Physics of Deflection

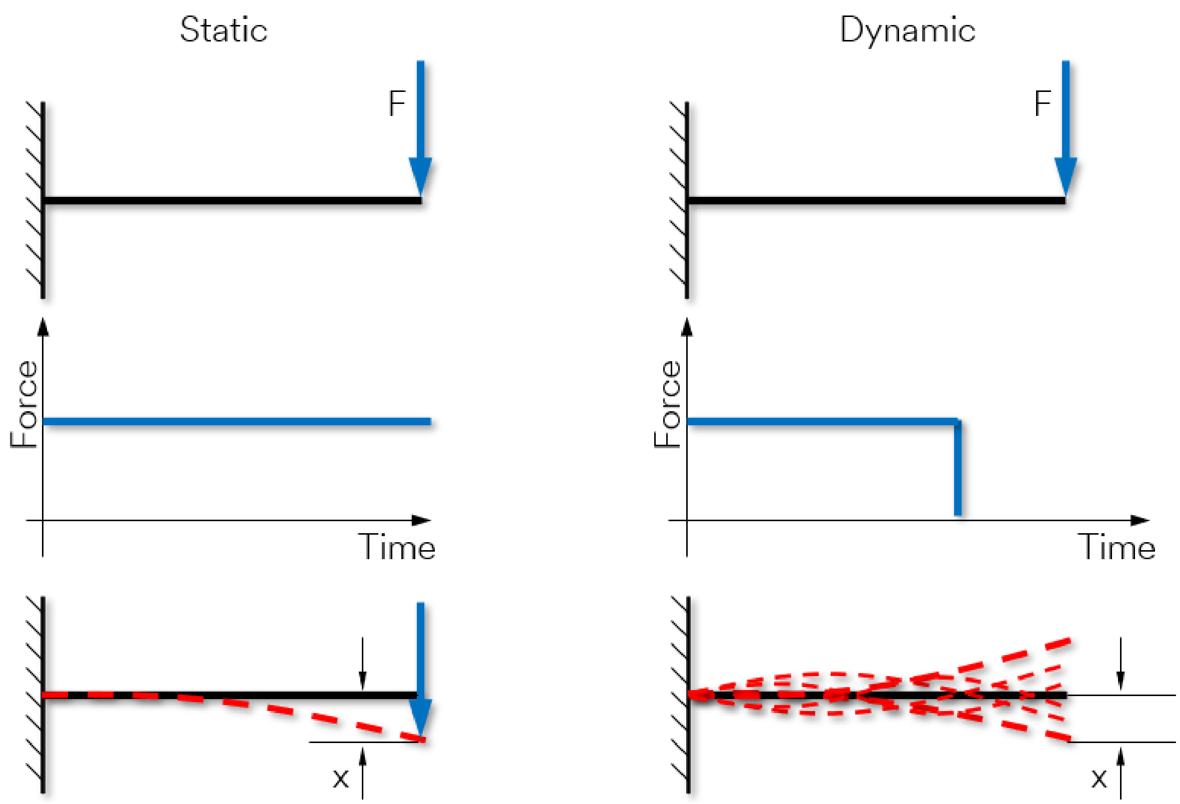
Grzegorz Świtek, Wroclaw October 2025

1. Abstract

Deflection remains one of the most critical and yet least intuitively understood phenomena in structural engineering. Newtonian mechanics provide consistent and powerful framework for static deflection analysis. However, Newtonian formalism falls short to fully explaining subtleties of dynamic vibrations and even worse, buckling. In the everyday life of a machine design engineer the triad of static, dynamic and buckling deflection analysis and optimization is fundamental. However, engineers typically stop at static because of a murky understanding of dynamics and buckling. Of the two misunderstood phenomena the buckling is the more elusive. Common explanations why buckling deflection does not follow the direction of applied force revolve around minute geometric and material imperfections. Experimental evidence contradicts this theory as the nature of displacement is predictable and can be calculated or simulated with great fidelity. To align with experimental data these small imperfections would have to occur every time with the same distribution and magnitude. The imperfections theory not only fails to resolve this contradiction but also lacks the capacity to foster accurate intuition about the actual behavior of structures under load. This theoretical research aims to develop a first principles understanding of the physics underlying deflection.

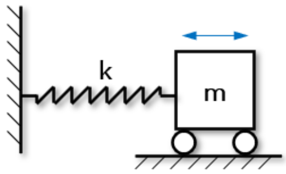
2. Potential and kinetic energy connection

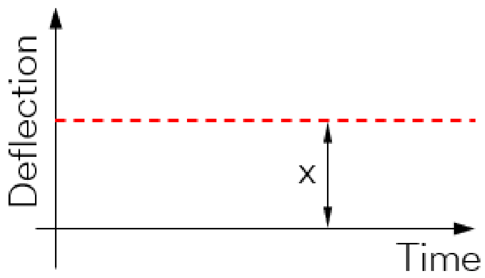
Let's consider simple cantilevered rod in static and dynamic load scenarios as follows.

