A General Pairwise-Markov GLMB Filter

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Abstract

The generalized labeled multi-Bernoulli (GLMB) filter is the first general and exact-closed-form approximation of the labeled multitarget recursive Bayes filter. It is, like almost all single-target tracking and multitarget tracking algorithms, based on hidden Markov models (HMMs). Unfortunately, HMM assumptions are physically unrealistic for many target tracking applications. For this reason, Pieczynski's generalization of the HMM, the pairwise-Markov model (PMM), has attracted interest in the target-tracking community. This paper answers in the affirmative the following question: Is there a theoretically rigorous PMM generalization of the GLMB filter? The update equation for this generalization is similar in form to the measurement-update equation for the GLMB filter, and the derivation of the former is closely patterned after a cleaner probability generating functional (PGFL) derivation of the latter.

1 Introduction

The *generalized labeled multi-Bernoulli* (GLMB) *filter* [23], [24], [25], [7, Chapt. 15] is the first general—and computationally tractable—exact-closed-form approximation[1] of the labeled multitarget recursive Bayes filter (LMRBF)—which is to say, of the labeled random finite set (LRFS) time-sequence of labeled multitarget probability densities (LMPDFs)

$$... \to \mathring{f}_{k-1|k-1}(\mathring{X}_{k-1}|Z_{1:k-1}) \to \mathring{f}_{k|k-1}(\mathring{X}_k|Z_{1:k-1}) \to \mathring{f}_{k|k}(\mathring{X}_k|Z_{1:k}) \to ...$$

where: $\mathring{X}_{0:k} : \mathring{X}_0, \mathring{X}_1, ..., \mathring{X}_k$ is the sequence of multitarget state-sets at times $t_0, t_1, ..., t_k$; $Z_{1:k} : Z_1, ..., Z_k$ is the sequence of multitarget measurement-sets collected by the sensor at times $t_1, ..., t_k$; and $f_{k|k-1}(\mathring{X}_k|Z_{1:k-1})$ resp. $f_{k|k}(\mathring{X}_k|Z_{1:k})$ are the predicted resp. posterior probabilities that the multitarget state at time t_k is $\mathring{X}_k$.

The latest Gibbs-based GLMB filter implementations [1], [4] can simultaneously track over a million 3D targets in real time in significant clutter using off-the-shelf computing equipment. GLMB-type filters have: a) quantifiable

[1] In the sense of [8].

approximation errors [24]; b) linear complexity in the number of measurements [24]; c) log-linear complexity in the number of hypothesized tracks [21]; and d) linear complexity in the number of scans in the multi-scan case [22].

An overview of the LRFS approach and GLMB filter can be found in [25].

The GLMB filter is, like almost all single-target tracking and multitarget tracking (MTT) algorithms, based on *hidden Markov models* (HMMs). That is, the states of targets can be determined only indirectly via measurements collected by a sensor, assuming Markovian target motion and statistically independent sensor and motion noise.

Unfortunately, in many target-tracking applications HMM assumptions are physically unrealistic [28, p. 1, col. 1]. For this reason, W. Pieczynski's clever generalization of the HMM, the *pairwise-Markov model* (PMM) [17], has attracted interest in the target-tracking community. This is because it a) has an underlying HMM structure—see Eq. (56)—and yet allows b) non-Markovian target motion; c) correlated ("colored") measurement noise; and d) estimation of the sensor measurement and target state-transition densities (which are usually assumed a priori) from the measurements.

The following question then naturally arises:

- Is there a theoretically rigorous PMM generalization of the GLMB filter?

PMM has been applied to single-target tracking [20], [28], as well as other applications [2], [3], [18]. As for MTT, the following publications are relevant:

1. Mahler, 2020: tracking $n = 0, 1$ targets using a PMM generalization of the Bernoulli filter [15], [9, Sec. 14.4].

2. Petetin and Desbouvries, 2013: a PMM generalization of the probability hypothesis density (PHD) filter [19].

3. Liu et al., 2019: A particle implementation of a PMM-PHD filter [5].

4. Zhou et al., 2025: a PMM generalization of an approximation of the GLMB filter, the labeled Bernoulli (LMB) filter [27].

5. Zhou et al., 2024: a PMM and jump-Markov generalization of the GLMB filter [26, Sec. 3.1].

The PMM-GLMB filter approach described in [26, Sec. 3.1] is, unfortunately, theoretically unclear. Like the GLMB filter, it consists of a sequence of time-update steps followed by measurement-update steps [26, Eqs. (16,17)].[2] But as will be noted in Section 4, a Pieczynski-consistent multitarget PMM filter must consist of a single step $\mathring{f}_{k-1|k-1} \rightarrow \mathring{f}_{k|k}$ (see Eq. (64)) rather than two steps $\mathring{f}_{k-1|k-1} \rightarrow \mathring{f}_{k|k-1} \rightarrow \mathring{f}_{k|k}$. Moreover, the theoretical basis of the derivations of the two steps $\mathring{f}_{k-1|k-1} \rightarrow \mathring{f}_{k|k-1}$ and $\mathring{f}_{k|k-1} \rightarrow \mathring{f}_{k|k}$ in [26, Sec. 3.1] is, at

[2] In the most recent GLMB filter implementations, the time-update and measurement-update steps are consolidated into a single step in order to improve computability.

least to me, unclear. In particular, it cannot rely on single-target PMMs since these require single-step updates, $\mathring{f}_{k-1|k-1}(\mathring{x}) \to \mathring{f}_{k|k}(\mathring{x})$, see Eq. (60).

The purpose of this paper is to propose a different approach. The original GLMB filter in [24] was derived directly by deducing an exact-closed-form solution of the sequence of LMPDFs. An alternative approach would have been to instead deduce an exact-closed-form solution of the corresponding sequence of probability generating functionals (PGFLs, see Eq. (16)):

$$... \to \mathring{G}_{k-1|k-1}[\mathring{h}_{k-1}|Z_{1:k-1}] \to \mathring{G}_{k|k-1}[\mathring{h}_k|Z_{1:k-1}] \to \mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] \to ...$$

This goal was accomplished in [16]. In this paper we will similarly adopt a PGFL approach, one based on a more polished version of the approach in [16, Sec. III-C], to derive a proposed "PMM-mGLMB filter." This is because it is far more algebraically straightforward than an LMPDF approach (though, of course, still rather complex).

1.1 Summary of Main Results

The following six claims will be demonstrated:

1. A cleaner version of the PGFL derivation of the measurement-update step of the GLMB filter in [16, Sect. III-C]—see Section 2.10. This offers two advantages: a) It permits apples-to-apples comparison of the GLMB and PMM-mGLMB filters; and b) examination of the derivation of the PGFL derivation of the measurement-update of the former in Section 6.2 should clarify the derivation of the latter in Section 5.2. This is because the latter is essentially just a more complex version of the former.

2. The proper form of the general multitarget PMM (MPMM) recursive Bayes filter is a direct generalization of Pieczynski's original single-target PMM recursive Bayes filter—see Eq. (64).

3. In particular, an MPMM transition function that is consistent with the "standard" GLMB multitarget motion and measurement models is given in Eq. (89).

4. This general MPMM filter is exact-closed-form with respect to the family of "modified GLMB" (mGLMB) LMPDFs (see Eq. (24)), thus resulting in a "PMM-mGLMB filter"(Section 5).

5. An mGLMB LMPDF is actually a GLMB LMPDF written in different form—see Section 6.1. This means that the general PMM-mGLMB filter is also exact-closed-form with respect to the family of GLMB LMPDFs.

6. Nevertheless, the PMM-mGLMB filter appears to be preferable to the corresponding PMM-GLMB filter since it is so similar in form to the GLMB filter's measurement-update step (compare Eq. (1) to Eq. (41)).

To briefly summarize (with terminology and notation to be explained later), in Section 5.1 we will demonstrate the following. Let the prior PGFL be mGLMB—that is, it has the form of Eq. (24):

$$\mathring{G}_{k-1}[\mathring{h}_{k-1}|Z_{1:k-1}] = \sum_{o\in O_{k-1}} \sum_{L\subseteq L_{k-1}} \omega^{o}_{k-1}(L) \prod_{l\in L} \sigma^{o}_{l,k-1}[\mathring{h}_{k-1}] \tag{1}$$

with label-set L_{k-1} and "Bernoulli spatial densities" $(q^{o}_{l,k-1}, s^{o}_{l,k-1})$ with corresponding PGFLs

$$\sigma^{o}_{l,k-1}[\mathring{h}_{k-1}] = 1 - q^{o}_{l,k-1} + q^{o}_{l,k-1} s^{o}_{l,k-1}[\mathring{h}_{k-1}]. \tag{2}$$

Then we demonstrate that, if Z_k is the new measurement-set at time t_k, the posterior PGFL is also mGLMB:

$$\mathring{G}_{k}[\mathring{h}_{k}|Z_{1:k}] = \sum_{(o,\alpha_k)\in O_{k}} \sum_{L\subseteq L_{k}} \omega^{o,\alpha_k}_{k}(L) \prod_{l\in L} \sigma^{o,\alpha_k}_{l,k}[\mathring{h}_{k}] \tag{3}$$

where:

1. $L_k = L^B_{k|k-1} \uplus L_{k-1}$, where $L^B_{k|k-1}$ is the set of labels of newly-appearing targets and "$\uplus$" is disjoint union (i.e., $L_1 \uplus L_2 = L_1 \cup L_2$ with $L_1 \cap L_2 = \emptyset$).

2. $O_k = O_{k-1} \times \mathcal{A}_k$, where $\mathcal{A}_k$ is the set of *label-to-measurement associations* (LMAs) i.e., functions $\alpha_k : L_k \to \{0, 1, ..., |Z_k|\}$ such that $\alpha_k(l_1) = \alpha_k(l_2) > 0$ implies $l_1 = l_2$.

3. The PGFL of the updated Bernoulli spatial density $(q^{o,\alpha_k}_{l,k}, s^{o,\alpha_k}_{l,k})$ is

$$\sigma^{o}_{l,k}[\mathring{h}_{k}] = 1 - q^{o}_{l,k} + q^{o}_{l,k} s^{o}_{l,k-1}[\mathring{h}_{k}] \tag{4}$$

   where ω^{o,α_k}_{k}, $q^{o}_{l,k}$ and $s^{o}_{l,k-1}$ are given in Section 5.1.

1.2 Organization of the Paper

The paper is organized as follows: labeled random finite sets (Section 2); single-target labeled PMMs (Section 3); multitarget labeled PMMs (Section 4); the PMM-mGLMB filter (Section 5); mathematical derivations (Section 6); and Conclusions (Section 7).

2 Labeled Random Finite Sets (LRFSs)

This section is organized as follows: basic concepts (Section 2.1); spatial densities and labeled integrals (Section 2.2); unlabeled set integrals (Section 2.3); labeled random finite sets (Section 2.4); labeled set integrals (Section 2.5); probability generating functionals (Section 2.6); functional derivatives (Section 2.7); the labeled multitarget Bayes recursive filter (Section 2.8); the "standard" LRFS multitarget motion and measurement models (Section 2.9); and the GLMB filter (Section 2.10).

2.1 Basic Concepts

In LRFS theory, the single-target state space is $\mathring{\mathbb{X}} = \mathbb{X} \times \mathbb{L}$ where $\mathbb{X}$ is the kinematic state space (typically, a region of a Euclidean space) and $\mathbb{L}$ is a countability infinite set of "labels" $l \in \mathbb{L}$ (typically, the Vo-Vo label space[3]). Labels are provisional stand-ins for a discrete state variable: *unique target identity*. The elements of $\mathring{\mathbb{X}}$ therefore have the form $\mathring{x} = (x, l)$, which is interpreted as the state of the target with label l. A single-target trajectory is therefore a time-sequence of single-target states that have the same label.

