Strictly-Signed-Diagonal Pay-off Matrix Based Bi-Matrix Games

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Abstract

We consider four types of games that together comprise "strictly-signed-diagonal pay-off matrix based bi-matrix games" and for each type we provide explicit formulas for the "unique completely mixed" equilibrium that we prove the existence of. If the pay-off matrix of the column player is a diagonal matrix with entries along the diagonal having sign exactly opposite to the sign of the diagonal entries in the pay-off matrix of the row player, then the "unique completely mixed" equilibrium is also the "unique equilibrium".

Keywords: strictly-signed-diagonal pay-off matrix, bi-matrix game, equilibrium, completely mixed

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1. Introduction: In this note, we are concerned with the decision-making process of an agent, who we refer to as a "decision maker" (DM). The scenario is interactive, in the sense, that the pay-off from an action chosen by the decision-maker, depends on the action chosen by a "representative agent" that the decision-maker is interacting with. It is customary to refer to this "representative agent" as "nature", though "the ghost" may be an equally (if not more) appropriate name for the representative agent. We will assume that the DM can choose an action from a non-empty finite set of actions and the ghost can also choose an action from a non-empty finite set of actions. The DM believes that the pay-off to the ghost from any action that the ghost chooses, depends on the action chosen by the DM. Such decision-making problems are generally referred to as bi-matrix games (https://en.wikipedia.org/wiki/Bimatrix_game).

The DM knows the pay-offs it will earn from each pair of actions chosen by it and the ghost. The DM would ideally wish to choose an action that maximizes its pay-off given the action chosen by the ghost and has reasons to believe the ghost would wish to do the same. What the DM does not know is the pay-off function of the ghost, and without this information, it cannot proceed with the decision-making problem it faces. Hence, the DM has to conjecture the pay-off function of the ghost. This kind of decision-making problem and its related solution concept (referred to as the "Ellsberg solution") has been discussed in Lahiri (2021).

Under such circumstances, a widely used recommendation is that the DM chooses a "strategy" that maximizes its minimum pay-off. This amounts to the DM assuming that its gain is the precise loss incurred by the ghost and conversely. Ferguson (2000) provides a detailed theory for such decision-making procedures. An alternative possibility is that DM assumes that the ghost's pay-off function is identical to its own pay-off function. Such decision-making problems are referred to as "common pay-off matrix games", whose formal genesis in a very general context can be traced back to the Marschak (1955) paper on the theory teams with notable subsequent contributions available in Takashi (2009) and Emmons, Oesterheld, Critch, Conitzer and Russell, S. (2022). In the kind of interactions we are interested in, some new results are available in Lahiri (2026).

In this note we are interested in those decision-making problems where the DM and the ghost has the same "number" of available actions to choose from. We refer to such problems as "square bi-matrix games", and its importance (as for instance in the context of several results noted in Chandrasekaran (n.d.)) cannot be over emphasized. In fact, we go a step further and assume that the pay-off matrix (derived from the pay-off function) of the DM is a diagonal matrix with either all entries along the diagonal or all entries along the diagonal being negative. We refer to a diagonal matrix with all entries along the diagonal being positive as a "positive-diagonal matrix", and we refer to a diagonal matrix with all entries along the diagonal being negative as a "negative-diagonal matrix". We refer to a matrix which is either a positive-diagonal matrix or a negative-diagonal matrix as a "strictly-signed-diagonal matrix".

In this note we are concerned with the pay-off matrix for the ghost that the DM conjectures arising out of a rearrangement of the pay-offs of the DM that in addition satisfies the requirement "no two distinct diagonal entries" in the pay-off matrix of the DM share either a row or a column in common after rearrangement. We refer to decision-making problems where the DM's pay-off matrix is a strictly-signed-diagonal pay-off matrix and the pay-off matrix that the DM conjecture for the ghost is of the kind that we mentioned above as a *strictly-signed-diagonal pay-off matrix based bi-matrix game*.

