizes the players maximum expected pay-off is a symmetric saddle-point. The existence result is a “well-known” special case of theorem 8 in von Stengel (2023), and our proof (which is different from the proof of theorem 8 in von Stengel (2023)) is applicable only for the special case we are concerned with.

Example 1.2 in Sorger (2015) is a special type of square common pay-off matrix game with a symmetric pay-off matrix. All such games have a symmetric equilibrium which is concurrently a symmetric saddle-point of the pay-off matrix. We provide a necessary and sufficient condition that a symmetric pay-off matrix must satisfy, so that the common pay-off matrix game with it as the pay-off matrix has a symmetric equilibrium which is concurrently a symmetric saddle-point of the pay-off matrix.

While the proof that we provide for the result is a straight forward application of Farkas lemma, the result- to the best of our knowledge- is new.

While common pay-off matrix games are a kind of coordination game in which the objectives of both players are similarly aligned, there are coordination games in which the objectives of the players may not be similarly aligned. This is best illustrated by a pay-off matrix for the row player which is anti-diagonal with all entries along the anti-diagonal being strictly positive but not necessarily equal to each other and the pay-off matrix for the column player being the transpose of the pay-off matrix for the row player. We refer to such games as anti-diagonal symmetric coordination games. If the top right hand corner entry in the pay-off matrix for the row player is greater than all pay-offs and the bottom right hand corner entry in this matrix is the least among all positive pay-offs, then while both players have an incentive to coordinate their actions, the row player would prefer the top right-hand corner entry to all other, where as in the pay-off matrix of the column player, the column player will prefer the bottom left-hand corner entry to all other pay-offs. Every entry on the anti-diagonal corresponds to a (pure strategy) equilibrium, and all such equilibria are necessarily asymmetric if the size of the pay-off matrices is an even integer. We also provide explicit formulas for calculating several symmetric equilibria that may assign positive probability to more than one pure strategy.

2. The Framework: Let A be a real-valued $m \times n$ matrix for some positive integers m and n with $m \geq 2$ and $n \geq 2$. We assume there are two-players- the row player and the column player- with the pure strategy set of the row player being the rows of the matrix A and the pure strategy set of the column player being the columns of the matrix A .

Let B be a " $m \times n$ matrix" which is either A or $-A$ or A^T , i.e., $B \in \{A, -A, A^T\}$

Since both A and B are $m \times n$ matrices, $B = A^T$ if and only if $m = n$.

We assume that when the row player chooses row $i \in \{1, \dots, m\}$ and the column player chooses column $j \in \{1, \dots, n\}$, the pay-off to the row player is a_{ij} and the pay-off to the column player is b_{ij} .

(1) If $B = A$, then the pair $(A, B) = (A, A)$ is said to be a **common pay-off matrix game**.

Common pay-off matrix games are a special case of the multi-player common pay-off games defined in Takashi (2009) and Emmons, Oesterheld, Critch, Conitzer, Russell (2022) as well as many others cited in these two papers.

(2) $B = -A$, then $(A, B) = (A, -A)$ is said to be a **two-person zero-sum matrix game**.

(3) If $m = n$, $B \in \{A, -A, A^T\}$, then the pair (A, B) will be referred to in this paper as a **square matrix game**.

(4) A square matrix game with $B = A^T$ is said to be a **symmetric bi-matrix game**.

(5) If A is an anti-diagonal matrix (i.e., $m = n$ and for all $i \in \{1, 2, \dots, n\}$, $a_{ij} = 0$ if $j \neq n - i + 1$) and $a_{ij} > 0$ for all $i \in \{1, 2, \dots, n\}$ and $j = n - i + 1$, then the symmetric bi-matrix game (A, A^T) will be referred to in this paper as an **anti-diagonal symmetric coordination game**.

All five types of games are defined on the basis of a single $m \times n$ matrix A , and thus may be referred to as **single matrix games**.

