Positive-Diagonal Bi-Matrix Games: Five Cases

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Abstract

We consider five types of positive-diagonal bi-matrix games, which are special cases of positive-diagonal pay-off matrix based bi-matrix games, and provide explicit formulas for each equilibrium that we propose. In all the five cases there are multiple equilibria and thus require coordination on the part of the agents.

Keywords: positive-diagonal bi-matrix game, diagonal reflection, anti-diagonal reflection, equilibrium, completely mixed

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1. Introduction: In this note, we are concerned with five types of positive-diagonal based bi-matrix games, which are cases of "positive-diagonal pay-off matrix based bi-matrix games". Positive-diagonal pay-off matrix based bi-matrix games are discussed in Lahiri (2026b) and in the same work a very general and reasonably strong "equilibrium existence result" is proved.

Let us briefly recapitulate the interactive decision-making problem faced by a decision-maker (DM) that is discussed along with general results in Lahiri (2026b). The scenario is interactive, in the sense, that the pay-off from an action chosen by the DM, depends on the action chosen by a "representative agent" that the DM is interacting with. We refer to this "representative agent" as "the ghost". The DM can choose an action from a non-empty finite set of actions and the ghost can also choose an action from a non-empty finite set of actions. The DM believes that the pay-off to the ghost from any action that the ghost chooses, depends on the action chosen by the DM. Such decision-making problems are generally referred to as bi-matrix games (https://en.wikipedia.org/wiki/Bimatrix_game).

The DM knows the pay-offs it will earn from each pair of actions chosen by it and the ghost. The DM would ideally wish to choose an action that maximizes its pay-off given the action chosen by the ghost and has reasons to believe the ghost would wish to do the same. What the DM does not know is the pay-off function of the ghost, and without this information, it cannot proceed with the decision-making problem it faces. Hence, the DM has to conjecture the pay-off function of the ghost. This kind of decision-making problem and its related solution concept (referred to as the "Ellsberg solution") has been discussed in Lahiri (2021).

Under such circumstances, a widely used recommendation is that the DM chooses a "strategy" that maximizes its minimum pay-off. This amounts to the DM assuming that its gain is the precise loss incurred by the ghost and conversely. Ferguson (2000) provides a detailed theory for such decision-making procedures. An alternative possibility is that the pay-off matrix of the ghost is identical to the pay-off matrix of the DM. Such decision-making problems are called "common-payoff games", some new results for which are available in Lahiri (2026a).

In Lahiri (2026b) as well as here, the DM and the ghost has the same "number" of available actions to choose from. We refer to such problems as "square bi-matrix games", and its importance (as for instance in the context of several results noted in Chandrasekaran (n.d.)) cannot be over emphasized. As in Lahiri (2026b), we go a step further and assume that the pay-off matrix (derived from the pay-off function) of the DM is a diagonal matrix with positive entries along the diagonal. We refer to such a pay-off matrix as a "positive-diagonal pay-off matrix". In Lahiri (2026b) it is assumed that the pay-off matrix of the ghost is obtained by rearranging the entries of the pay-off matrix of the DM, while ensuring that "no two" positive entries share a row or a column after the rearrangement. In this note, we will discuss a strict subset of such decision making problems, so that we can obtain more context specific results than the ones available in Lahiri (2026b).

To illustrate the kind of problems that we shall be concerned with consider a positive-diagonal pay-off matrix of size two, say $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where a and b are both positive real numbers. The two rows correspond to the two actions available to the DM, and the two columns correspond to the two actions available to the ghost. If both choose their first action the DM gets a pay-off of 'a'. If both choose their second action the DM gets a pay-off of 'b'. For all other pairs of actions, the pay-off to the DM is zero.

If the DM conjectures that the pay-off matrix of the ghost is $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then the decision-making problem is a common pay-off matrix game, which has been discussed in Lahiri (2026a). If the DM conjectures that the pay-off matrix of the ghost is $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$, then decision-making problem falls within the scope of what is discussed in Ferguson (2000), for which there are abundant results that are presented in this discussion. If the conjectured pay-off matrix of the ghost is $\begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix}$, then from the second sentence of theorem 1 in Lahiri (2026b), we know that there is a unique equilibrium and this equilibrium is completely mixed.

In this note we shall discuss five other conjectures which together with the three already mentioned leads to a class of interactive decision-making problem that we refer to as "positive-diagonal bi-matrix games". Such problems are a strict subset of "positive-diagonal pay-off matrix based bi-matrix games" that are discussed in Lahiri (2026b).