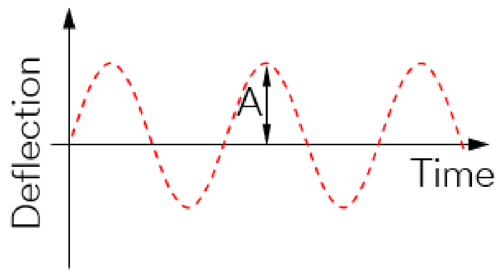


In astatic load scenario the force produces displacement X in the direction of the load. Similarly, in a dynamic load scenario comparable displacement occurs. When the force is suddenly removed, the rod does not exhibit simple harmonic oscillations, contrary to what might be intuitively expected. Instead, the motion is complex, with the rod bending into various shapes whose amplitudes decrease. However, this complexity is often negligible compared to the dominant motion, which resembles the initial bending shape. One explanation is that the rod's shape under bending does not fully conform to an eigen shape of vibration, resulting in the excitation of higher-order modes that help dissipate energy. A simpler explanation follows from first principles: oscillatory motion occurs only when the system is disturbed from its equilibrium position. As a result, the rod must initially bend into a complex shape to produce vibrations of complex shapes.

Let's consider analogic simple harmonic oscillator.







$$E_p = \frac{1}{2}kx^2 \quad (1)$$

$$E_k = \frac{1}{2}m\omega^2A^2 = 2\pi^2mf^2A^2 \quad (2)$$

E_p – potential energy,

E_k - kinetic energy

k – stiffness,

m – mass,

A, x – deflection

The following flows from the energy conservation law.

$$E_p = E_k$$

$$x = A$$

$$\frac{1}{2}kx^2 = 2\pi^2mf^2A^2$$

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad (3), \qquad \omega = \sqrt{\frac{k}{m}} \quad (4)$$

The obtained equation for frequency (angular velocity ω) governs the resonance of the simple harmonic oscillator, which serves as a suitable model for further analysis. Simple harmonic oscillators can only support one oscillation mode, so we need deeper definition of the resonance to understand the behavior of the cantilever rod. The resonance is both a temporal and a spatial phenomenon. The spatial aspect relates to the shape of the vibrating body and can be described using the concept of standing waves. For a cantilevered rod, the node (the point of no displacement) of the standing wave is at the fixed end, and the antinode (the point of maximum displacement) is at the free end. Other standing wave patterns can also satisfy this condition, but they require higher frequencies (i.e., higher angular velocities) to occur. These frequencies are integer multiples of the fundamental resonance frequency, corresponding to standing waves with an increasing number of nodes.

$$\omega_n = n\sqrt{\frac{k}{m}}, \qquad n = 1, 2, 3, \dots \quad (5)$$

Equation (2) suggests that, for a given kinetic energy, the system's vibration amplitude decreases as the angular velocity (or frequency) increases. This relationship holds for simple harmonic oscillators but cannot be universally applied, as it does not hold in all cases. A cantilever rod cannot be accurately modeled by a single massless spring and a concentrated mass. Instead, it must be represented as a set of harmonic oscillators—one for each degree of freedom or vibration mode. Modeling a cantilever rod using simple harmonic oscillators requires converting the distributed mass into an equivalent system consisting of a massless spring and a point mass—known as the effective mass. The effective mass represents the portion of the total mass that contributes to the dynamic response. It can be calculated by rearranging Equation (4), using the known values of angular velocity and stiffness.

$$m_{effective} = \frac{k}{\omega^2} \quad (6)$$

Modal stiffness is obtained by rearranging the equation (4) as well.

$$k_{modal} = \omega^2 m_{effective} \quad (7)$$

3. Stiffness connection

It is important to note that the units of stiffness obtained from equation (7) are kilograms per second squared, whereas static stiffness is defined as force divided by the deflection it causes, resulting in a unit of newtons per meter. The difference between the units is apparent, but it provides an important insight.

$$\frac{N}{m} = \frac{kgm}{s^2}\frac{1}{m} = \frac{kg}{s^2}$$

Simple vibration mode shapes correspond to simple static bending shapes, which represent static global stiffness. Global static stiffness is more meaningful when expressed in units of newtons per meter, as the deflection caused by a force can then be clearly defined. Complex vibration mode shapes correspond to local static stiffness, where stiffness cannot be defined using a simple static force and deflection. In such cases, stiffness is generalized and expressed in kilograms per second squared—often referred to as modal or effective stiffness. This is the second strong connection between static and dynamic structural behavior.