The single-target measurement space is $\mathbb{Z}$ with elements $y, z \in \mathbb{Z}$.

Labeled HMM theory requires the following items:

1. A measurement density (a.k.a. likelihood function) $f_k(z_k|\mathring{x}_k) = \mathring{L}_{z_k}(\mathring{x}_k)$.

2. A Markov transition density

$$\mathring{f}_{k|k-1}(x_k, l_k|x_{k-1}, l_{k-1}) = \delta_{\,l_k, l_{k-1}} f_{k|k-1}(x_k|x_{k-1}) \tag{5}$$

   where $\delta_{\,l_k, l_{k-1}}$ is the Kronecker delta.

3. A target probability of detection $\mathring{p}_D(\mathring{x}_k)$ with

$$\mathring{p}_D^c(\mathring{x}_k) \overset{\text{abbr.}}{=} 1 - \mathring{p}_D(\mathring{x}_k). \tag{6}$$

4. A target probability of survival $\mathring{p}_S(\mathring{x}_{k-1})$ with

$$\mathring{p}_S^c(\mathring{x}_{k-1}) \overset{\text{abbr.}}{=} 1 - \mathring{p}_S(\mathring{x}_{k-1}). \tag{7}$$

5. A clutter density distribution $\kappa_k(Z_k)$, which in this paper will be assumed to be Poisson (see Eq. (17)).

2.2 Spatial Densities and Labeled Integrals

Let $\mathring{f}(\mathring{x}) \geq 0$ with $\mathring{x} \in \mathring{\mathbb{X}}$ be a *labeled density function*. That is, the unit of measurement (UoM) of $\mathring{f}(\mathring{x})$ is $\iota_{\mathbb{X}}^{-1}$ if the UoM of $\mathbb{X}$ is $\iota_{\mathbb{X}}$; and $\int \mathring{f}(x, l)dx = 0$ for all but a finite number of $l \in \mathbb{L}$. The *labeled single-target integral* is

$$\int \mathring{f}(\mathring{x})d\mathring{x} = \sum_{l \in \mathbb{L}} \int \mathring{f}(x, l)dx < \infty. \tag{8}$$

Suppose that $\mathring{s}(\mathring{x})$ is a labeled density function such that, in addition,

$$\int s_l(x)dx = \int \mathring{s}(x, l)dx = 1 \tag{9}$$

[3] This is the space of pairs $l = (k, i)$ where $k = 0, 1, ...$ represents the time t_k that the target with label l appeared and $i = 1, 2, ...$ is the index of the specific target l (among possibly many targets) that appeared at that time.

whenever $\int \mathring{s}(x,l)dx > 0$, where

$$s_l(x) \stackrel{\text{def.}}{=} \mathring{s}(x,l). \tag{10}$$

Then $s_l(x)$ is called a *spatial density* of the target with label l.[4]

Let $0 \leq \mathring{h}(\mathring{x}) \leq l$ be a unitless function (a "test function" on $\mathring{\mathbb{X}}$). Then quantum-physics *functional notation* will be used in the sequel:

$$\mathring{f}[\mathring{h}] = \int \mathring{h}(x)\, \mathring{f}(\mathring{x})d\mathring{x}. \tag{11}$$

That is, $\mathring{f}[\mathring{h}]$ is a functional (a function whose arguments $\mathring{h}$ are ordinary functions), and brackets "$[\cdot]$" are used instead of parentheses "$(\cdot)$" to emphasize that fact. Note that $\mathring{f}[\mathring{h}]$ is unitless.

If $\mathring{s}(\mathring{x})$ is a spatial density then the following abbreviation will be employed in what follows:[5]

$$s_l[\mathring{h}] = \int \mathring{h}(x,l)\, s_l(x)dx. \tag{12}$$

In the measurement-space case, let $f(z) \geq 0$ and $0 \leq g(z) \leq 1$ be, respectively, a density function and a test function on $\mathbb{Z}$. Then

$$f[g] \stackrel{\text{def.}}{=} \int f(z)\, f(z)dz. \tag{13}$$

2.3 Unlabeled Set Integrals

A *multitarget measurement-set* is a finite subset $Z \subseteq \mathbb{Z}$. A *random measurement-set* $\Sigma \subseteq \mathbb{Z}$ is a random variable whose realizations are finite subsets of $\mathbb{Z}$. It has a probability distribution $f_\Sigma(Z) \geq 0$ such that the UoM of $f_\Sigma(Z)$ is $\iota_{\mathbb{Z}}^{-|Z|}$ if the UoM of $\mathbb{Z}$ is $\iota_{\mathbb{Z}}$. If $f(Z) \geq 0$ is any function of an finite-set variable $Z \subseteq \mathbb{Z}$ with these properties then its *set integral* is

$$\int f(Z)\delta Z = \sum_{m \geq 0} \frac{1}{m!} \int f(\{z_1, ..., z_m\})dz_1 \cdots dz_m \tag{14}$$

where if $m = 0$ then $f(\{z_1, ..., z_m\}) \stackrel{\text{def.}}{=} f(\emptyset)$. Thus $f(Z)$ is a multi-measurement probability distribution if and only if $\int f(Z)\delta Z = 1$.

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[4] Rigorously speaking, $\mathring{s}(\mathring{x})$ should be replaced by $\mathring{s}_l(x_1, l_1) \stackrel{\text{def.}}{=} \delta_{l,l_1}\, s_l(x_1)$, so that it is consistent with labeled integration:—i.e., so that $\int \mathring{s}_l(\mathring{x}_1)d\mathring{x}_1 = 1$. To avoid unnecessary notational complexity, the informal definition, Eq. (9), will be adopted hereafter.

[5] Functional notation is used for the following reason. In the LRFS literature, $\mathring{f}[\mathring{h}]$ is often notated as $\langle \mathring{f}, \mathring{h} \rangle$. Strictly speaking, this is misleading since "$\langle \cdot, \cdot \rangle$" is the preferred notation for a scalar (a.k.a. inner) product but $\mathring{f}$ and $\mathring{h}$ belong to very different function spaces. In particular, $\mathring{h}(x,l)$ is unitless whereas $\mathring{f}(x,l)$ is not.

2.4 Labeled Random Finite Sets

Let $\mathring{X} \subseteq \mathring{\mathbb{X}}$ be finite, let $|\mathring{X}|$ denote the number of elements in $\mathring{X}$, and, if $\mathring{X} = \{(x_1, l_1), ..., (x_n, l_n)\}$, let $\mathring{X}_{\mathbb{L}} = \{l_1, ..., l_n\}$ denote the set of labels in $\mathring{X}$. Then $\mathring{X}$ is a *labeled finite set* (LFS) if $|\mathring{X}_{\mathbb{L}}| = |\mathring{X}|$—i.e., if the elements of $\mathring{X}$ have distinct labels.

If $\mathring{X}$ is not an LFS then it is *physically impossible*—for example, $X = \{(x_1, l), (x_2, l)\}$ with $x_1 \neq x_2$ represents a target that is in two different locations at the same time. The states of a multitarget system are the LFSs.

A *labeled random finite set* (LRFS) $\mathring{\Xi}$ is a random variable whose realizations are LFSs. It is characterized by its probability distribution $\mathring{f}_{\mathring{\Xi}}(\mathring{X}) \geq 0$, which has the property that $\mathring{f}_{\mathring{\Xi}}(\mathring{X}) = 0$ if $|\mathring{X}_{\mathbb{L}}| \neq |\mathring{X}|$—i.e., if $\mathring{X}$ is physically impossible. If $l_1, ..., l_n$ are not distinct then $f_{\Xi}(\{(x_1, l_1), ..., (x_n, l_n)\}) = 0$. Also: the UoM of $\mathring{f}_{\mathring{\Xi}}(\mathring{X})$ is $\iota_{\mathbb{X}}^{-|\mathring{X}|}$ if $\iota_{\mathbb{X}}$ is the UoM of $\mathbb{X}$.

2.5 Labeled Set Integrals

Let $\mathring{f}(\mathring{X}) \geq 0$ be any function of an LFS variable $\mathring{X} \subseteq \mathring{\mathbb{X}}$ with these properties. Then its *labeled set integral* is

$$\begin{aligned} & \int \mathring{f}(\mathring{X}) \delta \mathring{X} \qquad (15) \\ = & \sum_{n \geq 0} \frac{1}{n!} \sum_{(l_1, ..., l_n) \in \mathbb{L}^n} \int \mathring{f}(\{(x_1, l_1), ..., (x_n, l_n)\}) dx_1 \cdots dx_n \end{aligned}$$

where if $n = 0$ then $\mathring{f}(\{(x_1, l_1), ..., (x_n, l_n)\}) \overset{\text{def.}}{=} f(\emptyset)$. Thus $\mathring{f}(\mathring{X})$ is a *labeled multitarget probability density function* (LMPDF) if and only if $\int \mathring{f}(\mathring{X}) \delta \mathring{X} = 1$.

2.6 Probability Generating Functionals

Let $\mathring{f}(\mathring{X})$ be an LMPDF and $\mathring{h}$ a test function. Then its *probability generating functional* (PGFL) is

$$0 \leq \mathring{G}[\mathring{h}] = \int \mathring{h}^{\mathring{X}} \mathring{f}(\mathring{X}) \delta \mathring{X} \leq \int \mathring{f}(\mathring{X}) \delta \mathring{X} = 1 \qquad (16)$$

where the *functional exponential* $\mathring{h}^{\mathring{X}}$ is defined by $\mathring{h}^{\mathring{X}} = 1$ if $\mathring{X} = \emptyset$ and $\mathring{h}^{\mathring{X}} = \prod_{\mathring{x} \in \mathring{X}} \mathring{h}(\mathring{x})$ otherwise.

PGFLs are important because many PGFL formulas are algebraically far simpler than their LMPDF counterparts—see, for example, Eqs. (35,37).

The following PGFLs will be important in what follows:

1. *Poisson* RFSs:
$$G[g] = e^{\kappa[g-1]} \qquad (17)$$
where $0 \leq g(z) \leq 1$ is a test function on $\mathbb{Z}$ and $\kappa(z)$ is a Poisson intensity function. These are commonly used to model clutter and/or false alarm processes.

2. *Labeled Bernoulli* RFSs:

$$\begin{aligned}\sigma_l[\mathring{h}] &= 1 - q_l + q_l s_l[\mathring{h}] && (18)\\ &= s_l[1 - q_l + q_l \mathring{h}] && (19)\\ &= 1 - q_l + q_l \int \mathring{h}(x,l)\, s_l(x) dx && (20)\end{aligned}$$

where $0 \leq q_l \leq 1$ is the *existence probability* of the target with label l and spatial density $s_l(x) = \mathring{s}(x,l)$. In what follows we will refer to a pair (q_l, s_l) with spatial density $s_l(x)$ and existence probability q_l as a "Bernoulli spatial density" and identify it with its Bernoulli PGFL $\sigma_l[\mathring{h}]$.