We refer to the matrix $\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$ as the "diagonal reflection" of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and to the matrix $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ as the "anti-diagonal reflection" of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. The diagonal reflection is symmetric but the transpose of the "anti-diagonal diagonal reflection" $\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$ is a third possible possible conjecture. Associated with these three conjectures are the three other possible conjectures, namely, $\begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix}$, $\begin{bmatrix} 0 & -a \\ -b & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -b \\ -a & 0 \end{bmatrix}$ of which the first matrix is already discussed exhaustively in Lahiri (2026b).

An equilibrium is said to be "completely mixed" if the randomizations chosen by the DM and the ghost assign positive probabilities to all actions available to them.

The main result in Lahiri (2026b) says that each "positive-diagonal pay-off matrix based bi-matrix game" has a "unique completely mixed" equilibrium, without restricting the possibility of other equilibria that may not be completely mixed. Further, if the pay-off matrix of the ghost is diagonal with negative entries along the diagonal, then there is a unique equilibrium and this equilibrium is completely mixed.

An important consequence of our results reported in section 3 of this note, is that results based on the assumption that the pay-off matrix of the ghost is the diagonal reflection of the pay-off matrix of the DM immediately imply corresponding results for "anti-diagonal symmetric coordination games" that are discussed in Lahiri (2026a).

If the the pay-off matrix of the ghost is the anti-diagonal reflection of the pay-off matrix of the DM or its negative, there are several symmetric equilibria including several symmetric equilibria in pure strategies.

For each and every case, we provide explicit formulas to calculate the equilibria proposed for it. In all the five cases discussed here, there are a muliplicity of equilibria, thereby leading to problems that require coordination between the DM and the ghost on the outcome of the decision-making problem.

2. The Framework of Analysis: We begin this section by introducing important notations and important matrices.

Important notations: (i) Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. For $z \epsilon \mathbb{R}^n$ and $i \epsilon \{1, \ldots, n\}$ let z_i denote the i^{th} coordinate of z. Let $\mathbb{R}^n_+ = \{z \epsilon \mathbb{R}^n | z_i \geq 0, i = 1, \ldots, n\}$. Unless otherwise mentioned, a point in $\mathbb{R}^n$ will be interpreted as a "*column vector*".

(ii) For all $i \epsilon \{1, \ldots, n\}$, let $E^{(n,i)}$ be the n-dimensional column vector whose j^{th} coordinate for $j \epsilon \{1, \ldots, n\}$ is equal to 1 if j = i, and is equal to 0 if $j \neq i$. $E^{(n,i)}$ is said to be the **n-dimensional i^{th} unit coordinate (column) vector**. Let $E^{(n)} = \sum_{j=1}^{n} E^{(n,j)}$ denote the **n-dimensional sum (column) vector**, i.e., the n-dimensional column vector with all coordinates equal to 1.

(iii) For $i \epsilon \{1, \ldots, n\}$, let f(i) = n - i + 1.

Clearly, f(f(i)) = i for all $i \epsilon \{1, \ldots, n\}$.

Important matrices: Given an n×n real-valued square matrix (*alternatively*, a square matrix of size 'n') C for some positive integer 'n' and $i,j \epsilon \{1, \ldots, n\}$, let c_{ij} denote the entry at the intersection of the i^{th} row and j^{th} column ($(i, j)^{th}$ entry) of the matrix C.

(i) Let C^T be the **transpose** of C, i.e., for $i,j \epsilon \{1, \ldots, n\}$, the $(i, j)^{th}$ entry of C^T is c_{ji}.

(ii) A square matrix C of size 'n' is said to be a **diagonal matrix** if $i \neq j$ implies $c_{ij} = 0$.

(iii) A square matrix C of size 'n' is said to be a **anti-diagonal matrix** if $j \neq f(i)$ implies $c_{ij} = 0$.

(iv) Given a diagonal matrix C of size n, its **diagonal reflection** is the diagonal matrix C^* of size 'n', such that for all $i \epsilon \{1, \ldots, n\}$, its $(i, i)^{th}$ term is $c_{f(i)f(i)}$.

(v) Given a diagonal matrix C of size n, its **anti-diagonal reflection** is the anti-diagonal matrix $\widehat{C}$ of size 'n', such that for all $i \epsilon \{1, \ldots, n\}$, its $(i, f(i))^{th}$ term is c_{ii}.