For $i \in \{1, \dots, m\}$ let A_i denote the i^{th} row of A and for $j \in \{1, \dots, n\}$ let A^j denote the j^{th} column of A .

Let $\Delta^{m-1} = \{x \in \mathbb{R}_+^m \mid \sum_{i=1}^m x_i = 1\}$ denote the set of mixed strategies of the row player and $\Delta^{n-1} = \{y \in \mathbb{R}_+^n \mid \sum_{j=1}^n y_j = 1\}$ denote the set of mixed strategies of the column player.

A pair $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ is said to be a **strategy profile**.

A strategy profile (x^*, y^*) is said to be a **saddle-point** of A if $A_i y^* \leq x^{*T} A y^* \leq x^{*T} A^j$ for all $i \in \{1, \dots, m\}$ and for all $j \in \{1, \dots, n\}$.

Let B be a " $m \times n$ matrix" which is either A or $-A$ or A^T .

A strategy profile (x^*, y^*) is said to be an **equilibrium** for (A, B) if $x^{*T} A y^* \geq A_i y^*$ for all $i \in \{1, \dots, m\}$ and $x^{*T} B y^* \geq x^{*T} B^j$ for all $j \in \{1, \dots, n\}$.

Clearly a saddle-point of A is an equilibrium for $(A, -A)$ and conversely.

The following are important concepts in the context of this paper.

If A is an $n \times n$ square matrix, then $x^* \in \Delta^{n-1}$ is said to a **symmetric saddle-point** of the square matrix A (of size n) if (x^*, x^*) is a saddle-point of A .

If A is an $n \times n$ square matrix and $B \in \{A, -A, A^T\}$, then $x^* \in \Delta^{n-1}$ is said to a **symmetric equilibrium** of (A, B) if (x^*, x^*) is an equilibrium of (A, B) .

3. Equilibrium of a Common Pay-off Matrix Game as Optimal Solution of Bi-

linear Programming Problem: We provide in this section an easy proof of the existence of equilibrium for common pay-off matrix games obtained by solving a bi-linear programming problem.

Proposition 1: $\operatorname{argmax}_{(x,y) \in \Delta^{m-1} \times \Delta^{n-1}} x^T A y \neq \emptyset$. If $(x^*, y^*) \in \operatorname{argmax}_{(x,y) \in \Delta^{m-1} \times \Delta^{n-1}} x^T A y$ then (x^*, y^*)

is an equilibrium for (A, A) .

Proof: Let $a_{i^*j^*} = \max \{a_{ij} \mid (i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}\}$ for some $(i^*, j^*) \in \{1, \dots, m\} \times \{1, \dots, n\}$.

Let $x^{(0)}$ be the m - dimensional column vector such that for all $i \in \{1, \dots, m\} \setminus \{i^*\}$, $x_i^{(0)} = 0$ and $x_{i^*}^{(0)} = 1$.

Let $y^{(0)}$ be the n - dimensional column vector such that for all $j \in \{1, \dots, n\} \setminus \{j^*\}$, $y_j^{(0)} = 0$ and $y_{j^*}^{(0)} = 1$.

Thus, $(x^{(0)}, y^{(0)}) \in \Delta^{m-1} \times \Delta^{n-1}$.

Further, for all $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$: $x^{*T}Ay^* = a_{i^*j^*} \geq A_i y$ for all $i \in \{1, \dots, m\}$ and $x^{*T}Ay^* = a_{i^*j^*} \geq x^T A_j$ for all $j \in \{1, \dots, n\}$.

Thus, $(x^{(0)}, y^{(0)}) \in \operatorname{argmax}_{(x,y) \in \Delta^{m-1} \times \Delta^{n-1}} x^T Ay$, whence $\operatorname{argmax}_{(x,y) \in \Delta^{m-1} \times \Delta^{n-1}} x^T Ay \neq \emptyset$.