Our main result in this note, is that every strictly-signed-diagonal pay-off matrix based bi-matrix game has a "unique completely mixed" equilibrium. This means that there exists a pair of randomizations chosen by the DM and the ghost such that every action is chosen with a positive probability and the cardinality of the set of such pairs is "one". There may be other equilibria, but then such equilibria cannot be completely mixed. However, if the pay-off matrix of the ghost is a diagonal matrix with entries along the diagonal having a sign that is exactly opposite to the sign of the entries along the diagonal of the pay-off matrix of the DM, then the "unique completely mixed" equilibrium is the "unique equilibrium". When all equilibria of a bi-matrix game are completely mixed, as in the case just mentioned, the game itself is said to be completely mixed.

2. Some notations and important matrices: We begin this section by introducing important notations and important matrices.

Important notations: (i) Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. For $z \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$ let z_i denote the i^{th} coordinate of z. Let $\mathbb{R}^n_+ = \{z \in \mathbb{R}^n | z_i \geq 0, i = 1, \ldots, n\}$. Unless otherwise mentioned, a point in $\mathbb{R}^n$ will be interpreted as a "*column vector*".

(ii) For all $i \in \{1, \ldots, n\}$, let $E^{(n,i)}$ be the n-dimensional column vector whose j^{th} coordinate for $j \in \{1, \ldots, n\}$ is equal to 1 if j = i, and is equal to 0 if j ≠ i. $E^{(n,i)}$ is said to be the **n-dimensional i^th^ unit coordinate (column) vector**. Let $E^{(n)} = \sum_{j=1}^{n} E^{(n,j)}$ denote the **n-dimensional sum (column) vector**, i.e., the n-dimensional column vector with all coordinates equal to 1.

Definition: (i)Given a positive integer 'n' a *one-to-one function* from {1, ..., n} to itself is said to be a **permutation** on {1, ..., n}.

(ii) The **identity permutation** on {1, …, n} denoted by id is such that id(i) = i, for all $i \in \{1, \ldots, n\}$.

Note 1: Let f be the permutation on {1, …, n} such that for all $i \in \{1, \ldots, n\}$, f(i) = n - i + 1. Clearly, $f(i) \in \{1, \ldots, n\}$ and f(f(i)) = i for all $i \in \{1, \ldots, n\}$.

Definition: A "*one-to-one function*" from {1, …, n}×{1, …, n} to {1, …, n}×{1, …, n} such that the image of (i, j)(∈{1, …, n}× {1, …, n}) is (π(i, j), ρ(i, j)) of for some functions π, ρ from {1, …, n}×{1, …, n} to {1, …, n} is said be a **n×n square-matrix permutation** and denoted by the pair (π, ρ). A n×n square-matrix permutation (π, ρ) is said to be **unique along the diagonal** if for all i, j∊{1, …, n}: [i ≠ j] implies [π (i, i) ≠π (j, j) and ρ (i, i) ≠ ρ (j, j)].

Important matrices: Given an n×n real-valued square matrix (*alternatively*, a square matrix of size 'n') C for some positive integer 'n' and i,j∊{1, …, n}, let c_{ij} denote the entry at the intersection of the i^{th} row and j^{th} column ($(i, j)^{th}$ entry) of the matrix C. For i∊{1, …, n} let C_i denote the i^{th} row (vector) of C and C^i denote the i^{th} column (vector) of C.

(i) Let C^T be the **transpose** of C, i.e., for i,j∊{1, …, n}, the $(i, j)^{th}$ entry of C^T is c_{ji}.

(ii) A square matrix C of size 'n' is said to be a **diagonal matrix** if i≠ j implies $c_{ij} = 0$.

(iii) Given a n×n square-matrix permutation (π, ρ), let $C^{(\pi,\rho)}$ be the square matrix of size 'n' whose $(\pi(i, j), \rho(i, j))^{th}$ entry for i, j∊{1, …, n} denoted by $c^{(\pi,\rho)}_{\pi(i,j)\rho(i,j)} = c_{ij}$ and $C^{*(\pi,\rho)}$ be the square matrix of size 'n' whose whose $(\rho(i, j), (\pi(i, j))^{th}$ entry for i, j∊{1, …, n} denoted by $c^{*(\pi,\rho)}_{\rho(i,j)\pi(i,j)} = c^{(\pi,\rho)}_{\pi(i,j)\rho(i,j)} = c_{ij}$.

Thus, $C^{*(\pi,\rho)} = C^{(\rho,\pi)}$

$C^{*(\pi,\rho)}$ is the transpose of $C^{(\pi,\rho)}$.