3. *Labeled Multi-Bernoulli* (LMB) RFSs:

$$\mathring{G}[\mathring{h}] = \prod_{l \in L} (1 - q_l + q_l s_l[\mathring{h}]) \tag{21}$$

where $L \subseteq \mathbb{L}$ is finite and where $0 \leq q_l \leq 1$ for all $l \in L$. LMB RFSs are the theoretically correct LRFS analogs of Poisson RFSs [11, Eqs. (49-51)].[6] They are commonly used in LRFS theory to model a group of newly-appearing targets with label-set L.[7]

4. *Generalized LMB* (GLMB) RFSs:

$$\mathring{G}[\mathring{h}] = \sum_{o \in O} \sum_{L \in \mathbb{L}} \omega^o(L) \prod_{l \in L} s_l^o[\mathring{h}] \tag{22}$$

where O is a finite index set; $0 \leq \omega^o(L) \leq 1$; $s_l^o[\mathring{h}]$ is as in Eq. (12); and $\sum_{o \in O} \sum_{L \in \mathbb{L}} \omega^o(L) = 1$ (which means that $\omega^o(L) = 0$ for all but a finite number of pairs o, L). The corresponding LMPDF is

$$\mathring{f}(\mathring{X}) = \delta_{|\mathring{X}|, |\mathring{X}_{\mathbb{L}}|} \sum_{o \in O} \omega^o(\mathring{X}_{\mathbb{L}})\, (\mathring{s}^o)^{\mathring{X}} \tag{23}$$

where $\mathring{s}^o(\mathring{x})$ is a spatial density and $(\mathring{s}^o)^{\mathring{X}}$ is as in Eq. (16). These LMPDFs are the basis for the GLMB filter. Note that if $\mathring{G}[\mathring{h}]$ is known then so are O, $\omega_o(L)$, and $\mathring{s}^o(\mathring{x})$ and thereby also $\mathring{f}(\mathring{X})$. It follows that derivations involving GLMB PDFs can be replaced by derivations involving their corresponding PGFLs.

[6] This is because LMB RFSs, like Poisson RFSs, are completely characterized by their first-order multitarget moment density (a.k.a. PHD): $\mathring{D}(x,l) = \frac{\delta \mathring{G}}{\delta(x,l)}[1] = \mathbf{1}_L(l)\; q_l s_l(x)$.

[7] Poisson RFSs on $\mathbb{X}$ are commonly used to model target appearances in unlabeled RFS theory. However, they cannot serve this purpose in LRFS theory since they are not LRFSs even when defined on the labeled state space $\mathring{\mathbb{X}} = \mathbb{X} \times \mathbb{L}$ [10]. Prior to LRFS theory [24], heuristically labeled RFS filters and Poisson RFSs were necessary as stopgaps to avoid computational intractability.

5. *Modified GLMB* (mGLMB) RFSs:

$$\mathring{G}[\mathring{h}] = \sum_{o\in O}\sum_{L\in\mathbb{L}} \omega^o(L) \prod_{l\in L}(1 - q_l^o + q_l^o s_l^o[\mathring{h}]). \tag{24}$$

GLMB RFSs are mGLMB RFSs and vice-versa (Section 6.1).

2.7 Functional Derivatives

The intuitive definition of the *functional derivative* of a PGFL $\mathring{G}[\mathring{h}]$ is:[8]

$$\frac{\delta\mathring{G}}{\delta\mathring{x}}[\mathring{h}] = \lim_{\varepsilon\searrow 0} \frac{\mathring{G}[\mathring{h}+\varepsilon\,\delta_{\mathring{x}}] - \mathring{G}[\mathring{h}]}{\varepsilon} \tag{25}$$

where $\delta_{(x,l)}(x_1,l_1) = \delta_{l_1,l}\,\delta_x(x_1)$ is the Dirac delta function concentrated at $\mathring{x} = (x,l)$.

If $\mathring{X} = \{\mathring{x}_1,...,\mathring{x}_n\}$ with $|\mathring{X}| = n$ then the *iterated functional derivative* is

$$\frac{\delta\mathring{G}}{\delta\mathring{X}}[\mathring{h}] = \frac{\delta^n\mathring{G}}{\delta\mathring{x}_1\cdots\delta\mathring{x}_n}[\mathring{h}] = \frac{\delta}{\delta\mathring{x}_n}\frac{\delta^{n-1}\mathring{G}}{\delta\mathring{x}_1\cdots\delta\mathring{x}_{n-1}}[\mathring{h}] \tag{26}$$

if $\mathring{X} \neq \emptyset$ and $\mathring{G}[\mathring{h}]$ if otherwise.

Remark 1 *As a simple example, consider the functional derivative of Eq. (13):*

$$\frac{\delta f}{\delta Z}[g] = \begin{cases} f[g] & \text{if} \quad Z=\emptyset \\ f(z_1) & \text{if} \quad Z=\{z_1\} \\ 0 & \text{if} \quad |Z|\geq 2 \end{cases} \tag{27}$$

where, if $G_z[g] \stackrel{def}{=} g(z)$ *then the* $Z=\{z_1\}$ *case follows from*

$$\frac{\delta G_z}{\delta z_1}[g] = \left(\frac{\delta}{\delta z_1}g\right)(z) = \delta_{z_1}(z) \tag{28}$$

and then from

$$\frac{\delta f}{\delta z_1}[g] = \frac{\delta}{\delta z_1}\int g(z)\,f(z)dz = \int \delta_{z_1}(z)\,f(z)dz = f(z_1). \tag{29}$$

The functional $f[g]$ *is said to be "first-order" since all of its functional derivatives of order* $n\geq 2$ *vanish. Eq. (27) will be employed pervasively throughout the paper.*

The PGFL and LMPDF of an RFS are related by:

$$\mathring{f}_{\mathring{\Xi}}(\mathring{X}) = \frac{\delta\mathring{G}_{\mathring{\Xi}}}{\delta\mathring{X}}[0] = \left[\frac{\delta\mathring{G}_{\mathring{\Xi}}}{\delta\mathring{X}}[\mathring{h}]\right]_{\mathring{h}=0}. \tag{30}$$

This is the converse of the relationship in Eq. (16), so that set integrals and set derivatives are inverse operations of each other.

Remark 2 *There is an extensive "toolbox" of "turn-the-crank" rules for set integrals and functional derivatives, see [13, pp. 383-389], [7, pp. 69-80].*

[8] For a rigorous definition see [14], [7, Appx. C].

2.8 Labeled Multitarget Bayes Recursive Filter

Given a time-sequence $Z_{1:k} : Z_1, ..., Z_k$ of collected measurement-sets from a single sensor, the labeled multitarget Bayes recursive filter (LMBRF) is defined by the equations

$$\begin{aligned}&\mathring{f}_{k|k-1}(\mathring{X}_k|Z_{1:k-1}) \\ = & \int \mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1})\, \mathring{f}_{k-1|k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}\end{aligned} \tag{31}$$

(time-update) and

$$\mathring{f}_{k|k}(\mathring{X}_k|Z_{1:k}) \propto f_k(Z_k|\mathring{X}_k)\, \mathring{f}_{k|k-1}(\mathring{X}_k|Z_{1:k-1}); \tag{32}$$

(measurement-update); and where $\mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1})$ is the multitarget state-transition density and $f_k(Z_k|\mathring{X}_k)$ is the sensor multitarget measurement density (multitarget likelihood function).

The corresponding PGFLs are

$$\mathring{G}_{k|k-1}[\mathring{h}_k|\mathring{X}_{k-1}] = \int \mathring{h}_k^{\mathring{X}_k}\, \mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1})\delta\mathring{X}_k \tag{33}$$

$$G_k[g_k|\mathring{X}_k] = \int g_k^{Z_k}\, f_k(Z_k|\mathring{X}_k)\delta Z_k. \tag{34}$$

2.9 "Standard" Multitarget Models

The PGFL of the "standard" multitarget Markov density is [13, Eq. (13.62)]

$$\mathring{G}_{k|k-1}[\mathring{h}_k|\mathring{X}_{k-1}] = \mathring{G}^B_{k|k-1}[\mathring{h}_k]\,(\mathring{p}^c_S + \mathring{p}_S\mathring{M}_{\mathring{h}_k})^{\mathring{X}_{k-1}} \tag{35}$$

where $\mathring{G}^B_{k|k-1}[\mathring{h}_k]$ is the PGFL of the target-appearance process (typically LMB, see Eq. (91)), and where

$$\mathring{M}_{\mathring{h}_k}(\mathring{x}_{k-1}) \overset{\text{def.}}{=} \int \mathring{h}_k(\mathring{x}_k)\, f_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1})d\mathring{x}_k. \tag{36}$$

The PGFL of the "standard" multitarget measurement model is

$$G_k[g_k|\mathring{X}_k] = G^\kappa_k[g_k]\,(\mathring{p}^c_D + \mathring{p}_D\mathring{L}_{g_k})^{\mathring{X}_k} \tag{37}$$

where $G^\kappa_k[g_k]$ is the PGFL of the clutter process and where

$$\mathring{L}_{g_k}(\mathring{x}_k) \overset{\text{def.}}{=} \int g_k(z_k)\, f_k(z_k|\mathring{x}_k)dz_k \tag{38}$$

and where it is typically assumed that $G^\kappa_k[g_k]$ is Poisson [13, Eq. (12.151)], [7, Eq. (7.19)]:

$$G_k[g_k|\mathring{X}_k] = e^{\kappa_k[g_k-1]}\,(\mathring{p}^c_D + \mathring{p}_D\mathring{L}_{g_k})^{\mathring{X}_k}. \tag{39}$$

2.10 The GLMB Filter

The PGFL GLMB filter has the form

$$... \rightarrow \mathring{G}_{k-1|k-1}[\mathring{h}_{k-1}|Z_{1:k-1}] \rightarrow \mathring{G}_{k|k-1}[\mathring{h}_k|Z_{1:k-1}] \rightarrow \mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] \rightarrow ...$$

where only the measurement-update step $\mathring{G}_{k|k-1} \rightarrow \mathring{G}_{k|k}$ is of interest in what follows. Assume that the predicted PGFL is GLMB, i.e.,

$$\mathring{G}_{k|k-1}[\mathring{h}_k|Z_{1:k-1}] = \sum_{o \in O_{k|k-1}} \sum_{L \subseteq L_{k|k-1}} \omega^o_{k|k-1}(L) \prod_{l \in L} s^o_{l,k|k-1}[\mathring{h}_k] \tag{40}$$

and let us be given the new measurement-set Z_k.

As previously noted, in [16, Eqs. (40-47)] it was shown, using PGFL methods only, that the posterior PGFL is GLMB. This section summarizes the results of the cleaner GLMB derivation presented below in Section 6.2. Specifically, the posterior PGFL is

$$\mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] = \sum_{(o,\alpha_k) \in O_k} \sum_{L \subseteq L_{k|k}} \omega^{o,\alpha_k}_{k|k}(L) \prod_{l \in L} s^{o,\alpha_k}_{l,k|k}[\mathring{h}_k] \tag{41}$$

where:

1. $L_{k|k} = L_{k|k-1}$.
2. $O_k = O_{k-1} \times \mathcal{A}_k$, where $\mathcal{A}_k$ is the set of label-to-measurement associations (LMAs) $\alpha_k : L_k \rightarrow \{0, 1, ..., |Z_k|\}$.
3. The spatial distributions are

$$s^{o,\alpha_k}_{l,k|k}(x_k) = \frac{\mathring{p}^c_D(x_k, l)\, s^o_{l,k|k-1}(x_k)}{s^o_{l,k|k-1}[\mathring{p}^c_D]} \tag{42}$$

   if $\alpha_k(l) = 0$ (i.e., target l was not detected); and

$$s^{o,\alpha_k}_{l,k|k}(x_k) = \frac{\mathring{p}_D(x_k, l)\, \mathring{L}_{z_{\alpha_k(l)}}(x_k, l)\, s^o_{l,k|k-1}(x_k)}{s^o_{l,k|k-1}[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}]} \tag{43}$$

   if $\alpha_k(l) > 0$ (i.e., target l was detected and generated $z_{\alpha_k(l)}$).