To prove the second part, if $(x^*, y^*) \in \operatorname{argmax}_{(x,y) \in \Delta^{m-1} \times \Delta^{n-1}} x^T Ay$, then $x^{*T}Ay^* \geq x^T Ay$ for all $y \in \Delta^{n-1}$ and $x^{*T}Ay^* \geq x^T Ay^*$ for all $x \in \Delta^{m-1}$.

From here it easily follows that (x^*, y^*) is an equilibrium for (A, A) . Q.E.D.

Example 1 (Example 1.2 in Sorger (2015)): Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be a 2×2 real valued matrix with $a > 0$ and $b > 0$. Note that $A = A^T$. The common-payoff game (A, A) has 3 equilibria, namely $((1, 0), (1, 0))$, $((0, 1), (0, 1))$ and $((\frac{b}{a+b}, \frac{a}{a+b}), (\frac{b}{a+b}, \frac{a}{a+b}))$.

Let $\alpha \in [0, 1]$ be the probability with which the row player chooses the first row and $\beta \in [0, 1]$ be the probability with which the column player chooses the first column.

Thus $(\alpha \mid 1-\alpha)A \begin{pmatrix} \beta \\ 1-\beta \end{pmatrix} = a\alpha\beta + b(1-\alpha)(1-\beta)$.

If $\beta = 1$, then the maximizing value of α is also equal to 1 with 'a' being the value of the objective function.

If $\beta = 0$, then the maximizing value of α is also equal to 0 with 'b' being the value of the objective function.

The same argument applies if we interchange the roles of α and β .

Now suppose $\beta \in (0, 1)$.

Then, $a\alpha\beta + b(1-\alpha)(1-\beta) = (a\beta - b(1-\beta))\alpha + b(1-\beta)$.

If $a\beta > b(1-\beta)$, then the maximizing value of α is 1 with the value of the objective function being $a\beta < a$. Thus, $\beta \in (0, 1)$ satisfying $a\beta > b(1-\beta)$ cannot be an optimal solution.

If $a\beta < b(1-\beta)$, then the maximizing value of α is 0 with the value of the objective function being $b(1-\beta) < b$. Thus, $\beta \in (0, 1)$ satisfying $a\beta < b(1-\beta)$ cannot be an optimal solution.

If $a\beta = b(1-\beta)$, i.e., $\beta = \frac{b}{a+b}$, then regardless of $\alpha \in [0, 1]$, the value of the objective function is $\frac{ab}{a+b} = a - \frac{a^2}{a+b} < a$. Thus, $\beta = \frac{b}{a+b}$ cannot be an optimal solution.

Thus, $\beta \in (0, 1)$ is incompatible with any optimal solution.

By a similar argument it can be shown that $\alpha \in (0, 1)$ is incompatible with any optimal solution.

Thus, $((1, 0), (1, 0))$ is the unique optimal solution if $a > b$ and $((0, 1), (0, 1))$ is the unique optimal solution if $b > a$.

If $a = b$, then both $((1, 0), (1, 0))$ and $((0, 1), (0, 1))$ are optimal.

However, A has a unique saddle-point and that is the equilibrium $((\frac{b}{a+b}, \frac{a}{a+b}), (\frac{b}{a+b}, \frac{a}{a+b}))$. This, as we will see later, is not a coincidence.

4. Two Equivalence Results: The proof of the first two propositions follow almost immediately from the definitions of equilibrium and saddle point.

Proposition 2: Suppose $y^* \in \Delta^{n-1}$ is completely mixed (i.e., $y_j^* > 0$ for all $j \in \{1, \dots, n\}$).

Then, (x^*, y^*) is an equilibrium for (A, A) if and only if it is a saddle point of A .

Proof: Suppose $y^* \in \Delta^{n-1}$ satisfies $y_j^* > 0$ for all $j \in \{1, \dots, n\}$.

Suppose (x^*, y^*) is an equilibrium for (A, A) . Thus, $A_i y^* \leq x^{*T} A y^*$ for all $i \in \{1, \dots, m\}$ and $x^{*T} A^j \leq x^{*T} A y^*$ for all $j \in \{1, \dots, n\}$.