Let (f, f) denote the n×n square-matrix permutation that maps (i, j)∊{1, …, n}×{1, …, n} to (f(i), f(j)). (f, f) is unique along the diagonal.

We will denote the square-matrix of size 'n' whose $(f(i), f(j))^{th}$ term is c_{ij} by $C^{(f, f)}$.

$C^{(f,f)T}$ is denoted by C^* and the $(f(j), f(i))^{th}$ entry of C^* is c_{ij}.

Let (id, f) denote the n×n square-matrix permutation that maps (i, j)∊{1, …, n}×{1, …, n} to (i, f(j)). (id, f) is unique along the diagonal.

$C^{(id,f)}$ is denoted by $\widehat{C}$ and for i, j∊{1, …, n}, the $(id(i), f(j))^{th}$ entry of $\widehat{C}$ is denoted by $\hat{c}_{if(j)}$ and is equal to c_{ij}.

Let (id, id) denote the n×n square-matrix permutation that maps (i, j)∊{1, …, n}×{1, …, n} to (i, j). (id, id) is unique along the diagonal and $C^{(id, id)} = C$.

The following example illustrates the different kinds of matrices discussed above.

Example 1: For n = 3, $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$, $C^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$, $C^* = \begin{bmatrix} c_{33} & c_{23} & c_{13} \\ c_{32} & c_{22} & c_{12} \\ c_{31} & c_{21} & c_{11} \end{bmatrix}$, $C^{*T} = C^{(f, f)} = \begin{bmatrix} c_{33} & c_{32} & c_{31} \\ c_{23} & c_{22} & c_{21} \\ c_{13} & c_{12} & c_{11} \end{bmatrix}$, $\widehat{C} = C^{(id, f)} = \begin{bmatrix} c_{13} & c_{12} & c_{11} \\ c_{23} & c_{22} & c_{21} \\ c_{33} & c_{32} & c_{31} \end{bmatrix}$, $\widehat{C}^T = \begin{bmatrix} c_{13} & c_{23} & c_{33} \\ c_{12} & c_{22} & c_{32} \\ c_{11} & c_{21} & c_{31} \end{bmatrix}$, $C^{(f, id)} = \begin{bmatrix} c_{31} & c_{32} & c_{33} \\ c_{21} & c_{22} & c_{23} \\ c_{11} & c_{12} & c_{13} \end{bmatrix}$, $C^{(f, id)T} = \begin{bmatrix} c_{31} & c_{21} & c_{11} \\ c_{32} & c_{22} & c_{12} \\ c_{33} & c_{23} & c_{13} \end{bmatrix}$.

3. The framework of analysis: We assume there are two-players- the "decision-maker" (DM), hereafter referred to as the row player and "the ghost", hereafter referred to as the column player- with the pure strategy set of the row player being the rows of the matrix A and the pure strategy set of the column player being the columns of the matrix A.

If for i, j∈{1, …, n}, if the row player chooses the i^{th} row and the column player chooses the j^{th} column, the pay-off to the row player is a_{ij}.

The payoff matrix for the column player (that is conjectured by the row player?) is a n×n real-valued square matrix B, i.e., if for i, j∈{1, …, n}, if the row player chooses the i^{th} row and the column player chooses the j^{th} column, the pay-off to the column player is b_{ij}.

The pair (A, B) is said to be a **square bi-matrix game**. If B = A, then the pair (A, A) is an example of a common pay-off matrix game. Common-payoff matrix games have been defined and discussed in Lahiri (2026) and in general such games allow the possibility of A not being a square matrix.

Let $\Delta^{n-1} = \{z \in \mathbb{R}^n_+ | \sum_{i=}^{n} z_i = 1\}$. If $z \in \Delta^{n-1}$, then z is said to be a **randomization**.

For a randomization z, the set of {i∈{1, …, n}| $z_i > 0$} is said to be **the support** of z and is denoted by "support (z)".

A randomization z is said to be **completely mixed** if support (z) = {1, …, n}.

A pair $(x, y) \in \Delta^{n-1} \times \Delta^{n-1}$ is said to be a **strategy profile**.

A strategy profile (x, y) is said to be **completely mixed** if support (x) = support (y) = {1, …, n}.