Note 2.1: The anti-diagonal reflection of (the diagonal reflection of C) C^* denoted $\widehat{C^*}$ is the anti-diagonal matrix of size 'n', such that for all $i \epsilon \{1, \ldots, n\}$, its $(i, f(i))^{th}$ is $c_{f(i)i}$.

$\widehat{C^*} = \widehat{C}^T$

Note 2.2: In Lahiri (2026b), C^* is written as $C^{(f, f)T}$ and $\widehat{C}$ is $C^{(id,f)}$, with "id" in the present context being the identity function from {1, ..., n} to itself.

Suppose A is a n×n real-valued square matrix for some positive integer $n \geq 2$.

We assume there are two-players- the "decision-maker" (DM), hereafter referred to as the row player and "the ghost", hereafter referred to as the column player- with the pure strategy set of the row player being the rows of the matrix A and the pure strategy set of the column player being the columns of the matrix A.

If for i, j∊{1, …, n}, if the row player chooses the i^{th} row and the column player chooses the j^{th} column, the pay-off to the row player is a_{ij}.

The payoff matrix for the column player (that is conjectured by the row player?) is a n×n real-valued square matrix B, i.e., if for i, j∊{1, …, n}, if the row player chooses the i^{th} row and the column player chooses the j^{th} column, the pay-off to the column player is b_{ij}.

The pair (A, B) is said to be a **square bi-matrix game**. If B = A, then the pair (A, A) is an example of a common pay-off matrix game. Common-payoff matrix games have been defined and discussed in Lahiri (2026a) and in general such games allow the possibility of A not being a square matrix.

For i∊{1, …, n}, let A_i, B_i denote the i^{th} row of A and B respectively and A^i, B^i denote the i^{th} column of A and B respectively.

Let $\Delta^{n-1} = \{z \in \mathbb{R}^n_+ | \sum_{i=}^{n} z_i = 1\}$. If $z \in \Delta^{n-1}$, then z is said to be a **randomization**. The set Δ^{n-1} is said to be the **set of all randomizations**.

For a randomization z, the set of {i∊{1, …, n}| $z_i > 0$} is said to be **the support** of z and is denoted by "support (z)".

A randomization z is said to be **completely mixed** if support (z) = {1, …, n}.

For i∊{1, …, n}, $E^{(n,i)}$ is said to be the **i^{th} pure strategy**.

A non-empty subset S of {1, …, n} is said to be a **symmetric set of pure-strategies** if i∊S implies f(i)∊S.

A pair $(x, y) \in \Delta^{n-1} \times \Delta^{n-1}$ is said to be a **strategy profile**.

A strategy profile (x, y) is said to be **completely mixed** if support (x) = support (y) = {1, …, n}.

A strategy profile (x^*, y^*) is said to be an **equilibrium** for the square bi-matrix game (A, B), if for all i∊{1, …, n: $A_i y^* \leq x^{*T}Ay^*$ and $x^{*T}B^i \leq x^{*T}By^*$.

If for some i,j∊{1, …, n}, $(E^{(n,i)}, E^{(n,j)})$ is an equilibrium then it said to be an **equilibrium in pure strategies** and (A, B) is said to **have an equilibrium in pure strategies**.

A randomization x^* is a **symmetric equilibrium** for (A, B) if (x^*, x^*) is an equilibrium for (A, B).

Assumption: In this note we shall assume that there exists an array $<\alpha_i|i\epsilon\{1, \ldots, n\}>$ of *strictly positive real numbers* such that for all i, j$\epsilon\{1, \ldots, n\}$: $a_{ij} = \alpha_i$ if i = j and a_{ij} = 0 if i ≠ j.

Thus, for all z, w$\epsilon\mathbb{R}^n$, $z^TAw = \sum_{i=1}^{n} \alpha_i z_i w_i$.

Such a pay-off matrix A may be referred to as a **positive-diagonal pay-off matrix**.

If B$\epsilon\{A, -A, A^*, -A^*, \widehat{A}, -\widehat{A}, \widehat{A}^T, -\widehat{A}^T\}$, then (A, B) is said to be a **positive-diagonal bi-matrix game**.

Results for B = A are substantially implied by results reported in Lahiri (2026a) and B = -A is within the scope of Ferguson (2000). B = $-A^*$ is a special case of positive-diagonal pay-off matrix based bi-matrix games in second sentence of theorem 1 in Lahiri (2026) and hence has a "unique equilibrium" that is completely mixed. In this note we shall therefore restrict our attention to the remaining five types of positive-diagonal bi-matrix games, namely, $\{(A, B)|B\epsilon\{A^*, \widehat{A}, -\widehat{A}, \widehat{A}^T, -\widehat{A}^T\}\}$.