Towards a contradiction suppose that for some $j \in \{1, \dots, n\}$, $x^{*T} A^j < x^{*T} A y^*$. Then since $y^* \in \Delta^{n-1}$ satisfies $y_j^* > 0$ for all $j \in \{1, \dots, n\}$, we get $x^{*T} A y^* < x^{*T} A y^*$ leading to a contradiction. Thus it must be the case that $x^{*T} A^j = x^{*T} A y^*$ for all $j \in \{1, \dots, n\}$.

Thus, (x^*, y^*) is a saddle point.

Now suppose, (x^*, y^*) is a saddle-point. Thus, $A_i y^* \leq x^{*T} A y^*$ for all $i \in \{1, \dots, m\}$ and $x^{*T} A^j \geq x^{*T} A y^*$ for all $j \in \{1, \dots, n\}$.

Towards a contradiction suppose that for some $j \in \{1, \dots, n\}$, $x^{*T} A^j > x^{*T} A y^*$. Then since $y^* \in \Delta^{n-1}$ satisfies $y_j^* > 0$ for all $j \in \{1, \dots, n\}$, we get $x^{*T} A y^* > x^{*T} A y^*$ leading to a contradiction. Thus it must be the case that $x^{*T} A^j = x^{*T} A y^*$ for all $j \in \{1, \dots, n\}$.

Thus, (x^*, y^*) is an equilibrium for (A, A) . Q.E.D.

Note 1: In example 1, at the equilibrium $((\frac{b}{a+b}, \frac{a}{a+b}), (\frac{b}{a+b}, \frac{a}{a+b}))$ both components of the column player's strategy are strictly positive. Hence, by proposition 2, $((\frac{b}{a+b}, \frac{a}{a+b}), (\frac{b}{a+b}, \frac{a}{a+b}))$ should be a saddle point, which it indeed is.

The proof of the next proposition is very similar to the proof of proposition 2.

Proposition 3: Let $(x^*, y^*) \in \Delta^{m-1} \times \Delta^{n-1}$ be an equilibrium for (A, A) (a saddle-point of A). Then, (x^*, y^*) is a saddle-point of A (an equilibrium for (A, A)) if and only if $x^{*T}A^j = x^{*T}Ay^*$ for all $j \in \{1, \dots, n\}$.

The following is the existence result for equilibrium of (A, A) .

5. Square common pay-off matrix games: If $m = n$, so that A is a square matrix, then (A, A) is said to be a **square common pay-off matrix game**.

Let $E^{(n)}$ denote the n -dimensional column vector all whose coordinates are equal to 1.

Proposition 4: Suppose A is a symmetric matrix of size n , (i.e., $A = A^T$) and $x^* \in \Delta^{n-1}$ is a symmetric saddle-point of A . Then, x^* is a symmetric equilibrium of (A, A) .

Proof: Suppose, x^* is a symmetric saddle-point of A . x^* is a symmetric saddle-point of A if and only if $x^T Ax^* \leq x^{*T} Ax^* \leq x^{*T} Ax$ for all $x \in \Delta^{n-1}$.

Thus, $x^{*T} A^T x \leq x^{*T} A^T x^* \leq x^T A^T x$ for all $x \in \Delta^{n-1}$.

Since $A = A^T$, it follows that $x^{*T} Ax \leq x^{*T} Ax^* \leq x^T Ax^*$ for all $x \in \Delta^{n-1}$.

This combined with $x^T Ax^* \leq x^{*T} Ax^* \leq x^{*T} Ax$ for all $x \in \Delta^{n-1}$ implies $x^T Ax^* = x^{*T} Ax^* = x^{*T} Ax$ for all $x \in \Delta^{n-1}$.

Thus, for all $i, j \in \{1, \dots, n\}$, $A_i x^* = x^{*T} Ax^* = x^{*T} A_j$.