A strategy profile (x^*, y^*) is said to be an **equilibrium** for the square bi-matrix game (A, B), if for all i∈{1, …, n: $A_i y^* \le x^{*T} A y^*$ and $x^{*T} B^i \le x^{*T} B y^*$.

An equilibrium that is completely mixed, is said to be a **completely mixed equilibrium**.

A diagonal matrix A of size n is said to be a **positive-diagonal matrix** if there exists an array <α_i|i∈{1, …, n}> of "*positive real numbers*" such that for all i, j∈{1, …, n}: $a_{ij} = \alpha_i$ if i = j and $a_{ij} = 0$ if i ≠ j.

A diagonal matrix A of size n is said to be a **negative-diagonal matrix** if there exists an array <α_i|i∈{1, …, n}> of "*negative real numbers*" such that for all i, j∈{1, …, n}: $a_{ij} = \alpha_i$ if i = j and $a_{ij} = 0$ if i ≠ j.

A diagonal matrix A of size n is said to be a **strictly-signed-diagonal matrix** if it is either a positive-diagonal matrix or it is a negative-diagonal matrix.

Assumption: In this note we shall assume that A is a strictly-signed-diagonal matrix. Thus, for all z, w$\epsilon\mathbb{R}^n$, $z^TAw = \sum_{i=1}^{n} \alpha_i z_i w_i$.

Such a pay-off matrix A may be referred to as a **strictly-signed-diagonal pay-off matrix**.

Let (π, ρ) be "n×n square matrix permutation that is unique along the diagonal".

$A^{(\pi,\rho)}$ is the square matrix of size n, whose $(\pi(i, j), \rho(i, j))^{th}$ term is α_i if i = j and is equal to '0' otherwise.

Since $A^{*(\pi,\rho)} = A^{(\rho,\pi)}$, in the general framework that allows π and ρ to vary, there are two distinct possibilities for B, namely B = $A^{(\pi,\rho)}$ and B = -$A^{(\pi,\rho)}$.

Let Bϵ{$A^{(\pi,\rho)}$, -$A^{(\pi,\rho)}$}.

If π = ρ, then it is easy to see that since (π, ρ) is a n×n square matrix permutation that is unique along the diagonal, it must be the case $A^{(\pi,\pi)}$ is a strictly-signed-diagonal matrix.

If Bϵ{$A^{(\pi,\rho)}$, -$A^{(\pi,\rho)}$}, then (A, B) is said to be a **strictly-signed-diagonal pay-off matrix based bi-matrix game**.

There are therefore "four" types of strictly-signed-diagonal pay-off matrix based bi-matrix games, two types corresponding to A being a positive-diagonal matrix and two types corresponding to A being a negative-diagonal matrix.

Results for B = A = $A^{(id, id)}$ are substantially implied by results reported in Lahiri (2026) and B = -A = -$A^{(id, id)}$ is within the scope of Ferguson (2000).

4. The main result: In this section we present two propositions that prove if (A, B) is a strictly-signed-diagonal pay-off matrix based bi-matrix game, then it has exactly one completely mixed equilibrium.

For what follows, recall that A is strictly-signed-diagonal matrix of size n.

For iϵ{1, ..., n}, let let $\gamma_i = \prod_{k\in\{1,\dots,n\}\setminus\{i\}} |\alpha_k| > 0$ and let $y^*\epsilon\Delta^{n-1}$ be such that $y_i^* = \frac{\gamma_i}{\sum_{j=1}^{n}\gamma_j}$ for iϵ{1, ..., n}. Thus, if A is a positive-diagonal matrix, then for all iϵ{1, ..., n}, $A_iy^* = \alpha_i\gamma_i = \prod_{k\in\{1,\dots,n\}} |\alpha_k| > 0$, and if A is a negative-diagonal matrix, then for all iϵ{1, ..., n}, $A_iy^* = \alpha_i\gamma_i = -\prod_{k\in\{1,\dots,n\}} |\alpha_k| < 0$.

Thus, for all $x\epsilon\Delta^{n-1}$: $x^TAy^* = A_iy^*$ for all iϵ{1, …, n}.