3. The Case B = A^*: A^* is the diagonal reflection of A. In this case B is the diagonal matrix whose (i, j)th term is $\alpha_{f(i)}$ if j = i and is '0' otherwise.

Example 3.1: For n = 2 and strictly positive real numbers a, b, let A = $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ so that B = $\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$. Clearly, $E^{(2,1)}$ and $E^{(2,2)}$ are symmetric equilibria for (A, B). So is (x^*, y^*), where $x_1^* = y_2^* = \frac{a}{a+b}$, $x_2^* = y_1^* = \frac{b}{a+b}$.

$x^{*T}Ay^* = \frac{ab}{a+b} = x^{*T}By^*$.

We shall omit the rather obvious proof of the following proposition.

Proposition 3.1: For all i$\epsilon\{1, \ldots, n\}$, $E^{(n,i)}$ is a symmetric equilibrium for (A, B).

For S any non-empty subset of {1, …, n} let f(S) = {f(i)| iϵS}.

Let S be any non-empty subset of {1, …, n} containing at least two elements.

For all iϵS, let $\gamma_i^S = \prod_{k\in S\setminus\{i\}} \alpha_k > 0$ and let $y^{(S)}\epsilon\Delta^{n-1}$ be such that $y_i^{(S)} = \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S}$ for iϵS and $y_i^{(S)} = 0$ for i$\epsilon\{1, \ldots, n\}\setminus$S.

For all iϵS, let $\delta_i^S = \prod_{k\in S\setminus\{i\}} \alpha_{f(k)} > 0$ and let $x^{(S)}\epsilon\Delta^{n-1}$ be such that $x_i^{(S)} = \frac{\delta_i^S}{\sum_{j\in S}\delta_j^S}$ for iϵS and $x_i^{(S)} = 0$ for i$\epsilon\{1, \ldots, n\}\setminus$S.

Proposition 3.2: $(x^{(S)}, y^{(S)})$ is an equilibrium for (A, B).

Proof: For i$\in$S, $A_i y^{(S)} = \alpha_i y_i^{(S)} = \alpha_i \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S} = \alpha_i \frac{\prod_{k\in S\setminus\{i\}}\alpha_k}{\sum_{j\in S}\gamma_j^S} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} > 0$ and $x^{(S)T}B^i =$

$\alpha_{f(i)} x_i^{(S)} = \alpha_{f(i)} \frac{\delta_i^S}{\sum_{j\in S}\delta_j^S} = \alpha_{f(i)} \frac{\prod_{k\in S\setminus\{i\}}\alpha_{f(k)}}{\sum_{j\in S}\delta_j^S} = \frac{\prod_{k\in S}\alpha_{f(k)}}{\sum_{j\in S}\delta_j^S} > 0.$

For i$\notin$S, $A_i y^{(S)} = 0 = x^{(S)T}B^i$.

Thus, $x^{(S)T}Ay^{(S)} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} \geq A_i y^{(S)}$ for all i$\in$\{1, …, n\} and $x^{(S)T}By^{(S)} = \frac{\prod_{k\in S}\alpha_{f(k)}}{\sum_{j\in S}\delta_j^S} \geq x^{(S)T}B^i$ for all i$\in$\{1, …, n\}.

Thus, $(x^{(S)}, y^{(S)})$ is an equilibrium for (A, B). Q.E.D.

Note 3.1: As in Lahiri (2026a), we refer to the pair the pair $(\widehat{A}, \widehat{A}^T)$ as an **anti-diagonal symmetric coordination game**.

For $y\in\mathbb{R}^n$ let $z(y)\in\mathbb{R}^n$ be such that for all i$\in$\{1, …, n\}, $z_i(y) = y_{f(i)}$.

Clearly $z(y)\in\Delta^{n-1}$ <u>if and only if</u> $y\in\Delta^{n-1}$.

Thus, for $x,y\in\mathbb{R}^n$, $x^T\widehat{A}y = \sum_{i=1}^n \alpha_i x_i y_{f(i)} = \sum_{i=1}^n \alpha_i x_i z_i(y) = x^T A z(y)$ and $x^T\widehat{A}^T y = \sum_{i=1}^n \alpha_{f(i)} x_i y_{f(i)} = \sum_{i=1}^n \alpha_{f(i)} x_i z_i(y) = x^T A^* z(y)$.