Hence, x^* is a symmetric equilibrium of (A, A) . Q.E.D.

Note 2: In example 1, $((\frac{b}{a+b}, \frac{a}{a+b}), (\frac{b}{a+b}, \frac{a}{a+b}))$ is a symmetric saddle point of A . By

proposition 4, it should be a symmetric equilibrium of (A, A) , which it indeed is.

However, $((1, 0), (1, 0)), ((0, 1), (0, 1))$ are also symmetric equilibria of the common pay-off game in example 1, but neither is a saddle-point of the pay-off matrix. This shows that the converse of proposition 4 is not true.

6. Existence of Symmetric Saddle-Points: Let A be a square matrix of size 'n'.

If A is skew-symmetric, i.e. $A = -A^T$, then $A + A^T = 0$.

Thus, $0 = x^T(A + A^T)x = x^T Ax + x^T A^T x = x^T Ax + x^T Ax = 2x^T Ax$ for all $x \in \mathbb{R}^n$.

Thus, $x^T Ax = 0$ for all $x \in \mathbb{R}^n$.

Proposition 5: Suppose A is skew-symmetric, i.e., $A = -A^T$ and the pair $(x^*, u^*) \in \Delta^{n-1} \times \mathbb{R}$ solves the linear programming problem: Minimize u , subject to $A_i x - u \leq 0$, $i =$

$1, \dots, n, \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \dots, n, u \in \mathbb{R}$. Then, x^* is a symmetric saddle point of A . If x^* is completely mixed (i.e., $x_i^* > 0$ for all $i \in \{1, \dots, n\}$), the x^* is a symmetric equilibrium of (A, A) .

Proof: From the assumption it follows that $(x^*, u^*) \in \Delta^{n-1} \times \mathbb{R}$ solves the linear programming problem: Maximize $-u$, subject to $A_i x - u \leq 0, i = 1, \dots, n, \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \dots, n, u \in \mathbb{R}$.

Thus, $Ax^* - u^*E^{(n)} \leq 0$ and hence $x^{*T}A^T - u^*E^{(n)T} \leq 0$.

Since $A = -A^T$, it follows that $-x^{*T}A - u^*E^{(n)T} \leq 0$ and hence $x^{*T}A + u^*E^{(n)T} \geq 0$.

The dual of the linear programming maximization problem invoked in this proof is:

Minimize v , subject to $\xi^T A^j + v \geq 0, j = 1, \dots, n, -\sum_{j=1}^n \xi_j = -1, \xi_j \geq 0, j = 1, \dots, n, v \in \mathbb{R}$.

Hence, the pair (x^*, u^*) satisfies the constraint of this problem.

Further (x^*, u^*) solves the linear programming maximization problem.

Thus, $u^* \geq -u^*$ and hence $u^* \geq 0$.

Let $(\xi^*, v^*) \in \Delta^{n-1} \times \mathbb{R}$ solve the dual minimization problem. Thus, $v^* = -u^*$.

Towards a contradiction suppose $u^* > 0$. Then, $v^* = -u^* < 0$ so that $-v^* > 0$.

$\xi^{*T} A^j + v^* \geq 0$ for all $j = 1, \dots, n$, and $v^* = -u^* < 0$ implies $\xi^{*T} A^j \geq -v^* > 0$ for all $j = 1, \dots, n$.

Since $\xi^* \in \Delta^{n-1}$: $\xi^{*T} A^j > 0$ for all $j = 1, \dots, n$ implies $\xi^{*T} A \xi^* > 0$, contradicting the consequence of our assumption that A is a skew-symmetric matrix, namely, $x^T A x = 0$ for all $x \in \mathbb{R}^n$.

Thus, it must be the case that $u^* = 0 = v^*$.

Thus, $A_i x^* \leq 0 = x^{*T} A x^*$ and $x^{*T} A^j \geq 0 = x^{*T} A x^*$ for all $j = 1, \dots, n$.