Lemma 1: There exists a unique $y\epsilon\Delta^{n-1}$ such that that $A_iy = A_jy$ for all i, jϵ{1, …, n}.

Proof: Suppose $A_iy = A_jy$ for all i, j∊{1, …, n}. Clearly, $A_iy = \alpha_i y_i$ for all i, j∊{1, …, n}.

Since y ≠ 0, there exists i∊{1, …, n} such that $y_i > 0$.

Thus, $\alpha_j y_j = A_jy = A_iy = \alpha_i y_i < 0$ for all i∊{1, …, n}.

Towards a contradiction suppose $y \neq y^*$.

Thus, there exists i∊{1, …, n}, such that $y_i \neq y_i^*$.

Suppose $y_i > y_i^*$.

If A is a positive-diagonal matrix, then, $\alpha_i y_i > \alpha_i y_i^*$ and hence $\alpha_j y_j > \alpha_j y_j^*$ implying $y_j > y_j^*$ for all j∊{1, …, n}.

Since, y, $y^* \epsilon \Delta^{n-1}$, we get $1 = \sum_{j=1}^{n} y_j > \sum_{j=1}^{n} y_j^* = 1$, which is not possible.

If A is a negative-diagonal matrix, then, $\alpha_i y_i < \alpha_i y_i^*$ and hence $\alpha_j y_j < \alpha_j y_j^*$ implying $y_j > y_j^*$ for all j∊{1, …, n}.

Since, y, $y^* \epsilon \Delta^{n-1}$, we get $1 = \sum_{j=1}^{n} y_j > \sum_{j=1}^{n} y_j^* = 1$, which is not possible.

A similar contradiction results if we assume $y_i < y_i^*$.

Thus, $y = y^*$. This proves the lemma. Q.E.D.

Lemma 2: Suppose that B is a square matrix of size 'n', such that for some permutation g on {1, ..., n} it is the case that for all i, j∊{1, …, n}, the (i, j)th entry b_{ij} of B is $\beta_j \neq 0$ if i = g(j) and '0' otherwise. If <u>either</u> $\beta_j > 0$ for all j∊{1, …, n} <u>or</u> $\beta_j > 0$ for all j∊{1, …, n}, then, there is at most one $x \epsilon \Delta^{n-1}$ such that $x^T B^k = x^T B^j$ for all j, k∊{1, …, n}.

Proof: Towards a contradiction suppose there exists x, $w \epsilon \Delta^{n-1}$, with x ≠ w such that $x^T B^k = x^T B^j$ for all j, k∊{1, …, n} and $w^T B^k = w^T B^j$ for all j, k∊{1, …, n}.

Note that for all j∊{1, …, n}, $x^T B^j = x_{g(j)} \beta_j$ and $w^T B^j = w_{g(j)} \beta_j$.

From here on, the proof replicates the corresponding steps in the proof of lemma 1. Q.E.D.

Note 2: B∊{$A^{(\pi,\rho)}$, -$A^{(\pi,\rho)}$} implies there exists a permutation g on {1, ..., n} such that <u>either</u> (i) For all i, j∊{1, …, n}, the (i, j)th entry b_{ij} of B is $\alpha_j \neq 0$ if i = g(j) and '0' otherwise, <u>or</u> (ii) For all i, j∊{1, …, n}, the (i, j)th entry b_{ij} of B is $-\alpha_j \neq 0$ if i = g(j) and '0' otherwise.

Proposition 1: Suppose that for some n×n square-matrix permutation (π, ρ) that is unique along the diagonal, let B = $A^{(\pi,\rho)}$. Let x^* be such that for i∊{1, …, n}, $x^*_{\pi(i,i)} = y_i^*$. Then, (x^*, y^*) is the "unique completely mixed equilibrium" for (A, B).