4. The GLMB weights are

$$\omega^{o,\alpha_k}_{k|k}(L) = \frac{\omega^o_{k|k-1}(L)\, C^{o,\alpha_k}_{L,k}}{\sum_{(o,\alpha_k) \in O_k} \sum_{L \subseteq L_k} \omega^o_{k|k-1}(L)\, C^{o,\alpha_k}_{L,k}} \tag{44}$$

where

$$C^{o,\alpha_k}_{L,k} \tag{45}$$

$$= \mathbf{1}^L_{L_{\alpha_k}} \left(\prod_{l \in L : \alpha_k(l) > 0} \kappa_k(z_{\alpha_k(l)}) \right) \left(\prod_{l \in L : \alpha_k(l) = 0} s^o_{l,k|k-1}[\mathring{p}^c_D] \right)$$
$$\cdot \left(\prod_{l \in L : \alpha_k(l) > 0} s^o_{l,k|k-1}[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}] \right).$$

and where $\mathbf{1}^L_{L_{\alpha_k}}$ is defined in Eq. (264) below.

3 Labeled Pairwise Markov Models

Let the single-target state space be labeled: $\mathring{\mathbb{X}} = \mathbb{X} \times \mathbb{L}$. Then a labeled HMM on $\mathring{\mathbb{X}}$ is defined as follows. The target is a time-evolving random variable $\mathring{\mathbf{X}}_k \in \mathring{\mathbb{X}}$ at discrete times $t_0, t_1, ..., t_k, ...$ with realizations $\mathring{\mathbf{X}}_k = \mathring{x}_k = (x_k, l_k)$; and is measured at each time t_k by a sensor that collects a realization $\mathbf{Z}_k = z_k$. At time t_k the target's state $\mathring{x}_k$ depends only on its previous state $\mathring{x}_{k-1}$, as specified by a labeled Markov state-transition density

$$\mathring{f}_{k|k-1}(x_k, l_k | x_{k-1}, l_k) = \delta_{l_k, l_{k-1}} \, f_{k|k-1}(x_k | x_{k-1}), \tag{46}$$

where the Kronecker delta $\delta_{l_k, l_{k-1}}$ ensures that targets do not change labels and where $f_{k|k-1}(x_k | x_{k-1})$ is a Markov density on $\mathbb{X}$.

Also, the current measurement z_k depends only on the current state $\mathring{x}_k$, as specified by the measurement density function (a.k.a. likelihood function) $f_k(z_k | \mathring{x}_k) = \mathring{L}_{z_k}(\mathring{x}_k)$.

In labeled PMM, the target and its measurements are regarded as a joint time-evolving system with a random joint state $(\mathring{\mathbf{X}}_k, \mathbf{Y}_k)$. The statistical correlation between $\mathring{\mathbf{X}}_k$ and $\mathbf{Y}_k$ is an unknown that must be estimated along with the unknown random state variable $\mathring{\mathbf{X}}_k$.

The dynamics of a PMM are specified by a Bayesian Markov transition function expressed in the following factorized form:

$$\mathring{f}_{k|k-1}(\mathring{x}_k, y_k | \mathring{x}_{k-1}, y_{k-1}) = \mathring{f}_{k|k-1}(\mathring{x}_k | \mathring{x}_{k-1}, y_{k-1}) \, f_k(y_k | \mathring{x}_k, \mathring{x}_{k-1}, y_{k-1}). \tag{47}$$

To ensure that targets never change labels, it must be the case that

$$\mathring{f}_{k|k-1}(x_k, l_k, y_k | x_{k-1}, l_{k-1}, y_{k-1}) = \delta_{l_k, l_{k-1}} \, f_{k|k-1}(x_k, y_k | x_{k-1}, y_{k-1}) \tag{48}$$

where $f_{k|k-1}(x_k, y_k | x_{k-1}, y_{k-1})$ is a PMM transition function on $\mathbb{X}$.

In Section 6.3 it is shown that the marginal densities of the PMM transition function are

$$\mathring{f}_{k|k-1}(\mathring{x}_k | \mathring{x}_{k-1}, y_{k-1}) = \delta_{l_k, l_{k-1}} \, f_{k|k-1}(x_k | x_{k-1}, y_{k-1}) \tag{49}$$

$$\mathring{f}_{k|k-1}(y_k | \mathring{x}_{k-1}, y_{k-1}) = f_{k|k-1}(y_k | x_{k-1}, y_{k-1}). \tag{50}$$

A labeled PMM reduces to a labeled HMM if, for $k \geq 1$, [15, Eq. (35)]

$$\mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1}, y_{k-1}) = \mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1}) \tag{51}$$

$$f_k(y_k|\mathring{x}_k, \mathring{x}_{k-1}, y_{k-1}) = f_k(y_k|\mathring{x}_k). \tag{52}$$

Let $Y \subseteq \mathbb{Z}$ be a measurement-set. Then in the sequel the following abbreviations will be employed:

$$\mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, Y) = \frac{1}{|Y|}\sum_{y\in Y} \mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, y) \tag{53}$$

$$f_{k|k-1}(x_k, y_k|x_{k-1}, Y) = \frac{1}{|Y|}\sum_{y\in Y} f_{k|k-1}(x_k, y_k|x_{k-1}, y). \tag{54}$$

Note that if the labeled PMM is a labeled HMM then

$$\mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, Y) = \mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1})\, f_k(y_k|\mathring{x}_k) \tag{55}$$

is independent of Y.

Remark 3 *A labeled PMM can be recast as a labeled HMM if its measurement density (likelihood function) is defined in the obvious manner as*

$$f_k(z_k|\mathring{x}_k, y_k) = \mathring{L}_{z_k}(\mathring{x}_k, y_k) = \delta_{y_k}(z_k), \tag{56}$$

where $\delta_y(z)$ is the Dirac delta concentrated at y [15, p. 168232, col. 2.].

Remark 4 *[10, Remark 2] The notations z_k and y_k differ in that y_k is an unknown realization $\mathbf{Y}_k = y_k$ of the random state variable $\mathbf{Y}_k$ whereas z_k is a known realization $\mathbf{Y}_k = z_k$ collected by the sensor. The PMM literature typically conflates the two so that only the notation y_k is used. Hereafter, both notations will be used as appropriate to avoid confusion.*

The Bayesian PMM update equation is [15, Eq. (32)]

$$\mathring{f}_k(\mathring{x}_k, y_k|z_{1:k-1}) = \frac{\int \mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, z_{k-1})\, \mathring{f}_{k-1}(\mathring{x}_{k-1}, z_{k-1}|z_{1:k-2})d\mathring{x}_{k-1}}{f_{k-1}(z_{k-1}|z_{1:k-2})} \tag{57}$$

where

$$f_{k-1}(z_{k-1}|z_{1:k-2}) = \int \mathring{f}_{k-1}(\mathring{x}_{k-1}, z_{k-1}|z_{1:k-2})d\mathring{x}_{k-1}. \tag{58}$$

The initial PMM distribution can be chosen as $\mathring{f}_0(\mathring{x}_0, y_0) = f_0(y_0|\mathring{x}_0)\, \mathring{f}_0(\mathring{x}_0)$ where $\mathring{f}_0(\mathring{x}_0)$ is an initial target distribution and $\mathring{f}_0(y_0|\mathring{x}_0)$ is an initial measurement density.

The measurement density at time t_k can be estimated as [15, Eq. (33)]

$$f_k(y_k|\mathring{x}_k, z_{1:k-1}) = \frac{\mathring{f}_k(\mathring{x}_k, y_k|z_{1:k-1})}{\int \mathring{f}_k(\mathring{x}_k, y_k|z_{1:k-1})dy_k}. \tag{59}$$

Note that this has the form of a general Bayes measurement density in the sense of [13, Eq. (3.56)]. The Markov density at time t_k can be estimated similarly [15, Eq. (34)].

The posterior distribution $\mathring{f}_k(\mathring{x}_k|z_{1:k})$ of a single target with state $\mathring{x}_k$ and PMM dynamics can be propagated using the Bayesian recursion [19, Eq. (12)], [15, Eq. (37)]:

$$\mathring{f}_k(\mathring{x}_k|z_{1:k}) \tag{60}$$

$$= \frac{\int \mathring{f}_{k|k-1}(\mathring{x}_k, z_k|\mathring{x}_{k-1}, z_{k-1})\, \mathring{f}_{k-1}(\mathring{x}_{k-1}|z_{1:k-1})d\mathring{x}_{k-1}}{\int f_{k|k-1}(z_k|\mathring{x}_{k-1}, z_{k-1})\, \mathring{f}_{k-1}(\mathring{x}_{k-1}|z_{1:k-1})d\mathring{x}_{k-1}} \tag{61}$$

where

$$f_{k|k-1}(z_k|\mathring{x}_{k-1}, z_{k-1}) = \int \mathring{f}_{k|k-1}(\mathring{x}_k, z_k|\mathring{x}_{k-1}, z_{k-1})d\mathring{x}_k. \tag{62}$$

4 Multitarget PMM (MPMM)

This section is organized as follows: multitarget pairwise Markov models (Section 4.1); the "standard model" MPMM transition function (Section 4.2); the GLMB-consistent LMPMM transition function (Section 4.3); and the PGFL of the LMPMM update (Section 4.4).

4.1 Multitarget PMMs

LMPMM is a direct LRFS generalization of single-target PMM. The multitarget state and multitarget measurements are a joint time-evolving system with a random joint state $(\mathring{\Xi}_k, \Sigma_k)$ where $\mathring{\Xi}_k \subseteq \mathring{\mathbb{X}}$ is the multitarget-state LRFS and $\Sigma_k \subseteq \mathbb{Z}$ the measurement RFS. The dynamics of the LMPMM are specified by a labeled multitarget Bayesian Markov transition function:

$$\mathring{f}_{k|k-1}(\mathring{X}_k, Y_k|\mathring{X}_{k-1}, Y_{k-1}) \tag{63}$$

$$= \mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1}, Y_{k-1})\, f_{k|k-1}(Y_k|\mathring{X}_k, \mathring{X}_{k-1}, Y_{k-1}).$$

The posterior distribution $\mathring{f}_k(\mathring{X}_k|Z_{1:k})$ of the multitarget state $\mathring{X}_k$ with LMPMM dynamics is determined by the obvious generalization of Eq. (60):

$$\mathring{f}_k(\mathring{X}_k|Z_{1:k}) = \frac{\left(\begin{array}{c} \int \mathring{f}_{k|k-1}(\mathring{X}_k, Z_k|\mathring{X}_{k-1}, Z_{k-1}) \\ \cdot \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1} \end{array} \right)}{\left(\begin{array}{c} \int f_{k|k-1}(Z_k|\mathring{X}_{k-1}, Z_{k-1}) \\ \cdot \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1} \end{array} \right)} \tag{64}$$

where

$$f_{k|k-1}(Z_k|\mathring{X}_{k-1}, Z_{k-1}) = \int \mathring{f}_{k|k-1}(\mathring{X}_k, Z_k|\mathring{X}_{k-1}, Z_{k-1})\delta\mathring{X}_k. \tag{65}$$

An obvious question is:

- What is an appropriate formula for the LMPMM transition function

$$\mathring{f}_{k|k-1}(\mathring{X}_k, Y_k|\mathring{X}_{k-1}, Y_{k-1})?$$

4.2 "Standard Model" LMPMM Transition Function

The purpose of this subsection is to answer this question. A basic property of $\mathring{f}_{k|k-1}(\mathring{X}_k, Y_k|\mathring{X}_{k-1}, Y_{k-1})$ should be the following. If

$$\mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1}, y_{k-1}) = \mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1}) \tag{66}$$

$$f_{k|k-1}(y_k|\mathring{x}_k, \mathring{x}_{k-1}, y_{k-1}) = f_k(y_k|\mathring{x}_k) \tag{67}$$

(i.e., if the underlying single-target PMM is an HMM) then

$$\mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1}, Y_{k-1}) = \mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1}) \tag{68}$$

$$f_{k|k-1}(Y_k|\mathring{X}_k, \mathring{X}_{k-1}Y_{k-1}) = \mathring{f}_k(Y_k|\mathring{X}_k) \tag{69}$$

(i.e., the LMPMM must be an LMHMM).