Thus, x^* is a symmetric saddle point of A .

If x^* is completely mixed, then it follows from proposition 2, that x^* is a symmetric equilibrium of (A, A) . Q.E.D.

Note 4: The above proposition is a special case of a general existence result for bi-matrix games of the form (A, A^T) and in this special case, there is a simple proof using linear programming that we have provided above.

The kind of square common-payoff matrix game considered in example 1 belongs to a special class of games of the form (A, A) , where $A \in \mathcal{A} = \{B \mid B \text{ is a symmetric square matrix of size } n \text{ and there does not exist } y \in \mathbb{R}^n \text{ satisfying both } y^T E^{(n)} = 0 \text{ and } y^T B^j > 0 \text{ for all } j \in \{1, \dots, n\}\}$.

Proposition 6: Let A be symmetric square matrix of size 'n'. Then, A has a symmetric saddle point which is concurrently a symmetric equilibrium point of (A, A) if and only if $A \in \mathcal{A}$.

Proof: Suppose $A \in \mathcal{A}$ and towards a contradiction suppose there does not exist $x \in \mathbb{R}_+^n$, and non-negative real numbers u, v satisfying $Ax + E^{(n)}u - E^{(n)}v = 0, E^{(n)T}x = 1$.

Then, by Farkas lemma, there exists $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $y^T A^j + \alpha \geq 0$ for all $j \in \{1, \dots, n\}, y^T E^{(n)} \geq 0, -y^T E^{(n)} \geq 0, \alpha < 0$.

Thus, $y^T A^j \geq -\alpha > 0$ for all $j \in \{1, \dots, n\}, y^T E^{(n)} = 0$, contradicting $A \in \mathcal{A}$.

Thus, there exists $x^* \in \mathbb{R}_+^n$, and non-negative real numbers u, v satisfying $Ax^* + E^{(n)}u - E^{(n)}v = 0, E^{(n)T}x^* = 1$.

Let $u^* = v - u$. Thus, $x^* \in \Delta^{n-1}$ and $Ax^* = u^* E^{(n)}$.

Since $A^T = A$, we get $x^{*T}A = x^{*T}A^T = u^* E^{(n)}$.

Thus, $A_{ix^*} = u^* = x^{*T}Ax^*$ for all $i \in \{1, \dots, n\}$ and $x^{*T}A^j = u^* = x^{*T}Ax^*$ for all $j \in \{1, \dots, n\}$.

Thus, x^* is a symmetric saddle point of A and x^* is a symmetric equilibrium point of (A, A) .

Conversely, suppose x^* is a symmetric saddle point of A and x^* is a symmetric equilibrium point of (A, A) .

Thus, $A_{ix^*} \leq x^{*T}Ax^* = x^{*T}A^j$ for all $i, j \in \{1, \dots, n\}$. Let $u^* = x^{*T}Ax^*$. Thus, $A^T x^* = u^* E^{(n)}$.

Since $A^T = A$, $Ax^* = u^* E^{(n)}$, i.e., $Ax^* - u^* E^{(n)} = 0$.

Thus, $[Ax + E^{(n)}u - E^{(n)}v = 0, E^{(n)T}x = 1, x \in \mathbb{R}_+^n, u$ and v non-negative real numbers] has a solution $x = x^*, u = \max\{-u^*, 0\}, v = \max\{u^*, 0\}$.

Thus, by Farkas lemma *there does not exist* $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ satisfying $y^T A^j + \alpha \geq 0$ for all $j \in \{1, \dots, n\}, y^T E^{(n)} \geq 0, -y^T E^{(n)} \geq 0, \alpha < 0$.

Towards a contradiction suppose there exists $y \in \mathbb{R}^n$ satisfying both $y^T E^{(n)} = 0$ and $y^T A^j > 0$ for all $j \in \{1, \dots, n\}$.