Thus, $(x, y)\in\Delta^{n-1}\times\Delta^{n-1}$ is an equilibrium for $(\widehat{A}, \widehat{A}^T)$ <u>if and only if</u> [(x, z(y)) is an equilibrium for (A, A^*)]. Hence, the results concerning anti-diagonal symmetric coordination games in Lahiri (2026a), follow immediately from the results noted in this section.

4. The Case B = $\widehat{A}$: $\widehat{A}$ is the anti-diagonal reflection of A. In this case B is the anti-diagonal matrix whose (i, j)th term is α_i if j = f(i) and is equal to 0 otherwise.

Example 4.1: For n = 2, and strictly positive real numbers a, b, let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ so that $B = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$. It is easy to see that (A, B) has no equilibrium in pure strategies.

However (x^*, y^*) with $x_1^* = y_1^* = \frac{b}{a+b}$, $x_2^* = y_2^* = \frac{a}{a+b}$ is an equilibrium for (A, B).

$x^{*T}Ay^* = \frac{ab}{a+b}$ and $x^{*T}By^* = \frac{ab}{a+b}$. Since $x^* = y^*$, x^* is a symmetric equilibrium for (A, B).

Let S be a symmetric set of pure strategies. For all i$\in$S, let $\gamma_i^S = \prod_{k\in S\setminus\{i\}}\alpha_k > 0$ and let $x^{(S)}\in\Delta^{n-1}$ be such that $x_i^{(S)} = \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S}$ for i$\in$S and $x_i^{(S)} = 0$ for i$\in$\{1, …, n\}\S.

Proposition 4.1: Let S be a symmetric set of pure strategies. Then, (i)$x^{(S)}$ is a symmetric equilibrium for (A, B). (ii) $(x^{(S)}, x^{(S)})$ is the unique equilibrium with both both radomizations having S as its support.

Proof: (i) For $i\in S$, $A_i x^{(S)} = \alpha_i x_i^{(S)} = \alpha_i \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S} = \alpha_i \frac{\prod_{k\in S\setminus\{i\}}\alpha_k}{\sum_{j\in S}\gamma_j^S} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} > 0$ and $x^{(S)T}B^{(i)} =$

$\alpha_{f(i)} x_{j(i)}^{(S)} = \alpha_{f(i)} \frac{\gamma_{f(i)}^S}{\sum_{j\in S}\gamma_j^S} = \alpha_{f(i)} \frac{\prod_{k\in S\setminus\{f(i)\}}\alpha_k}{\sum_{j\in S}\gamma_j^S} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S}$.

For $i\notin\{1, ..., n\}\setminus S$, $x_i^{(S)} = 0 = x_{f(i)}^{(S)}$ and hence $A_i x^{(S)} = 0 = x^{(S)T}B^{(i)}$.

Thus, $x^{(S)T}Ax^{(S)} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} = x^{(S)T}Bx^{(S)} > 0$.

Further, for all $i\in\{1, \ldots, n\}$: $A_i x^{(S)} \leq x^{(S)T}Ax^{(S)}$ and $x^{(S)T}B^i \leq x^{(S)T}Bx^{(S)}$.

Thus, $x^{(S)}$ is a symmetric equilibrium for (A, B).

(ii) The proof of part (ii) is very similar to the proof of uniqueness in proposition 4.1.

Q.E.D.

5. The Case B = - $\widehat{A}$: In this case B is the anti-diagonal matrix whose $(i, j)^{th}$ term is $-\alpha_i$ if $j = f(i)$ and is equal to 0 otherwise. This is a version of what is called a "coordination game" with the row player being "more focused" than the column player for $n > 2$.

Example 5.1: For n = 3, and strictly positive real numbers a, b and c, let A = $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, so that B = $\begin{bmatrix} 0 & 0 & -a \\ 0 & -b & 0 \\ -c & 0 & 0 \end{bmatrix}$. Clearly $(E^{(3,1)}, E^{(3,1)})$ and $(E^{(3,2)}, E^{(3,2)})$ are pure strategy equilibria for (A, B), but $(E^{(3,2)}, E^{(3,2)})$ is not. However, x^* satisfying $x_1^* = \frac{bc}{ab+ac+bc}$, $x_2^* = \frac{ac}{ab+ac+bc}$, $x_3^* = \frac{ab}{ab+ac+bc}$ is a randomization that satisfies $Ax^* = (\frac{abc}{ab+ac+bc})E^{(3)}$ and $x^{*T}B = (\frac{-abc}{ab+ac+bc})E^{(3)T}$, and hence x^* is a symmetric equilibrium for (A, B).