Proof: Since (π, ρ) is a $n \times n$ square matrix permutation that is unique along the diagonal, for all i, jϵ {1, …, n}, there exists a unique kϵ {1, …, n}, such that $\rho(k, k) = \rho(i, j)$ and the $(\pi(k, k), \rho(k, k))^{th}$ term of B is $a_{kk} = \alpha_k$, and further $[k' \epsilon \{1, …, n\} \backslash \{k\}]$ implies $[\pi(k', k') \neq \pi(k, k), \rho(k', k') \neq \rho(k, k)$ and the $(\pi(k', k'), \rho(k', k'))^{th}$ term of B is $a_{k'k'} = \alpha_{k'}]$.
If A is a positive-diagonal matrix, then for all kϵ {1, …, n}, $x^{*T}B^{\rho(k,k)} = x^*_{\pi(k,k)}a_{kk} = y^*_k\alpha_k = \alpha_k\prod_{i\in\{1,…,n\}\backslash\{k\}}|\alpha_i| = \prod_{i\in\{1,…,n\}}|\alpha_i| > 0$.
Since $\{\rho(k, k)| k\epsilon\{1, …, n\}\} = \{\pi(k, k)| k\epsilon\{1, …, n\}\} = 1, …, n\}$, it follows that $x^{*T}B^k = \prod_{i\in\{1,…,n\}}|\alpha_i| > 0$ for all kϵ{1, …, n}.
Thus, $x^{*T}By^* = \prod_{i\in\{1,…,n\}}|\alpha_i| = x^{*T}B^{\rho(k,k)}$ for all kϵ{1, …, n}.
Further, $x^{*T}Ay^* = A_iy^*$ for all kϵ{1, …, n}.
Thus, (x^*, y^*) is an equilibrium for (A, B).
If A is a negative-diagonal matrix, then for all kϵ {1, …, n}, $x^{*T}B^{\rho(k,k)} = x^*_{\pi(k,k)}a_{kk} = y^*_k\alpha_k = \alpha_k\prod_{i\in\{1,…,n\}\backslash\{k\}}|\alpha_i| =- \prod_{i\in\{1,…,n\}}|\alpha_i| < 0$.
Since $\{\rho(k, k)| k\epsilon\{1, …, n\}\} = \{\pi(k, k)| k\epsilon\{1, …, n\}\} = 1, …, n\}$, it follows that $x^{*T}B^k = - \prod_{i\in\{1,…,n\}}|\alpha_i| < 0$ for all kϵ{1, …, n}.
Thus, $x^{*T}By^* = - \prod_{i\in\{1,…,n\}}|\alpha_i| = x^{*T}B^{\rho(k,k)}$ for all kϵ{1, …, n}.
Further, $x^{*T}Ay^* = A_iy^*$ for all kϵ{1, …, n}.
Thus, (x^*, y^*) is an equilibrium for (A, B).
Since for all iϵ{1, …, n}, $x^*_i > 0$ and $y^*_i > 0$, (x^*, y^*) is a completely mixed equilibrium for (A, B).
Thus by lemmas 1 and 2, it follows that (x^*, y^*) is the unique completely mixed equilibrium for (A, B). Q.E.D.
Proposition 2: Suppose that for some $n \times n$ square-matrix permutation (π, ρ) that is unique along the diagonal, let $B = -A^{(\pi,\rho)}$. Let x^* be such that for iϵ{1, …, n}, $x^*_{\pi(i,i)} = y^*_i$. Then, (x^*, y^*) is the "unique completely mixed equilibrium" for (A, B).
Proof: As in the proof of proposition 1, for kϵ {1, …, n}, $x^{*T}B^{\rho(k,k)} = -x^*_{\pi(k,k)}a_{kk} = -y^*_k\alpha_k = -\alpha_k\prod_{i\in\{1,…,n\}\backslash\{k\}}|\alpha_i|$
If A is a positive-diagonal matrix then $-\alpha_k\prod_{i\in\{1,…,n\}\backslash\{k\}}|\alpha_i| = -\prod_{i\in\{1,…,n\}}|\alpha_i| < 0$.
If A is a negative-diagonal then $-\alpha_k\prod_{i\in\{1,…,n\}\backslash\{k\}}|\alpha_i| = \prod_{i\in\{1,…,n\}}|\alpha_i| > 0$.

Since $\{\rho(k, k)| k\epsilon\{1, \dots, n\}\} = \{\pi(k, k)| k\epsilon\{1, \dots, n\}\} = 1, \dots, n\}$, it follows that either $x^{*T}B^k = - \prod_{i\in\{1,\dots,n\}} |\alpha_i| < 0$ for all $k\epsilon\{1, \dots, n\}$ or $x^{*T}B^k = \prod_{i\in\{1,\dots,n\}} |\alpha_i| > 0$, for all $k\epsilon\{1, \dots, n\}$.