Since $\min\{y^T A^j | j \in \{1, \dots, n\}\} > 0$, there exists $\varepsilon > 0$ satisfying $\min\{y^T A^j | j \in \{1, \dots, n\}\} - \varepsilon > 0$. Let $\alpha = -\varepsilon < 0$.

Thus, $y \in \mathbb{R}^n, \alpha \in \mathbb{R}, y^T A^j + \alpha > 0$ for all $j \in \{1, \dots, n\}, y^T E^{(n)} \geq 0, -y^T E^{(n)} \geq 0, \alpha < 0$ contradicting the non-existence of such a y and α .

Thus, there does not exist $y \in \mathbb{R}^n$ satisfying both $y^T E^{(n)} = 0$ and $y^T A^j > 0$ for all $j \in \{1, \dots, n\}$ and hence $A \in \mathcal{A}$. Q.E.D.

7. Equilibria of Anti-Diagonal Symmetric Coordination Games: Suppose (A, A^T) is an anti-diagonal symmetric coordination game. Thus, there exists an array of strictly positive real numbers $\langle \alpha_j | j = 1, 2, \dots, n \rangle$ such that for all $i, j \in \{1, \dots, n\}$: $a_{ij} = \alpha_i$ if $i + j = n + 1$ and $a_{ij} = 0$, otherwise.

For $i \in \{1, \dots, n\}$, let $j(i) = n - i + 1$. Thus, for all $i, j \in \{1, \dots, n\}$: $a_{ij} = \alpha_i$ if $j = j(i)$ and $a_{ij} = 0$ if $j \neq j(i)$.

Note 5: For all $i \in \{1, \dots, n\}$, $j(j(i)) = i$. Further, $k = j(i)$ if and only if $i = j(k)$.

For all $x \in \Delta^{n-1}$ and $i \in \{1, \dots, n\}$, $A_i x = a_{ij(i)} x_{j(i)} = \alpha_i x_{j(i)}$, and $x^T (A^T)^i = x^T (A_i)^T = A_i x = a_{ij(i)} x_{j(i)} = \alpha_i x_{j(i)}$.

For all $i \in \{1, \dots, n\}$, let $E^{(n,i)}$ be the n -dimensional column vector whose j^{th} coordinate for $j \in \{1, \dots, n\}$ is equal to 1 if $j = i$, and is equal to 0 if $j \neq i$.

For all $i \in \{1, \dots, n\}$, $A_k E^{(n,i)} = \alpha_k > 0$ if $k = j(i)$ and $A_k E^{(n,i)} = 0$ if $k \neq j(i)$.

Proposition 7: Suppose (A, A^T) is an anti-diagonal symmetric coordination game.

There for all $i \in \{1, \dots, n\}$, $(E^{(n,i)}, E^{(n,j(i))})$ is an equilibrium for (A, A^T) .

Proof: Let $i \in \{1, \dots, n\}$. Thus, $E^{(n,i)T} A E^{(n,j(i))} = A_i E^{(n,j(i))} = \alpha_i$.

$E^{(n,i)T} A^T E^{(n,j(i))} = E^{(n,j(i))T} A E^{(n,i)} = E^{(n,j(i))T} A^i = \alpha_{j(i)}$.

For $k \in \{1, \dots, n\} \setminus \{i\}$, $A_k E^{(n,j(i))} = 0$ and hence, $A_k E^{(n,j(i))} < E^{(n,i)T} A E^{(n,j(i))}$.

For $k \in \{1, \dots, n\} \setminus \{j(i)\}$, $E^{(n,i)T} (A^T)^k = E^{(n,i)T} (A_k)^T = A_k E^{(n,i)} = 0$ and hence $E^{(n,i)T} (A^T)^k < E^{(n,i)T} A^T E^{(n,j(i))}$.

Thus, $(E^{(n,i)}, E^{(n,j(i))})$ is an equilibrium for (A, A^T) . Q. E. D.

In addition to the ones mentioned in proposition 7, there are symmetric equilibria for anti-diagonal coordination games such that i^{th} coordinate of the randomization is positive if and only if its $(n-i+1)^{\text{th}}$ coordinate is positive.