The proof of the following proposition is quite straight forward and hence will be omitted.

Proposition 5.1: For all $i\in\{1, \ldots, n\}$ such that $i\neq \frac{n+1}{2}$, $(E^{(n,i)}, E^{(n,i)})$ is a pure strategy equilibrium for (A, B).

For $i\in\{1, \ldots, n\}$, let $\gamma_i = \prod_{k\in\{1,\ldots,n\}\setminus\{i\}}\alpha_k$ and let $x^*\in\Delta^{n-1}$ be such that $x_i^* = \frac{\gamma_i}{\sum_{j=1}^n \gamma_j}$ for all $i\in N$.

The next proposition which is a generalization of the result in example 5.1 is implied by proposition 2 in Lahiri (2026b) applied to $B = -A^{(id,f)}$.

Proposition 5.2: x^* is a completely mixed symmetric equilibrium for (A, B).

Let S be a non-empty subset of {1, ..., n} such that S is "not" a singleton and i$\in$S implies f(i)$\notin$S (Note that the converse is not required). Further, f(f(i)) = i for all i$\in${1, …, n}and thus [i$\notin$S implies f(i)$\notin$S]. For all i$\in$S, let $\gamma_i^S = \prod_{k\in S\setminus\{i\}} \alpha_k > 0$ and let $x^{(S)}\in\Delta^{n-1}$ be such that $x_i^{(S)} = \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S}$ for i$\in$S and $x_i^{(S)} = 0$ for i$\in${1, …, n}\S.

Clearly, the existence of a non-empty subset S which is not a singleton and satisfies [i$\notin$S implies f(i)$\notin$S] requires $n \geq 4$. For n = 4, the subsets {1, 2} and {3, 4} satisfy the requirement i$\in$S implies f(i)$\notin$S. If n = 4, then f(1) = 4, f(2) = 3, f(3) = 2 and f(4) = 1.

Proposition 5.3: Suppose $n \geq 4$. Then $x^{(S)}$ is a symmetric equilibrium for (A, B).

Proof: For i$\in$S, $A_i x^{(S)} = \alpha_i x_i^{(S)} = \alpha_i \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S} = \alpha_i \frac{\prod_{k\in S\setminus\{i\}}\alpha_k}{\sum_{j\in S}\gamma_j^S} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} > 0$ and $x^{(S)T}B^i = x^{(S)T}B^{j(j(i))} = -\alpha_{f(i)} x_{f(i)}^{(S)} = 0$, since i$\in$S implies f(i)$\notin$S and hence $x_{f(i)}^{(S)} = 0$.

For i$\notin$S, $A_i x^{(S)} = 0$, and $x^{(S)T}B^i = x^{(S)T}B^{f(f(i))} = -\alpha_{f(i)} x_{f(i)}^{(S)} = 0$, since i$\notin$S implies f(i)$\notin$S and hence $x_{f(i)}^{(S)} = 0$.

Thus, $x^{(S)T}B^i = 0$ for all i$\in${1, …, n}, $x^{(S)T}Ax^{(S)} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S}$ and $x^{(S)T}Bx^{(S)} = 0$.

Hence, $A_i x^{(S)} \leq x^{(S)T}Ax^{(S)}$ and $x^{(S)T}B^i \leq x^{(S)T}Bx^{(S)}$ for all i$\in${1, …, n}.

Thus, $x^{(S)}$ is a symmetric equilibrium for (A, B). Q.E.D.

Note 5.1: Alternatively, if $x^{*(S)}\in\Delta^{n-1}$ be such that support $(x^{*(S)}) \subset S$ (so that $x_i^{*(S)} = 0$ for i$\in${1, …, n}\S), then $x^{*(S)T}B^i = x^{*(S)T}B^{f(f(i))} = -\alpha_{f(i)} x_{f(i)}^{*(S)} = 0$, since i$\in$S implies f(i)$\notin$S and hence $x_{f(i)}^{*(S)} = 0$. In view of this the following proposition stands validated.

Proposition 5.4: Suppose $n \geq 4$. $(x^{*(S)}, y^{(S)})$ is a equilibrium for (A, B).