An additional important observation can be made in the discussed example. The cantilever rod with a circular cross-section has only one simple bending shape: a force acting on its tip in a plane perpendicular to the rod's axis, with its line of action passing through the center of the cross-section, will always produce the same deflection at any angle. Hence, there is only one stiffness in all directions. However, there is another stiffness related to pure tension, occurring when the force acts axially on the rod's tip. Similarly, different stiffnesses arise when a force acts eccentrically to the axis or its line of action does not pass through the center of the cross-section—and many more such cases exist. If these stiffnesses have a much higher value than the simple bending stiffness, their contribution to the vibration motion is often small or negligible. Nonetheless, they still exist.

In any structural system, deflection analysis should begin with the determination of the directions of minimum stiffness. Stiffness is not a vector, but it has implicit directionality. Working with scalar quantities such as stiffness enables the translation of complex geometric problems into an algebraic framework, allowing the use of advanced analytical methods. Nonetheless, the implicit directionality must be preserved throughout the analysis to ensure accurate results.

Stiffness is used in finite element analysis software for vibration modes and buckling simulations. In both cases, it is a linear algebra problem where the solution involves finding the eigenvalues and eigenvectors of a system of matrices.

$$(K - M\omega^2)\phi = 0 \quad (8)$$ vibration modes analysis

$$(K - \lambda K_G)\phi = 0 \quad (9)$$ buckling analysis

$$Ku = f \quad (10)$$ static Hook's law in matrix form

Where,

K – stiffness matrix

K_G – geometric stiffness matrix (depends on pre-stress)

M – mass matrix

ω - natural frequency (eigen value)

ϕ – shape vector (eigen vector)

λ – critical load factor (eigen value)

u – displacement vector

f – force vector

In mechanical dynamics, stiffness plays a crucial role and is closely related to three key concepts: dynamic stiffness, mechanical impedance, and inertance.

Dynamic stiffness $$K(\omega) = \frac{F(\omega)}{x(\omega)} = \boldsymbol{k} - \omega^2 m + j\omega c \quad (11)$$

Mechanical impedance $$Z(\omega) = \frac{F(\omega)}{v(\omega)} = c + j\left(\omega m - \frac{\boldsymbol{k}}{j\omega}\right) \quad (12)$$

Inertance $$M(\omega) = \frac{F(\omega)}{a(\omega)} = m - \frac{\boldsymbol{k}}{\omega^2} - j\frac{c}{\omega} \quad (13)$$

F(ω) – Force in frequency domain

x(ω) – Displacement response in frequency domain

v(ω) – Velocity response in frequency domain

a(ω) – Acceleration response in frequency domain

k – stiffness

m – mass

c – damping

ω - angular velocity

j – imaginary unit $j = \sqrt{-1}$

Stiffness is an essential parameter in the physics of displacement. It has a perfect level of abstraction—low enough to remain practical, yet high enough to avoid the distraction of cumbersome details such as material properties, geometry, and boundary conditions of the system under study.

4. Principle of the least action

The energy conservation law discussed earlier coincidentally resembles the Lagrangian used in the principle of least action.

$$\mathcal{L} = T - V \quad (14)$$

T – Kinetic energy

V – Potential energy

$$\text{If} \quad T - V = 0, \quad \text{then} \quad T = V$$

The system is in a state where kinetic and potential energies balance perfectly, making the Lagrangian vanish.

This coincidence touches on the deeper symmetry: when energies are equal, the system is at a kind of "equipartition" point, which often corresponds to interesting dynamics (like in harmonic oscillators at mid-position).

The principle of least action finds important applications in analyzing motion. If we recognize that static deflection results from a transition between an initial and a final equilibrium state, we can then explore the stationary action path connecting them. As before, let's consider a simple harmonic oscillator to determine the generic value of action for the transition from the neutral position to the final displacement caused by a static force.

Using equations (1) and (2) for potential and kinetic energy, the Lagrangian takes a well-known form.

$$\mathcal{L} = \frac{1}{2}mV^2 - \frac{1}{2}kx^2 \quad (15)$$

The path satisfying stationary action is obtained from Lagrange – Euler equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad (16)$$

The obtained equation of motion takes the familiar form.

$$m\ddot{x} + kx = F \quad (17)$$

General solution to this equation of motion is

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{F}{k} \quad (18)$$

A and B depend on the initial conditions.