Such a formula was proposed in [15, Sec. V-C] for the case of a single disappearing and reappearing target (specifically, for application to the PMM Bernoulli filter). It can be generalized to the general multitarget case—see Eq. (89)—as follows.

First, the PGFL of the LMPMM transition function is

$$\mathring{G}_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}] = \int \mathring{h}_k^{\mathring{X}_k}\, g_k^{Y_k}\, \mathring{f}_{k|k-1}(\mathring{X}_k, Y_k|\mathring{X}_{k-1}, Y_{k-1})\delta\mathring{X}_k\delta Y_k. \tag{70}$$

Second, momentarily assume that the LMPMM is LMHMM and abbreviate

$$\mathring{T}_{g_k,\mathring{h}_k}(\mathring{x}_k) = \mathring{h}_k(\mathring{x}_k)\left\{\mathring{p}_D^c(\mathring{x}_k) + \mathring{p}_D(\mathring{x}_k)\,\mathring{L}_{g_k}(\mathring{x}_k)\right\}. \tag{71}$$

Then employ the obvious generalizations of the unlabeled equations in [12, Eqs. (11-15)] to the labeled case, which results in:

$$\begin{aligned} & \mathring{G}_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}] && (72) \\ = & \int \mathring{h}_k^{\mathring{X}_k}\, g_k^{Y_k}\, \mathring{f}_{k|k-1}(\mathring{X}_k, Y_k|\mathring{X}_{k-1}, \mathring{X}_{k-1})\delta\mathring{X}_k\delta Y_k && \\ = & \int \mathring{h}_k^{\mathring{X}_k}\, g_k^{Y_k}\, \mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1})\, f_{k|k-1}(Y_k|\mathring{X}_k)\delta\mathring{X}_k\delta Y_k && (73) \\ = & \int \mathring{h}_k^{\mathring{X}_k}\, G_k[g_k\,|\mathring{X}_k]\, \mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1})\delta\mathring{X}_k && (74) \end{aligned}$$

$$= \quad G^{\kappa}_k[g_k] \int \left(\mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D L_{g_k}) \right)^{\mathring{X}_k} \mathring{f}_{k|k-1}(\mathring{X}_k|\mathring{X}_{k-1}) \delta\mathring{X}_k \tag{75}$$

$$= \quad G^{\kappa}_k[g_k] \, G_{k|k-1}[\mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D L_{g_k})|\mathring{X}_{k-1}] \tag{76}$$

$$= \quad G^{\kappa}_k[g_k] \, \mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] \, \mathring{M}^{\mathring{X}_{k-1}}_{\mathring{p}^c_D + \mathring{p}_D \mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D \mathring{L}_{g_k})} \tag{77}$$

$$= \quad \mathring{G}_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}]. \tag{78}$$

Eq. (75) follows from the standard multitarget measurement model, Eq. (37), Eq. (76) follows from the definition of a PGFL, Eq. (16), and Eq. (77) follows from the standard multitarget motion model, Eq. (35).

Third, define [15, Eqs. (54)]:

$$\begin{aligned} & \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S \mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D g_k)}(\mathring{x}_{k-1}, y_{k-1}) \qquad (79) \\ = \quad & \int \left\{ \mathring{p}^c_S(\mathring{x}_{k-1}) + \mathring{p}_S(\mathring{x}_{k-1}) \, \mathring{h}_k(\mathring{x}_k) \, (\mathring{p}^c_D(\mathring{x}_k) + \mathring{p}_D(\mathring{x}_k) \, g_k(y_k)) \right\} \\ & \cdot \mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, y_{k-1}) d\mathring{x}_k dy_k \end{aligned}$$

where

$$\ddot{M}_{\ddot{h}_k}(\mathring{x}_{k-1}, y_{k-1}) \overset{\text{def.}}{=} \int \ddot{h}_k(\mathring{x}_k, y_k) \, \mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, y_{k-1}) d\mathring{x}_k dy_k \tag{80}$$

and where $0 \leq \ddot{h}_k(\mathring{x}_k, y_k) \leq 1$ is a test function on $\mathring{\mathbb{X}} \times \mathbb{Z}$.

Note that:

- Eq. (79) is first-order in both g_k and $\mathring{h}_k$ in the sense of Eq. (27).

Fourth, if the single-target PMM is HMM then it is shown in Section 6.4 that Eq. (79) reduces as

$$\ddot{M}_{\mathring{p}^c_S + \mathring{p}_S \mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D g_k)}(\mathring{x}_{k-1}, y_{k-1}) = \mathring{M}_{\mathring{p}^c_S + \mathring{p}_S \mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D \mathring{L}_{g_k})}(\mathring{x}_{k-1}), \tag{81}$$

where $\mathring{M}_{\mathring{h}_k}(\mathring{x})$ was defined in Eq. (36) and $\mathring{L}_{g_k}(\mathring{x})$ in Eq. (38).

Fifth, define the "evolution PGFL" by

$$\begin{aligned} & \mathring{G}^E_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}] \qquad (82) \\ = \quad & \prod_{\mathring{x}_{k-1} \in \mathring{X}_{k-1}} \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S \mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D g_k)}(\mathring{x}_{k-1}, Y_{k-1}) \\ = \quad & \mathring{H}^{\mathring{X}_{k-1}}_{\mathring{h}_k, g_k, Y_{k-1}} \qquad (83) \end{aligned}$$

where

$$\begin{aligned} & \mathring{H}_{\mathring{h}_k, g_k, Y_{k-1}}(\mathring{x}_{k-1}) \qquad (84) \\ \overset{\text{abbr.}}{=} \quad & \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S \mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D g_k)}(\mathring{x}_{k-1}, Y_{k-1}) \\ \overset{\text{def.}}{=} \quad & \frac{1}{|Y_{k-1}|} \sum_{y_{k-1} \in Y_{k-1}} \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S \mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D g_k)}(\mathring{x}_{k-1}, y_{k-1}) \qquad (85) \end{aligned}$$

is first-order in both g_k and $\mathring{h}_k$ in the sense of Eq. (27), and where $\ddot{M}_{\ddot{h}_k}(\mathring{x}_{k-1}, Y_{k-1})$ was defined from $\mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, Y_{k-1})$ in Eq. (80).

Remark 5 *In principle, any definition of* $f_{k|k-1}(x_k, y_k|x_{k-1}, Y_{k-1})$ *will satisfy the property described in Eqs. (66-69), provided that it reduces to*

$$f_k(y_k|\mathring{x}_k)\, f_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1})$$

if the single-target PMM is an HMM. But as noted in [15, Eq. (64)], Eq. (84) and thus $\mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, Y_{k-1})$ *have an intuitive physical interpretation—namely, that the MPMM state-transition* $(\{x_{k-1}\}, Y_{k-1}) \to (\{x_k\}, \{y_k\})$ *is the average of the MPMM state-transitions* $(\{x_{k-1}\}, \{y_{k-1}\}) \to (\{x_k\}, \{y_k\})$*, taken over all* $y_{k-1} \in Y_{k-1}$*. Given this, it is unclear what a plausible alternative to Eq. (85) might be.*

Sixth, in Eqs. (77,78), i.e., in

$$\mathring{G}_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}] = G_k^\kappa[g_k]\, \mathring{G}^B_{k|k-1}[T_{g_k,\mathring{h}_k}]\, \mathring{M}^{\mathring{X}_{k-1}}_{\mathring{p}_S^c + \mathring{p}_S \mathring{h}_k (\mathring{p}_D^c + \mathring{p}_D \mathring{L}_{g_k})}, \tag{86}$$

substitute

$$\mathring{M}_{\mathring{p}_S^c + \mathring{p}_S \mathring{h}_k (\mathring{p}_D^c + \mathring{p}_D \mathring{L}_{g_k})}(\mathring{x}_{k-1}) \quad \hookrightarrow \quad \ddot{M}_{\mathring{p}_S^c + \mathring{p}_S \mathring{h}_k (\mathring{p}_D^c + \mathring{p}_D g_k)}(\mathring{x}_{k-1}, Y_{k-1}) \tag{87}$$

$$\mathring{G}_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}] \quad \hookrightarrow \quad \mathring{G}_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}] \tag{88}$$

which results, finally, in:

- *the definition of the PGFL of the "standard model" LMPMM transition function*:

$$\begin{aligned} & G_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}] \\ = \;& G_k^\kappa[g_k]\, G^B_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}]\, \mathring{G}^E_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}] \end{aligned} \tag{89}$$

  where $G_k^\kappa[g_k]$ is the PGFL of the clutter process; $G^B_{k|k-1}[\mathring{h}_k]$ is the PGFL of the target-appearance process; $\mathring{G}^E_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}] = \mathring{H}^{\mathring{X}_{k-1}}_{\mathring{h}_k, g_k, Y_{k-1}}$ is the PGFL of the LMPMM evolution process; and where $\mathring{T}_{g_k,\mathring{h}_k}$ was defined in Eq. (71) and $\mathring{H}^{\mathring{X}_{k-1}}_{\mathring{h}_k, g_k, Y_{k-1}}$ in Eq. (84) and

$$\mathring{G}^E_{k|k-1}[\mathring{h}_k, g_k|\mathring{X}_{k-1}, Y_{k-1}]$$

  in Eq. (82).

Because of Eq. (81), it follows that if Eqs. (66,67) are true—i.e., if the single-target PMM transition density is HMM—then Eqs. (68,69) are true—i.e., the "standard model" LMPMM transition density reduces to the corresponding MHMM density and thus, as claimed, defines a proper LMPMM.

4.3 The LMPMM-GLMB Transition Function

In order to be applicable to GLMB filtering, the formulas for the PGFLs $G_k^\kappa[g_k]$ and $\mathring{G}^B_{k|k-1}[\mathring{h}_k]$ in Eq. (89) must conform to GLMB assumptions:

1. The clutter/false alarm process is Poisson:
$$G_k^\kappa[g_k] = e^{\kappa_k[g_k-1]} \tag{90}$$
where $\kappa_k[g] = \int g(z)\,\kappa_k(z)dz$ and $\kappa_k(z)$ is the intensity function of the Poisson clutter process.

2. The PGFL of the target-appearance process is LMB:
$$\mathring{G}^B_{k|k-1}[\mathring{h}_k] = \prod_{l \in L^B_{k|k-1}} \mathring{G}^B_{l,k|k-1}[h_k] \tag{91}$$
where, if $L^B_{k|k-1} \subseteq \mathbb{L}$ is the finite set of labels of the newly-appearing targets at time t_k,
$$\mathring{G}^B_{l,k|k-1}[\mathring{h}_k] = 1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{h}_k] \tag{92}$$
is the PGFL of the target with label $l \in L^B_{k|k-1}$.