Let S be a non-empty subset of $\{1, 2, \dots, n\}$ satisfying the property: $i \in S$ if and only if

$j(i) \in S$. For all $i \in S$, let $\beta_i^S = \prod_{k \in S \setminus \{j(i)\}} \alpha_k$ and let $x^{(S)} \in \Delta^{n-1}$ be such that $x_i^{(S)} = \frac{\beta_i^S}{\sum_{j \in S} \beta_j^S}$ for

$i \in S$ and $x_i^{(S)} = 0$ for $i \in \{1, \dots, n\} \setminus S$.

Proposition 8: Suppose (A, A^T) is an anti-diagonal symmetric coordination game and

let S be a non-empty subset of $\{1, 2, \dots, n\}$ satisfying the property: $i \in S$ if and only if

$j(i) \in S$. Then, $x^{(S)}$ is a symmetric equilibrium of (A, A^T) .

Proof: For $i \in S$, $A_i x^{(S)} = \alpha_i x_{j(i)} = \alpha_i \frac{\beta_{j(i)}^S}{\sum_{j \in S} \beta_j^S} = \alpha_i \frac{\prod_{k \in S \setminus \{i\}} \alpha_k}{\sum_{j \in S} \beta_j^S}$, since $j(j(i)) = i$.

Thus, $A_i x^{(S)} = \frac{\prod_{k \in S} \alpha_k}{\sum_{j \in S} \beta_j^S}$ for all $i \in S$.

For $i \in \{1, \dots, n\} \setminus S$, $A_i x^{(S)} = \alpha_i x_{j(i)} = 0$, since $i \in \{1, \dots, n\} \setminus S$ implies $j(i) \in \{1, \dots, n\} \setminus S$ and thus $x_{j(i)} = 0$.

Thus, $x^{(S)T} A x^{(S)} = \frac{\prod_{k \in S} \alpha_k}{\sum_{j \in S} \beta_j^S} = A_i x^{(S)}$ for all $i \in S$ and $x^{(S)T} A x^{(S)} > A_i x^{(S)} = 0$ for all $i \notin S$.

For $i \in S$, $x^{(S)T} (A^T)^i = x^{(S)T} (A_i)^T = A_i x^{(S)} = \frac{\prod_{k \in S} \alpha_k}{\sum_{j \in S} \beta_j^S}$.

For $i \in \{1, \dots, n\} \setminus S$, $x^{(S)T} (A^T)^i = x^{(S)T} (A_i)^T = A_i x^{(S)} = \alpha_i x_{j(i)} = 0$.

Thus, $x^{(S)T} (A^T)^i = \frac{\prod_{k \in S} \alpha_k}{\sum_{j \in S} \beta_j^S} = x^{(S)T} A^T x^{(S)}$ for all $i \in S$ and $x^{(S)T} (A^T)^i = 0 < x^{(S)T} A^T x^{(S)}$ for all

$i \in \{1, \dots, n\} \setminus S$.

Thus, $x^{(S)}$ is a symmetric equilibrium for (A, A^T) . Q. E. D.

Example 2: Let $A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ be a 2×2 real valued matrix with $a > 0$ and $b > 0$. Thus,

$A^T = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$. Let x^* be the 2-dimensional column vector $(\frac{b}{a+b}, \frac{a}{a+b})$. It is easily easily

verified that x^* is a symmetric equilibrium of (A, A^T) , with $x^{*T} A x^* = \frac{ab}{a+b}$. Since $A x^* =$

$\begin{bmatrix} \frac{a^2}{a+b} \\ \frac{b^2}{a+b} \end{bmatrix}$ if $a \neq b$, either $\frac{a^2}{a+b} > \frac{ab}{a+b}$ or $\frac{b^2}{a+b} > \frac{ab}{a+b}$ and hence if $a \neq b$, then x^* is not a saddle-

point of A .

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