Thus, $x^{*T}By^* = x^{*T}B^k$, for all $k\epsilon\{1, \dots, n\}$.

Further, $x^{*T}Ay^* = A_iy^*$ for all $i\epsilon\{1, \dots, n\}$.

Thus, (x^*, y^*) is an equilibrium for (A, B).

Since for all $i\epsilon\{1, \dots, n\}$, $x_i^* > 0$ and $y_i^* > 0$, (x^*, y^*) is a completely mixed equilibrium for (A, B).

Thus by lemmas 1 and 2, it follows that (x^*, y^*) is the unique completely mixed equilibrium for (A, B).Q.E.D.

The following theorem is the main result of this note. The two propositions above lead to the first part of the theorem.

Theorem 1: If (A, B) is a strictly-signed-diagonal pay-off matrix based bi-matrix game, then it has a unique completely mixed equilibrium. If in addition $\pi = \rho$ and $B = - A^{(\pi, \pi)}$ then the "unique completely mixed" equilibrium of (A, B) is the "unique" equilibrium of (A, B).

Proof: The first part is a direct consequence of propositions 1 and 2. Hence, suppose $\pi = \rho$ and $B = - A^{(\pi, \pi)}$. By note 2, there exists a permutation g on $\{1, \dots, n\}$, such that for all $(i, j) \epsilon \{1, \dots, n\}\times\{1, ..., n\}$, $b_{ij} = 0$ if $i\neq j$ and $b_{g(i)g(i)} = \alpha_i$ if $i = j$.

Suppose (x, y) is an equilibrium for (A, B).

First suppose A is a positive-diagonal matrix so that $\alpha_i > 0$ for all $i\epsilon\{1, \dots, n\}$.

If $x^TAy = 0$, then $x_iy_i = 0$ for all $i\epsilon\{1, \dots, n\}$. Since $y\epsilon\Delta^{n-1}$, there exists $k\epsilon\{1, \dots, n\}$ such that $y_k > 0$. Then, $E^{(n,k)T}Ay = \alpha_ky_k > 0 = x^TAy$, contradicting (x, y) is an equilibrium for (A, B).

Thus, $x^TAy > 0$. Thus, $\{i\epsilon\{1, \dots, n\}|x_iy_i > 0\}\neq \phi$. Hence, $x^TBy = - \sum_{i=1}^{n} \alpha_i x_{g(i)} y_{g(i)} < 0$.

It is easy to see that $x^TB^{g(i)} = -x_{g(i)}\alpha_i \leq 0$ and $A_iy = \alpha_iy_i \geq 0$ for all $i\epsilon\{1, \dots, n\}$.

Towards a contradiction suppose $x^TB^{g(i)} = 0$ for some $i\epsilon\{1, \dots, n\}$. Then, $x^TBE^{(n,g(i))} = 0 > x^TBy$ contradicting (x, y) is an equilibrium for (A, B).

Thus, $x^TB^{g(i)} = -x_{g(i)}\alpha_i < 0$ for all $i\epsilon\{1, \dots, n\}$.

Thus, $x_i > 0$, and hence $x^TB^{g(i)} = x^TBy < 0$, for all $i \epsilon\{1, \dots, n\}$

Now, towards a contradiction suppose $y_i = 0$ for some $i \epsilon\{1, \dots, n\}$.

Since $x_i > 0$, $\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}}(x_j + \frac{x_i}{n-1})y_j\alpha_j = \sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_j y_j \alpha_j + \frac{x_i}{n-1}\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} y_j\alpha_j > \sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_j y_j\alpha_j$, since $\frac{x_i}{n-1} > 0$ and $y_i = 0$ implies $\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} y_j\alpha_j > 0$.

Since $y_i = 0$, $\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_j y_j\alpha_j = \sum_{j=1}^{n} x_j y_j \alpha_j = x^TAy$.

Let $z\epsilon\mathbb{R}^n$ be such that $z_j = x_j + \frac{x_i}{n-1}$ for $j\epsilon\{1, \ldots, n\}\backslash\{i\}$, $z_j = 0$.