It follows from Eqs. (71,92) that
$$\mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] = \prod_{l \in L^B_{k|k-1}} \mathring{G}^B_{l,k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] \tag{93}$$

where
$$\begin{aligned} & \mathring{G}^B_{l,k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] & (94)\\ = & \; 1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] & \\ = & \; 1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D \mathring{L}_{g_k})] & (95) \end{aligned}$$

and so
$$\mathring{G}^B_{l,k|k-1}[\mathring{T}_{0_k,\mathring{h}_k}] = 1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D] \tag{96}$$
$$\frac{\delta}{\delta z}\mathring{G}^{B,l}_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] = q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{h}_k \mathring{p}_D \mathring{L}_z]. \tag{97}$$

4.4 PGFL of the LMPMM Update

The updated MPMM tracking distribution was given in Eq. (64). Its PGFL is [15, Eq. (42)]:

$$\begin{aligned} & \mathring{G}_k[\mathring{h}_k|Z_{1:k}] & (98)\\ = & \frac{\int \mathring{G}_{k|k-1}[\mathring{h}_k, Z_k|\mathring{X}_{k-1}, Z_{k-1}]\, \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}}{\int \mathring{f}_{k|k-1}(Z_k|\mathring{X}_{k-1}, Z_{k-1})\, \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}} & \\ = & \frac{\int \mathring{G}_{k|k-1}[\mathring{h}_k, Z_k|\mathring{X}_{k-1}, Z_{k-1}]\, \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}}{\int \mathring{G}_{k|k-1}[1, Z_k|\mathring{X}_{k-1}, Z_{k-1}]\, \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}} & (99) \end{aligned}$$

and thus from Eq. (89):

$$= \frac{\int \left[\left(\frac{\delta}{\delta Z_k} G_k^\kappa[g_k]\, \mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,h_k}]\, \mathring{H}^{\mathring{X}_{k-1}}_{\mathring{h}_k,g_k,Z_{k-1}}\right)\right]_{g_k=0} \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}}{\int \left[\frac{\delta}{\delta Z_k}\left(G_k^\kappa[g_k]\, \mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,1}]\, \mathring{H}^{\mathring{X}_{k-1}}_{1,g_k,Z_{k-1}}\right)\right]_{g_k=0} \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}} \tag{100}$$

$$= \frac{\left[\frac{\delta}{\delta Z_k}\left(G_k^\kappa[g_k]\, \mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] \int \mathring{H}^{\mathring{X}_{k-1}}_{\mathring{h}_k,g_k,Z_{k-1}}\, \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}\right)\right]_{g_k=0}}{\left[\frac{\delta}{\delta Z_k}\left(G_k^\kappa[g_k]\, \mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,1}] \int \mathring{H}^{\mathring{X}_{k-1}}_{1,g_k,Z_{k-1}}\, \mathring{f}_{k-1}(\mathring{X}_{k-1}|Z_{1:k-1})\delta\mathring{X}_{k-1}\right)\right]_{g_k=0}} \tag{101}$$

and so, from the definition of a PGFL,

$$\begin{aligned} & \mathring{G}_k[\mathring{h}_k|Z_{1:k}] \\ = \;& K_k^{-1}\left[\frac{\delta}{\delta Z_k}\left(G_k^\kappa[g_k]\, \mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}]\, \mathring{G}_{k-1}[\mathring{H}_{\mathring{h}_k,g_k,Z_{k-1}}|Z_{1:k-1}]\right)\right]_{g_k=0} \end{aligned} \tag{102}$$

where

$$K_k = \left[\frac{\delta}{\delta Z_k}\left(G_k^\kappa[g_k]\, \mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,1}]\, \mathring{G}_{k-1}[\mathring{H}_{1,g_k,Z_{k-1}}|Z_{1:k-1}]\right)\right]_{g_k=0}. \tag{103}$$

5 The PMM-mGLMB Filter

The purpose of this section is to determine the formula for the updated PGFL $\mathring{G}_k[h_k|Z_{1:k}]$ given that the prior PGFL $\mathring{G}_{k-1}[\mathring{h}_{k-1}|Z_{1:k-1}]$ is mGLMB. It is organized as follows: a summary of the main result (Section 5.1); and its demonstration (Section 5.2).

5.1 Summary of the Main Result

Suppose that at time t_{k-1} the LMPMM PGFL is mGLMB as in Eq. (24):

$$\mathring{G}_{k-1}[\mathring{h}_{k-1}|Z_{1:k-1}] = \sum_{o\in O_{k-1}} \sum_{L\subseteq L_{k-1}} \omega^o_{k-1}(L) \prod_{l\in L} \sigma^o_{l,k-1}[\mathring{h}_{k-1}] \tag{104}$$

where L_{k-1} is the set of labels for the targets present at time t_{k-1} and where

$$\begin{aligned} \sigma^o_{l,k}[\mathring{h}_{k-1}] &= 1 - q^o_{l,k-1} + q^o_{l,k-1} s^o_{l,k-1}[\mathring{h}_{k-1}] \qquad (105) \\ &= 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \mathring{h}_{k-1}(x_{k-1},l)\, s^o_{l,k-1}(x_{k-1},l) dx_{k-1} \qquad (106) \end{aligned}$$

with $l\in L_{k-1}$ are the PGFLs of the Bernoulli spatial densities $(q^o_{l,k-1}, s^o_{l,k-1})$ at time t_{k-1}.

Further suppose that a new measurement-set Z_k has been collected and that new targets have appeared with label set $L^B_{k|k-1}$ and Bernoulli spatial densities $(q^B_{l,k|k-1}, s^B_{l,k|k-1})$. Then the PGFL at time t_k is also mGLMB:

$$\mathring{G}_k[\mathring{h}_k|Z_{1:k}] = \sum_{(o,\alpha_k)\in O_k} \sum_{L\subseteq L_k} \omega^{o,\alpha_k}_k(L) \prod_{l\in L} \sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k], \tag{107}$$

with label set $L_k = L^B_{k|k-1} \uplus L_{k-1}$; index set $O_k = O_{k-1} \times \mathcal{A}_k$ where $\mathcal{A}_k$ is the set of label-to-measurement associations (LMAs) $\alpha_k : L_k \to \{0, 1, ..., |Z_k|\}$; and $\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k]$ is the PGFL of the Bernoulli spatial density $(q^{o,\alpha_k}_{l,k}, s^{o,\alpha_k}_{l,k})$. The formulas for these items are:

Updated GLMB weights. These are

$$\omega^{o,\alpha_k}_k(L) = \frac{\omega^o_{k-1}(L)\, C^{o,\alpha_k}_{L,k}}{\sum_{(o,\alpha_k)\in O_k} \sum_{L\subseteq L_k} \omega^o_{k-1}(L)\, C^{o,\alpha_k}_{L,k}} \tag{108}$$

where[9]

$$\begin{aligned} C^{o,\alpha_k}_{L,k} &= \mathbf{1}^L_{L_{\alpha_k}} \left(\prod_{l:\alpha_k(l)>0} \kappa_k(z_{\alpha_k(l)}) \right) \\ &\quad \cdot \left(\prod_{l:\alpha_k(l)=0} \mathring{F}^o_{l,k}[0,1] \right) \left(\prod_{l:\alpha_k(l)>0} \frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0,1] \right); \end{aligned} \tag{109}$$

where $\mathbf{1}^L_{L_{\alpha_k}}$ is defined in Eq. (264) below; where (see Eqs. (163,167))

$$\mathring{F}^o_{l,k}[0,1] = \begin{cases} 1 - q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}_D] & \text{if} \quad l \in L^B_{k|k-1} \\ \begin{pmatrix} 1 - q^o_{l,k-1} + q^o_{l,k-1} \\ \cdot \int \mathring{P}_l(x_{k-1}, x_k) \\ \cdot f_{k|k-1}(x_k|x_{k-1}, Z_{k-1}) \\ \cdot s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{pmatrix} & \text{if} \quad l \in L_{k-1} \end{cases}; \tag{110}$$

where

$$\mathring{P}_l(x_{k-1}, x_k) = 1 - \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(x_k, l) \tag{111}$$

$$\begin{aligned} f_{k|k-1}(x_k|x_{k-1}, Z_{k-1}) &= \frac{1}{|Z_{k-1}|} \\ &\quad \cdot \sum_{z_k\in Z_{k-1}} \int f_{k|k-1}(x_k, y_k|x_{k-1}, z_{k-1}) dy_k; \end{aligned} \tag{112}$$

[9] The items $\omega^{o,\alpha_k}_{k-1}(L)$ and $C^{o,\alpha_k}_{L,k}$ should not be confused with $\omega^{o,\alpha_k}_{k-1}(L)$ and $C^{o,\alpha_k}_{L,k}$ in Section 2.10. Their respective definitions are clear from context.

and where (see Eqs. (179,183))

$$\frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0,1] \tag{113}$$

$$= \left\{ \begin{array}{cl} q^B_{l,k|k-1} s^B_{l,k|k-1} \left[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}} \right] & \text{if} \quad l \in L^B_{k|k-1} \\ \left(\begin{array}{c} q^o_{l,k-1} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(x_k, l) \\ \cdot f_{k|k-1}(x_k, z_{\alpha_k(l)} | x_{k-1}, Z_{k-1}) \\ \cdot s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{array} \right) & \text{if} \quad l \in L_{k-1} \end{array} \right. .$$

Updated Bernoulli spatial densities. The definition of the Bernoulli spatial density $(q^{o,\alpha_k}_{l,k}, s^{o,\alpha_k}_{l,k})$ with PGFL

$$\sigma^o_{l,k}[\mathring{h}_k] = 1 - q^{o,\alpha_k}_{l,k} + q^{o,\alpha_k}_{l,k} s^{o,\alpha_k}_{l,k}[\mathring{h}_k] \tag{114}$$

has four cases:

Case 1: *Undetected newly-appearing target* ($\alpha_k(l) = 0$ and $l \in L^B_{k|k-1}$) see Eqs. (191,192)).

$$q^{o,\alpha_k}_{l,k} = \frac{q^B_{l,k|k-1}(1 - s^B_{l,k|k-1}[\mathring{p}_D])}{1 - q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}_D]} \tag{115}$$

$$s^{o,\alpha_k}_{l,k}(x_{k-1}) = \frac{\mathring{p}^c_D(x_{k-1}, l)\; s^B_{l,k|k-1}(x_{k-1})}{s^B_{l,k|k-1}[\mathring{p}^c_D]}. \tag{116}$$

If $\mathring{p}_D$ is constant then

$$q^{o,\alpha_k}_{l,k} = \frac{q^B_{l,k|k-1}(1 - \mathring{p}_D)}{1 - q^B_{l,k|k-1}\mathring{p}_D} \tag{117}$$

$$s^{o,\alpha_k}_{l,k}(x_{k-1}) = s^B_{l,k|k-1}(x_{k-1}). \tag{118}$$

Case 2: *Undetected surviving target* ($\alpha_k(l) = 0$ and $l \in L_{k-1}$), see Eqs. (211,220).

$$q^{o,\alpha_k}_{l,k} = \frac{\left(\begin{array}{c} q^o_{l,k-1} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}^c_D(x_k, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{array} \right)}{\left(\begin{array}{c} 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \{1 - \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(x_k, l)\} \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\; s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{array} \right)} \tag{119}$$

$$s^o_{l,k}(x_k) = \frac{\left(\begin{array}{c} \mathring{p}^c_D(x_k, l) \int \mathring{p}_S(x_{k-1}, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_{k-1} \end{array} \right)}{\left(\begin{array}{c} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}^c_D(x_k, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{array} \right)} \tag{120}$$

where

$$f_{k|k-1}(x_k | x_{k-1}, Z_{k-1}) = \frac{1}{|Z_{k-1}|} \sum_{z_k \in Z_{k-1}} \int f_{k|k-1}(x_k, y_k | x_{k-1}, z_{k-1}) dy_k. \tag{121}$$

If $\mathring{p}_S$ and $\mathring{p}_S$ are constant then

$$q^{o,\alpha_k}_{l,k} = \frac{q^o_{l,k-1}\mathring{p}_S(1-\mathring{p}_D)}{1-q^o_{l,k-1}\mathring{p}_S\mathring{p}_D} \quad (122)$$

$$s^o_{l,k}(x_k) = \int f_{k|k-1}(x_k|x_{k-1},Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}. \quad (123)$$

Case 3: *Detected newly-appearing target* ($\alpha_k(l)>0$ and $l\in L^B_{k|k-1}$), see Eq. (225).

$$q^{o,\alpha_k}_{l,k} = 1 \quad (124)$$

$$s^{o,\alpha_k}_{l,k}(x_k) = \frac{\mathring{p}_D(x_k,l)\,\mathring{L}_{z_{\alpha_k(l)}}(x_k,l)\, s^B_{l,k|k-1}(x_k)}{s^B_{l,k|k-1}[\mathring{p}_D\mathring{L}_{z_{\alpha_k(l)}}]}. \quad (125)$$

If $\mathring{p}_D$ is constant then

$$s^{o,\alpha_k}_{l,k}(x_k) = \frac{\mathring{L}_{z_{\alpha_k(l)}}(x_k,l)\, s^B_{l,k|k-1}(x_k)}{s^B_{l,k|k-1}[\mathring{L}_{z_{\alpha_k(l)}}]}, \quad (126)$$

which is a conventional single-target Bayes' rule measurement-update.