6. The Case B = $\hat{A}^T$: In this case for all i, j$\in${1, …, n}, the (i, j)th term of B is $\alpha_{f(i)}$ if j = f(i), and is equal to 0, otherwise.

Example 6.1: For n = 2, and strictly positive real numbers a, b, let A = $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ so that B = $\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$. It is easy to see that (A, B) has no equilibrium in pure strategies.

However (x^*, y^*) with $x_1^* = y_2^* = \frac{a}{a+b}$, $x_2^* = y_1^* = \frac{b}{a+b}$ is an equilibrium for (A, B).

$x^{*T}Ay^* = \frac{ab}{a+b}$ and $x^{*T}By^* = \frac{ab}{a+b}$. Note that $x^* \neq y^*$.

Example 6.2: For n = 3, and strictly positive real numbers a, b and c, let A = $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, so that B = $\begin{bmatrix} 0 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{bmatrix}$. Clearly ($E^{(3,2)}$, $E^{(3,2)}$) is an equilibrium in pure strategies.

Let S be a symmetric set of pure strategies. If S is a singleton, say {i}, then let $x^{(S)}$ = $y^{(S)}$ = $E^{(n,i)}$. If S is not a singleton, then for i∊S, let let $\gamma_i^S = \prod_{k\in S\setminus\{i\}} \alpha_k > 0$ and let $y^{(S)}\epsilon\Delta^{n-1}$ be such that $y_i^{(S)} = \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S}$ for i∊S and $y_i^{(S)} = 0$ for i∊{1, …, n}\S. Let $x^{(S)}\epsilon\Delta^{n-1}$ be such that $x_i^{(S)} = y_{f(i)}^{(S)}$ for i∊S and $x_i^{(S)} = 0$ for i∊{1, …, n}\S.

Proposition 6.1: ($x^{(S)}$, $y^{(S)}$) is an equilibrium for (A, B).

Proof: If S is a singleton {i}, then S is a symmetric set of pure strategies implies f(i) = { }, i.e., i = $\frac{n+1}{2}$ = f(i).

Thus, $E^{(n,i)T}AE^{(n,i)} = a_{ii} = \alpha_i \geq a_{ki} = A_kE^{(n,i)}$ for all k∊{1, …, n} since for k∊{1, …, n}\{i}, $a_{ki} = 0$.

$E^{(n,i)T}BE^{(n,i)} = E^{(n,i)T}\hat{A}^TE^{(n,i)} = a_{f(i)f(i)} = a_{ii} = \alpha_i \geq a_{ik} = E^{(n,i)T}B^k$, for all k∊{1, …, n}, since for k∊{1, …, n}\{i}, $a_{f(k)j} = 0$.

Thus, suppose S <u>is not</u> a singleton.

For i∊S, $A_iy^{(S)} = \alpha_i y_i^{(S)} = \alpha_i\frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S} = \alpha_i\frac{\prod_{k\in S\setminus\{i\}}\alpha_k}{\sum_{j\in S}\gamma_j^S} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} > 0$ and $x^{(S)T}B^i = \alpha_{f(i)}x_i^{(S)} = \alpha_{f(i)}y_{f(i)}^{(S)} = \alpha_{f(i)}\frac{\gamma_{f(i)}^S}{\sum_{j\in S}\gamma_j^S} = \alpha_{f(i)}\frac{\prod_{k\in S\setminus\{f(i)\}}\alpha_k}{\sum_{j\in S}\gamma_j^S} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} > 0$.

For i∉S, $A_iy^{(S)} = \alpha_i y_i^{(S)} = 0$, since $y_i^{(S)} = 0$ and $x^{(S)T}B^i = \alpha_{f(i)}x_i^{(S)} = 0$, since $y_i^{(S)} = 0$.

Thus, $x^{(S)T}Ay^{(S)} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} \geq A_iy^{(S)}$ for all i∊{1, …, n} and $x^{(S)T}By^{(S)} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} \geq x^{(S)T}B^i$ for all i∊{1, …, n}.

Thus, ($x^{(S)}$, $y^{(S)}$) is an equilibrium for (A, B). Q.E.D.

7. The Case B = - $\hat{A}^T$: In this case for all i, j∊{1, …, n}, the (i, j)th term of B is - $\alpha_{f(i)}$ if j = f(i), and is equal to 0, otherwise.