Initial equilibrium condition:

for $t_0 = 0$ displacement is $x(t) = 0$ and velocity $\dot{x}(t) = 0$

therefore $A = -\frac{F}{k}$ and $B = 0$

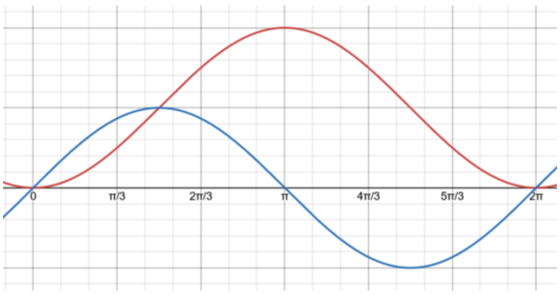
The equation of motion for given initial conditions is:

$$x(t) = -\frac{F}{k}\cos(\omega t) + \frac{F}{k}$$

The velocity:

$$\dot{x}(t) = \omega\frac{F}{k}\sin(\omega t)$$

The equation of motion (red) and velocity (blue) chart for $F = k = 1$, which results in $\omega = 1$



The maximum displacement (momentary static equilibrium) is achieved when velocity is equal to zero at time $t = \pi$.

The action is

$$S = \int_0^t \mathcal{L}dt \quad (19)$$

Given the equations of motion and velocity the upper bound of integration is $\omega t = \pi$, then $t = \frac{\pi}{\omega}$

Which after substitutions give:

$$S = \int_0^{\frac{\pi}{\omega}} \left(\frac{1}{2} m\omega^2 \frac{F^2}{k^2} (sin(\omega t))^2 - \frac{F^2}{2k} (1 - 2\, cos(\omega t) + (cos(\omega t))^2) \right) \mathrm{dt}$$

Which finally takes the following form:

$$S = -\frac{\pi E_p}{\omega} \quad (20)$$

$$\text{or } \; S = -\pi E_p \sqrt{\frac{m}{k}} \quad (21)$$

The obtained general action equation provides further insight into the nature of deflection. Action decreases with the potential energy stored in the system but increases with the natural frequency. Consequently, systems with higher resonant frequencies will produce greater action. The action is linear with negative slope if stiffness remains constant. These insights are fully consistent with experimental observations. As previously discussed, any system can be modeled as a set of simple harmonic oscillators—one for each degree of freedom. The system will reach equilibrium by exhibiting deflections in the degrees of freedom according to the least action principle. Furthermore, the resulting deflections do not directly depend on the direction of the applied force, but rather on the potential energy stored and the system's stiffness configuration. This offers a new approach to deflection analysis and the understanding of buckling, where the observed deflection does not follow the direction of the applied force. Linearity of action is also supported by everyday experience. Deflection follows changes in force linearly, up to the point where stiffness is no longer constant. This phenomenon is referred to in civil engineering as stiffness loss, which can lead to catastrophic structural failure. The same applies to buckling, where the system initially behaves linearly but becomes nonlinear at certain force level, leading to failure at the critical load. On the other hand, systems under very small buckling loads and high stiffness do not exhibit buckling deflection, because the potential energy stored is close to zero and the action is approximately equal in all directions.

5. General law of deflection

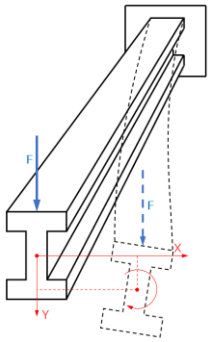
The body of arguments provided supports the following postulate, which governs the deflection behavior of systems subjected to loading.

The shape and magnitude of deflection depend on the system's stiffness configuration and its evolution toward static equilibrium.

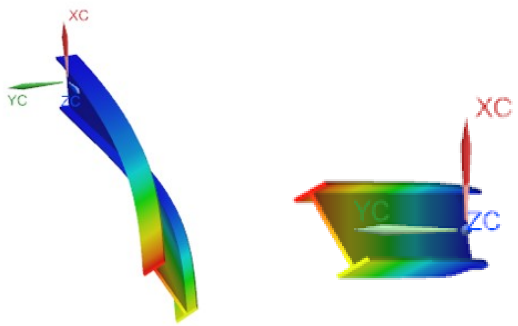
The system's stiffness configuration can be conveniently obtained from modal analysis, which provides modal stiffness, effective mass, and modal shapes. The evolution of such a system depends on the magnitude and direction of the applied force, in accordance with the principle of least action. Deflection that occurs precisely in the direction of the applied force represents a special case. To effectively control deflection, the system should be designed so that the applied force aligns with the direction of least action.