Case 4: *Detected surviving target* ($\alpha_k(l)>0$ and $l\in L_{k-1}$), see Eq. (233).

$$q^{o,\alpha_k}_{l,k} = 1 \quad (127)$$

$$s^{o,\alpha_k}_{l,k}(x_k) = \frac{\left(\begin{array}{c}\mathring{p}_D(x_k,l)\int \mathring{p}_S(x_{k-1},l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}\end{array}\right)}{\left(\begin{array}{c}\int \mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(x_k,l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_k dx_{k-1}\end{array}\right)} \quad (128)$$

If $\mathring{p}_S$ and $\mathring{p}_D$ are constant then

$$s^{o,\alpha_k}_{l,k}(x_k) = \frac{\int f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}}{\int f_{k|k-1}(z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}}. \quad (129)$$

If the single-target PMM is an HMM then this reduces to

$$s^{o,\alpha_k}_{l,k}(x_k) = \frac{f_k(z_{\alpha_k(l)}|x_k)\int f_{k|k-1}(x_k|x_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}}{\int f_k(z_{\alpha_k(l)}|x_k)\, f_{k|k-1}(x_k|x_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}dx_k}. \quad (130)$$

This is the consolidation, into a single step, of the single-target Bayes filter's time-update and measurement-update steps. Thus the PMM-mGLMB filter is consistent with conventional single-target tracking.

5.2 Demonstration of Main Result

For future reference recall the following:

1. The following special cases of Eqs. (84):

$$s^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,0,Z_{k-1}}] \tag{131}$$
$$= \int \ddot{M}_{\mathring{p}^c_S+\mathring{p}_S\mathring{h}_k\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}$$
$$s^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,\mathring{L}_z,Z_{k-1}}] \tag{132}$$
$$= \int \ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D\mathring{L}_z}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}.$$

2. The formula for the single-target labeled PMM transition density, Eq. (48); and

3. The definition a labeled integral, Eq. (8).

As previously noted, the derivation is closely patterned after the PGFL derivation of the GLMB filter's measurement-update step in Section 6.2. It has the following major steps:

1. Section 5.2.1: From the PGFL formula for the generic PMM GLMB filter update, Eq. (102), and the formula for the "standard model" LMPMM transition function, Eq. (89), derive a set-theoretic formula for the PGFL of the PMM-mGLMB filter update: Eqs. (133-135).

2. Section 5.2.2: Recast this second version into a version that is amenable for application of the general product rule for functional derivatives: Eqs. (136-146).

3. Section 5.2.3: Recast this third version into a more intuitive LMA version: Eqs. (147-156).

4. Section 5.2.4: Recast this fourth version into a version that is similar in form to the PGFL form of the GLMB filter measurement-update in Section 2.10: Eqs. (159-160).

5. Section 5.2.5: Derive formulas for specific factors in this fifth version: Eqs. (161-233).

5.2.1 Set-Theoretic Version

From Eqs. (102,91,92,89) the updated PGFL is

$$\mathring{G}_k[\mathring{h}_k|Z_{1:k}] \tag{133}$$
$$\propto \left[\frac{\delta}{\delta Z_k}\left(\begin{array}{c} G^\kappa_k[g_k]\,\mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] \\ \cdot\sum_{o\in O_{k-1}}\sum_{L\subseteq L_{k-1}}\omega^o_{k-1}(L)\prod_{l\in L}\sigma^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,g_k,Z_{k-1}}]\end{array}\right)\right]_{g_k=0}$$

$$= \sum_{o\in O_{k-1}}\sum_{L\subseteq L_{k-1}}\omega^o_{k-1}(L)\left[\frac{\delta}{\delta Z_k}\left(\begin{array}{c} G^\kappa_k[g_k]\,\mathring{G}^B_{k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] \\ \cdot\prod_{l\in L}\sigma^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,g_k,Z_{k-1}}]\end{array}\right)\right]_{g_k=0} \tag{134}$$

$$= \sum_{o\in O_{k-1}} \sum_{L\subseteq L_{k-1}} \omega^o_{k-1}(L) \left[\frac{\delta}{\delta Z_k} \left(\begin{array}{c} G^\kappa_k[g_k] \\ \cdot \left(\prod_{l\in L^B_{k|k-1}} \mathring{G}^B_{l,k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}]\right) \\ \cdot \left(\prod_{l\in L} \sigma^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,g_k,Z_{k-1}}]\right) \end{array} \right) \right]_{g_k=0} . \tag{135}$$

5.2.2 Product Rule Version

Eq. (133) can therefore be rewritten as

$$\begin{aligned} \mathring{G}_k[\mathring{h}_k|Z_{1:k}] \quad &\propto \quad \sum_{o\in O_{k-1}} \sum_{L\subseteq L_{k-1}} \omega^o_{k-1}(L)\, \mathring{F}^o_k[\mathring{h}_k] \\ &\quad \cdot \left[\frac{\delta}{\delta Z_k} \left(G^\kappa_k[g_k] \prod_{l\in L} \hat{F}^o_{l,k}[g_k,\mathring{h}_k] \right) \right]_{g_k=0} \end{aligned} \tag{136}$$

where $L_k = L^B_{k|k-1} \uplus L_{k-1}$ is the set of labels for the targets present at time t_k; and where

$$\hat{F}^o_{l,k}[g_k,\mathring{h}_k] \quad = \quad \frac{\mathring{F}^o_{l,k}[g_k,\mathring{h}_k]}{\mathring{F}^o_{l,k}[0,\mathring{h}_k]} \tag{137}$$

$$\mathring{F}^o_{l,k}[g_k,\mathring{h}_k] \quad = \quad \left\{ \begin{array}{cl} G^B_{l,k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}] & \text{if} \quad l\in L^B_{k|k-1} \\ \sigma^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,g_k,Z_{k-1}}] & \text{if} \quad l\in L_{k-1} \end{array} \right. \tag{138}$$

$$\mathring{F}^o_k[\mathring{h}_k] \quad = \quad \prod_{l\in L} \mathring{F}^o_{l,k}[0,\mathring{h}_k] \tag{139}$$

so that

$$\hat{F}^o_{l,k}[g_k,\mathring{h}_k] = \left\{ \begin{array}{cl} \frac{G^B_{l,k|k-1}[\mathring{T}_{g_k,\mathring{h}_k}]}{G^B_{l,k|k-1}[\mathring{T}_{0,\mathring{h}_k}]} & \text{if} \quad l\in L^B_{k|k-1} \\ \frac{\sigma^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,g_k,Z_{k-1}}]}{\sigma^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,0,Z_{k-1}}]} & \text{if} \quad l\in L_{k-1} \end{array} \right. . \tag{140}$$

Thus since $G^\kappa_k[g_k] = e^{\kappa_k[g_k-1]}$, we must derive the formula for the factor

$$\mathring{F}^o_k[\mathring{h}_k] \frac{\delta}{\delta Z_k} \left(e^{\kappa_k[g_k-1]} \prod_{l\in L} \hat{F}^o_{l,k}[g_k,\mathring{h}_k] \right) \tag{141}$$

in Eq. (136).

First, note that $\hat{F}^o_{l,k}[g_k,\mathring{h}_k]$ is first-order in g_k (in the sense of Eq. (27)) and thus

$$\begin{aligned} &\left[\frac{\delta}{\delta Z} \hat{F}^o_{l,k}[g_k,\mathring{h}_k] \right]_{g_k=0} \\ = \quad &\left\{ \begin{array}{cl} 1 & \text{if} \quad Z=\emptyset \\ \left[\frac{\delta}{\delta Z} \hat{F}^o_{l,k}[g_k,\mathring{h}_k] \right]_{g_k=0} & \text{if} \quad Z=\{z\} \\ 0 & \text{if} \quad |Z|\geq 2 \end{array} \right. . \end{aligned} \tag{142}$$

Second, for a fixed $L = \{l_1, ..., l_n\}$ with $|L| = n$, note that by the general product rule for functional derivatives, [13, Eq. (11.274)],

$$\mathring{F}^o_k[\mathring{h}_k]\frac{\delta}{\delta Z_k}\left(e^{\kappa_k[g_k-1]}\prod_{i=1}^n \hat{F}^o_{l_i,k}[g_k,\mathring{h}_k]\right) \tag{143}$$

$$= e^{\kappa_k[g_k-1]}\mathring{F}^o_k[\mathring{h}_k]\sum_{W_0\uplus W_1\uplus ...\uplus W_n = Z_k} \kappa_k^{W_0}\prod_{i=1}^n \frac{\delta \hat{F}^o_{l_i,k}}{\delta W_i}[g_k,\mathring{h}_k]$$

$$= \kappa_k^{Z_k} e^{\kappa_k[g_k-1]}\mathring{F}^o_k[\mathring{h}_k]\sum_{W_0\uplus W_1\uplus ...\uplus W_n = Z_k} \prod_{i=1}^n \frac{1}{\kappa_k^{W_i}}\frac{\delta \hat{F}^o_{l_i,k}}{\delta W_i}[g_k,\mathring{h}_k] \tag{144}$$

where the summation is taken over all possibly empty and mutually disjoint subsets $W_0, W_1, ..., W_n \subseteq Z_k$ such that $W_0 \uplus W_1 \uplus ... \uplus W_n = Z_k$.

By Eq. (142), any factor involving W_i with $|W_i| \geq 2$ and $i \geq 1$ vanishes, so after setting $g_k = 0$ we get

$$= e^{-\kappa_k[1]}\kappa_k^{Z_k}\mathring{F}^o_k[\mathring{h}_k]\sum_{W_1,...,W_n\subseteq Z_k}\prod_{i=1}^n \frac{1}{\kappa_k^{W_i}}\left[\frac{\delta \hat{F}^o_{l_i,k}}{\delta W_i}[g_k,\mathring{h}_k]\right]_{g_k=0} \tag{145}$$

where

$$\frac{1}{\kappa_k^{W_i}}\left[\frac{\delta \hat{F}^o_{l_i,k}}{\delta W_i}[g_k,\mathring{h}_k]\right]_{g_k=0} = 1 \tag{146}$$

if $W_i = \emptyset$.