Example 7.1: For n = 2, and strictly positive real numbers a, b, let A = $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ so that B = $\begin{bmatrix} 0 & -b \\ -a & 0 \end{bmatrix}$. It is easy to see that (A, B) has two in pure strategies, ($E^{(2,1)}$, $E^{(2,1)}$) and ($E^{(2,2)}$, $E^{(2,2)}$). If a = b, then $\frac{1}{2}E^{(2)}$ is a symmetric equilibrium for (A, B).

As in the case of proposition 5.1, the proof of the following proposition is quite straight forward and hence will be omitted. In fact the the two (obvious) proofs are identical.

Proposition 7.1: For all i$\epsilon\{1, \ldots, n\}$ such that i$\neq \frac{n+1}{2}$, $(E^{(n,i)}, E^{(n,i)})$ is a pure strategy equilibrium for (A, B).

For i$\epsilon\{1, \ldots, n\}$, let $\gamma_i = \prod_{k\in\{1,\ldots,n\}\setminus\{i\}} \alpha_k$ and let $y^*\epsilon\Delta^{n-1}$ be such that $y_i^* = \frac{\gamma_i}{\sum_{j=1}^n \gamma_j}$ for all iϵN. Let $x^*\epsilon\Delta^{n-1}$ be such that for all i$\epsilon\{1, \ldots, n\}$, $x_i^* = y_{f(i)}^*$.

The following result is an immediate consequence of proposition 2 in Lahiri (2026b) applied to B = $-A^{(f,id)}$.

Proposition 7.2: (x^*, y^*) is a completely mixed equilibrium for (A, B).

As in section 6, let S be a non-empty subset of {1, ..., n} such that S is "not" a singleton and iϵS implies f(i)$\notin$S (Note that the converse is not required). Clearly, the existence of such a non-empty subset which is not a singleton requires n $\geq$ 4. Further such a set S $\neq$ {1, …, n}.

Note that f(f(i)) = i for all i$\epsilon\{1, \ldots, n\}$and thus [i$\notin$S implies f(i)$\notin$S]. For all iϵS, let $\gamma_i^S = \prod_{k\in S\setminus\{i\}} \alpha_k > 0$ and let $y^{(S)}\epsilon\Delta^{n-1}$ be such that $y_i^{(S)} = \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S}$ for iϵS and $y_i^{(S)} = 0$ for i$\epsilon\{1, \ldots, n\}\setminus S$. Let, $x^{*(S)}\epsilon\Delta^{n-1}$ be such that support $(x^{*(S)}) \subset S$. Thus, $x_i^{*(S)} = 0$ for i$\epsilon\{1, \ldots, n\}\setminus S$.

Proposition 7.3: Suppose n $\geq$ 4. Then, $(x^{(S)}, y^{(S)})$ is an equilibrium for (A, B).

Proof: For iϵS, $A_i y^{(S)} = \alpha_i y_i^{(S)} = \alpha_i \frac{\gamma_i^S}{\sum_{j\in S}\gamma_j^S} = \alpha_i \frac{\prod_{k\in S\setminus\{i\}}\alpha_k}{\sum_{j\in S}\gamma_j^S} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} > 0$ and $x^{*(S)T}B^i = x^{*(S)T}B^{f(f(i))} = -\alpha_{f(f(i))}x_{f(i)}^{*(S)} = 0$, since given support $(x^{*(S)}) \subset S$ and [iϵS implies f(i)$\notin$S] we get $x_{f(i)}^{*(S)} = 0$.

For i$\notin$S, $A_i y^{(S)} = \alpha_i y_i^{(S)} = 0$. Further, i$\notin$S, f(f(i)) = i and the definition of S, implies f(i)$\notin$S. Thus, $x^{*(S)T}B^i = x^{*(S)T}B^{f(f(i))} = -\alpha_{f(f(i))}x_{f(i)}^{*(S)} = 0$, since given support $(x^{*(S)}) \subset S$ and f(i)$\notin$S, we get $x_{f(i)}^{*(S)} = 0$.

Thus, $x^{*(S)T}Ay^{(S)} = \frac{\prod_{k\in S}\alpha_k}{\sum_{j\in S}\gamma_j^S} \geq A_i y^{(S)}$ for all i$\epsilon\{1, \ldots, n\}$ and $x^{*(S)T}By^{(S)} = 0 = x^{*(S)T}B^i$, for all i$\epsilon\{1, \ldots, n\}$.

Thus, $(x^{(S)}, y^{(S)})$ is an equilibrium for (A, B). Q.E.D.

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