Thus, $z^TAy = \sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}}(x_j + \frac{x_i}{n-1})y_j\alpha_j > x^TAy$, contradicting (x, y) is an equilibrium for (A, B).

Thus, $y_i > 0$ for all $i \epsilon\{1, \ldots, n\}$.

Thus, (x, y) is completely mixed.

By proposition 2 such an equilibrium must be unique.

Thus, (A, B) has a unique equilibrium, and this equilibrium is completely mixed.

Now suppose A is a negative-diagonal matrix so that $\alpha_i < 0$ for all $i\epsilon\{1, \ldots, n\}$..

Thus, B is a positive-diagonal matrix.

If $x^TBy = 0$, then $x_iy_i = 0$ for all $i\epsilon\{1, \ldots, n\}$. Since $x\epsilon\Delta^{n-1}$, there exists $k\epsilon\{1, \ldots, n\}$ such that $x_{g(k)} > 0$. Then, $x^TBE^{(n, g(k))} = -\alpha_k x_{g(k)} > 0 = x^TBy$, contradicting (x, y) is an equilibrium for (A, B).

Thus, it must be the case that $x^TBy > 0$. Thus, $\{i\epsilon\{1, \ldots, n\}|x_iy_i > 0\} \neq \phi$. Hence, $x^TAy = \sum_{i=1}^{n} \alpha_i x_i y_i < 0$.

$x^TB^{g(i)} = -x_{g(i)}\alpha_i \geq 0$ and $A_iy = \alpha_iy_i \leq 0$ for all $i\epsilon\{1, \ldots, n\}$.

Towards a contradiction, suppose $A_iy = 0$ for some $i\epsilon\{1, \ldots, n\}$.

Then, $E^{(n,i)T}Ay = 0$ contradicting (x, y) is an equilibrium for (A, B).

Thus, $A_iy < 0$ for all $i\epsilon\{1, \ldots, n\}$ and hence $y_i > 0$ for all $i\epsilon\{1, \ldots, n\}$.

Now, towards a contradiction suppose $x_{g(i)} = 0$ for some $i \epsilon\{1, \ldots, n\}$.

Thus, $x^TBy = -\sum_{j\epsilon\{1,\ldots,n\}} x_{g(j)}y_{g(j)}\alpha_j = -\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_{g(j)}y_{g(j)}\alpha_j$

Since $y_{g(i)} > 0$, $-\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}}(y_{g(j)} + \frac{y_{g(i)}}{n-1})x_{g(j)}\alpha_j = -\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_{g(j)}y_{g(j)}\alpha_j - \frac{y_{g(i)}}{n-1}\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_{g(j)}\alpha_j > -\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_{g(j)}y_{g(j)}\alpha_j$, since $\frac{y_{g(i)}}{n-1} > 0$ and $x_{g(i)} = 0$ implies $-\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_{g(j)}\alpha_j > 0$.

Let $w\epsilon\mathbb{R}^n$ be such that $w_{g(j)} = y_{g(j)} + \frac{y_{g(i)}}{n-1}$ for $j\epsilon\{1, \ldots, n\}\backslash\{i\}$, $w_{g(i)} = 0$.

Thus, $x^TBw = -\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}}(y_{g(j)} + \frac{y_{g(i)}}{n-1})x_{g(j)}\alpha_j > -\sum_{j\epsilon\{1,\ldots,n\}\backslash\{i\}} x_{g(j)}y_{g(j)}\alpha_j = x^TBy$, contradicting (x, y) is an equilibrium for (A, B).

Thus, $x_{g(i)} > 0$ for all $i \in \{1, \ldots, n\}$ and hence $x_i > 0$ for all $i \in \{1, \ldots, n\}$.

Thus, (x, y) must a completely mixed equilibrium.

By proposition 2 such an equilibrium must be unique.

Thus, (A, B) has a unique equilibrium, and this equilibrium is completely mixed.

Q.E.D.

Note 3: None of the propositions claim uniqueness of equilibrium. We prove that every strictly-signed-diagonal pay-off matrix based bi-matrix game has a completely mixed equilibrium and the cardinality of the set of such equilibria is "one". It is only in the case where -B is a diagonal matrix, that the "unique completetely mixed equilibrium of (A, B) is also the "unique equilibrium" of (A, B).

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