6. Lateral – Torsional Buckling (LTB).

Lateral–torsional buckling occurs most severely in long cantilevered I-beams whose cross-section has two weakest stiffness directions that are perpendicular to each other. To distinguish between two practical loading scenarios for I-beams, these two directions are referred to as the strong and weak axes. The LTB occurs only when strong axis is loaded as illustrated below.



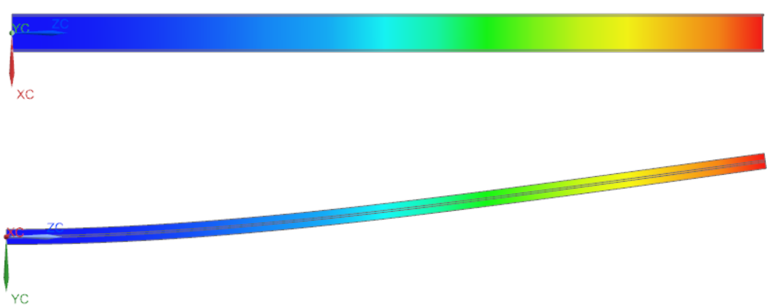
The deflection shape of the system in static equilibrium is too complex to be explained by a simple load scenario. Deflection consists of three primary components: simple bending along the strong and weak axes, and simple torsion. Static finite element analysis (FEA) would only result in deflection in the direction of strong axis. An analyst unfamiliar with the LTB phenomenon might misinterpret such a result, leading to a significant error. The following result is obtained from linear buckling FEA (NX Nastran).



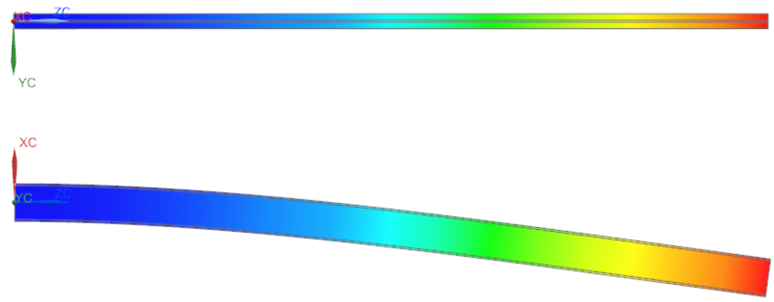
Only two components are represented in linear buckling FEA. The strong axis deflection is missing. Also, the simple torsion is presented without a warp which is always present for such a cross-section shapes.

Now let’s review modal FEA (NX Nastran).

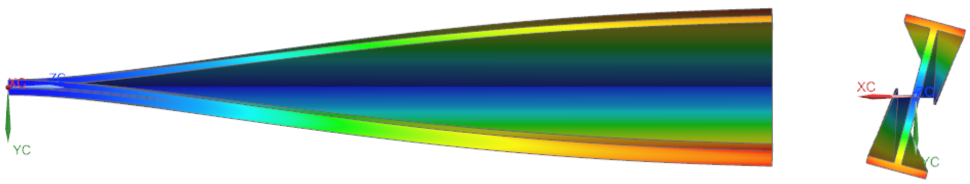
Mode 1 – deflection of the weak axis



Mode 2 - deflection of the strong axis



Mode 3 – torsion with warp



The superposition of the first three modes from the modal FEA matches the actual deflection shape of lateral–torsional buckling (LTB). The modal FEA magnitudes are typically normalized and therefore do not represent the actual deflection magnitude caused by the applied load. Modal stiffness in this case is approximately the same as the corresponding static bending stiffness and can be extracted for use in further analysis.

The LTB phenomenon is better understood in light of the general law of deflection. Static equilibrium in the first mode, which has the lowest natural frequency, is achieved with the least action. This results in the highest magnitude of deflection, even if the force is applied in the direction of the second mode. The second and third modes will follow, each with progressively smaller deflection magnitudes. These three modes are fundamental given the system's boundary conditions. Higher-order modes are more complex versions of these three fundamental modes, characterized by an increasing number of nodes. They require significantly more action to be activated, making their contribution negligible. The LTB phenomenon would not occur if the force were applied to the weak axis. In such a case all other modes require higher action, so they occur with negligible magnitude. A force applied along the weak axis would follow the direction of least action.

7. Conclusion

The examples discussed are trivial and have been thoroughly analyzed by many researchers. In practice, real engineering problems are rarely so straightforward. Understanding special, simplified cases can offer limited insight when addressing complex systems. Modal analysis is a suitable candidate to play a central role in understanding both static and dynamic behavior. Stress–strain, buckling, and other types of analysis should not be disregarded, but they become secondary once the system's fundamental nature is well understood.