5.2.3 LMA Version

Third, recall the procedure used to construct the multitarget likelihood function for the L/RFS standard measurement model (originally presented in [13, pp. 742-745] and later summarized in [10, p. 6, Sec. 9]). We emulate the version of it presented in the PGFL derivation of the measurement-update step of the GLMB filter in Section 6.2 below.

As before, fix $L = \{l_1, ..., l_n\}$ with $|L| = n$. Then given each W_i in Eq. (145) define a label-to-measurement association (LMA) $\alpha_{L,k} : L \rightarrow \{0, 1, ..., |Z_k|\}$ by $\alpha_{L,k}(l_i) = 0$ if $W_i = \emptyset$ (target undetected) and $\alpha_{L,k}(l_i) = j$ if $W_i = \{z_j\}$ (target detected). Conversely, given an LMA $\alpha_{L,k}$, if $\alpha_{L,k}(l_i) > 0$ define $W_i = \{z_{\alpha_{L,k}(l_i)}\}$ if $\alpha_{L,k}(l_i) > 0$ and $W_i = \emptyset$ otherwise. It follows that there is a one-to-one correspondence between LMAs on L and lists of mutually disjoint subsets $W_1, ..., W_n$ of Z_k such that $|W_i| \leq 1$ for $i = 1, ..., n$.

Let $\mathcal{A}_{L,k}$ denote the set of all LMAs on L at time t_k.

Given this, the right side of Eq. (145) can be rewritten in LMA form as

$$e^{-\kappa_k[1]}\kappa_k^{Z_k}\mathring{F}^o_k[\mathring{h}_k]\sum_{\alpha_{L,k}\in\mathcal{A}_{L,k}}\prod_{l:\alpha_{L,k}(l)>0}\frac{1}{\kappa_k(z_{\alpha_{L,k}(l)})}\frac{\delta \hat{F}^o_{l,k}}{\delta z_{\alpha_{L,k}(l)}}[0,\mathring{h}_k] \tag{147}$$

where the product is taken over all $l \in L$ such that $\alpha_{L,k}(l) > 0$.

Note from Eqs. (137-139) that the rightmost part of Eq. (136) becomes

$$= \quad e^{-\kappa_k[1]} \kappa_k^{Z_k} \left(\prod_{l \in L} \mathring{F}^o_{l,k}[0, \mathring{h}_k] \right) \tag{148}$$

$$\cdot \sum_{\alpha_{L,k} \in \mathcal{A}_{L,k}} \prod_{l: \alpha_{L,k}(l) > 0} \frac{1}{\kappa_k(z_{\alpha_{L,k}(l)})} \frac{\delta \hat{F}^o_{l,k}}{\delta z_{\alpha_{L,k}(l)}}[0, \mathring{h}_k].$$

This formula is problematic because it involves LMAs $\alpha_{L,k} : L \to \{0, 1, ..., |Z_k|\}$ rather than, as should be the case, LMAs $\alpha_k : L_k \to \{0, 1, ..., |Z_k|\}$.

Define [6, Eq. (69)]

$$\alpha_k(l) = \left\{ \begin{array}{ccc} \alpha_{L,k}(l) & \text{if} & l \in L, \, \alpha_{L,k}(l) > 0 \\ 0 & \text{if} & \text{otherwise} \end{array} \right. . \tag{149}$$

As noted in Section 6.2.4, this α_k is distinguished from other $\alpha : L_k \to \{0, 1, ..., |Z_k|\}$ by Eq. (149) and, in particular, by the identity[10]

$$L_\alpha \overset{\text{def.}}{=} \{l \in L_k : \alpha(l) > 0\} = \{l \in L : \alpha_{L,k}(l) > 0\} \overset{\text{def.}}{=} \alpha_{L,k}(L). \tag{150}$$

This, as noted in Eq. (264) below, is equivalent to the identity $\mathbf{1}^L_{L_{\alpha_k}} = 1$.

Given this, let $\mathcal{A}_k$ denote the set of all LMAs $\alpha_k : L_k \to \{0, 1, ..., |Z_k|\}$. Then Eq. (136) becomes

$$\mathring{G}_k[\mathring{h}_k | Z_{1:k}] \tag{151}$$

$$\propto \quad e^{-\kappa_k[1]} \kappa_k^{Z_k} \sum_{o \in O_{k-1}} \sum_{L \subseteq L_{k-1}} \omega^o_{k-1}(L) \left(\prod_{l \in L} \mathring{F}^o_{l,k}[0, \mathring{h}_k] \right)$$

$$\cdot \sum_{\alpha_k \in \mathcal{A}_k} \mathbf{1}_L^{L_{\alpha_k}} \prod_{l \in L : \alpha_k(l) > 0} \left(\frac{1}{\kappa_k(z_{\alpha_k(l)})} \frac{\delta \hat{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0, \mathring{h}_k] \right)$$

$$= \quad e^{-\kappa_k[1]} \kappa_k^{Z_k} \sum_{o \in O_{k-1}} \sum_{\alpha_k \in \mathcal{A}_k} \sum_{L \subseteq L_{k-1}} \omega^o_{k-1}(L) \left(\prod_{l \in L : \alpha_k(l) = 0} \mathring{F}^o_{l,k}[0, \mathring{h}_k] \right) \tag{152}$$

$$\cdot \mathbf{1}_L^{L_{\alpha_k}} \prod_{l \in L : \alpha_k(l) > 0} \left(\frac{1}{\kappa_k(z_{\alpha_k(l)})} \frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0, \mathring{h}_k] \right)$$

[10] *Errata*: Eqs. (70,71) in [6] have typos. The correct definition is in the line immediately following Eq. (69).

$$
\begin{aligned}
= \quad & e^{-\kappa_k[1]}\kappa_k^{Z_k} \sum_{o\in O_{k-1}} \sum_{\alpha_k\in\mathcal{A}_k} \sum_{L\subseteq L_{k-1}} \omega_{k-1}^{o}(L) \\
& \cdot \left(\prod_{l\in L:\alpha_k(l)>0} \kappa_k(z_{\alpha_k(l)}) \right) \left(\prod_{l\in L:\alpha_k(l)=0} \mathring{F}_{l,k}^{o}[0,\mathring{h}_k] \right) \\
& \cdot \mathbf{1}_L^{L_{\alpha_k}} \prod_{l\in L:\alpha_k(l)>0} \frac{\delta \mathring{F}_{l,k}^{o}}{\delta z_{\alpha_k(l)}}[0,\mathring{h}_k].
\end{aligned} \tag{153}
$$

Consequently,

$$
\begin{aligned}
& \mathring{G}_k[\mathring{h}_k|Z_{1:k}] \\
\propto \quad & e^{-\kappa_k[1]} \sum_{o\in O_{k-1}} \sum_{\alpha_k\in\mathcal{A}_k} \sum_{L\subseteq L_{k-1}} \omega_{k-1}^{o}(L) \\
& \cdot \mathbf{1}_L^{L_{\alpha_k}} \left(\prod_{l\in L:\alpha_k(l)>0} \kappa_k(z_{\alpha_k(l)}) \right) \left(\prod_{l\in L:\alpha_k(l)=0} \mathring{F}_{l,k}^{o}[0,1] \right) \\
& \cdot \left(\prod_{l\in L:\alpha_k(l)>0} \frac{\delta \mathring{F}_{l,k}^{o}}{\delta z_{\alpha_k(l)}}[0,1] \right) \\
& \cdot \left(\prod_{l\in L:\alpha_k(l)=0} \frac{\mathring{F}_{l,k}^{o}[0,\mathring{h}_k]}{\mathring{F}_{l,k}^{o}[0,1]} \right) \left(\prod_{l\in L:\alpha_k(l)>0} \frac{\frac{\delta \mathring{F}_{l,k}^{o}}{\delta z_{\alpha_k(l)}}[0,\mathring{h}_k]}{\frac{\delta \mathring{F}_{l,k}^{o}}{\delta z_{\alpha_k(l)}}[0,1]} \right).
\end{aligned} \tag{154}
$$

Define

$$
\begin{aligned}
C_{L,k}^{o,\alpha_k} \quad = \quad & \mathbf{1}_L^{L_{\alpha_k}} \left(\prod_{l:\alpha_k(l)>0} \kappa_k(z_{\alpha_k(l)}) \right) \left(\prod_{l:\alpha_k(l)=0} \mathring{F}_{l,k}^{o}[0,1] \right) \\
& \cdot \left(\prod_{l:\alpha_k(l)>0} \frac{\delta \mathring{F}_{l,k}^{o}}{\delta z_{\alpha_k(l)}}[0,1] \right)
\end{aligned} \tag{155}
$$

and

$$
\sigma_{l,k}^{o,\alpha_k}[\mathring{h}_k] = \begin{cases} \frac{\mathring{F}_{l,k}^{o}[0,\mathring{h}_k]}{\mathring{F}_{l,k}^{o}[0,1]} & \text{if} \quad \alpha_k(l)=0 \\ \frac{\frac{\delta \mathring{F}_{l,k}^{o}}{\delta z_{\alpha_k(l)}}[0,\mathring{h}_k]}{\frac{\delta \mathring{F}_{l,k}^{o}}{\delta z_{\alpha_k(l)}}[0,1]} & \text{if} \quad \alpha_k(l)>0 \end{cases}. \tag{156}
$$

Then

$$
\mathring{G}_k[\mathring{h}_k|Z_{1:k}] \propto \sum_{o\in O_{k-1}} \sum_{\alpha_k\in\mathcal{A}_k} \sum_{L\subseteq L_k} \omega_{k-1}^{o}(L)\, C_{L,k}^{o,\alpha_k} \prod_{l\in L} \sigma_{l,k}^{o,\alpha_k}[\mathring{h}_k]. \tag{157}
$$

5.2.4 GLMB Version

From Eq. (136) the updated PGFL becomes

$$\mathring{G}_k[\mathring{h}_k|Z_{1:k}] \tag{158}$$

$$= \frac{\sum_{o\in O_{k-1}}\sum_{\alpha_k\in\mathcal{A}_k}\sum_{L\subseteq L_k}\omega^o_{k-1}(L)\ C^{o,\alpha_k}_{L,k}\prod_{l\in L}\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k]}{\sum_{o\in O_{k-1}}\sum_{\alpha_k\in\mathcal{A}_k}\sum_{L\subseteq L_k}\omega^o_{k-1}(L)\ C^{o,\alpha_k}_{L,k}}$$

$$= \sum_{(o,\alpha_k)\in O_k}\sum_{L\subseteq L_k}\omega^{o,\alpha_k}_k(L)\prod_{l\in L}\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k] \tag{159}$$

where $O_k = O_{k-1}\times\mathcal{A}_k$ and where

$$\omega^{o,\alpha_k}_k(L) = \frac{\omega^o_{k-1}(L)\,C^{o,\alpha_k}_{L,k}}{\sum_{(o,\alpha_k)\in O_k}\sum_{L\subseteq L_k}\omega^o_{k-1}(L)\,C^{o,\alpha_k}_{L,k}}. \tag{160}$$

As will be seen momentarily, $\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k]$ is the PGFL of a Bernoulli LRFS, Eq. (18). Thus the right side of Eq. (159) is not only a mGLMB PGFL, it is quite similar to the measurement-update step of the conventional GLMB filter, see Eq. (41). The main conceptual differences are that Bernoulli spatial densities $(q^{o,\alpha_k}_{l,k}, s^{o,\alpha_k}_{l,k})$ have been substituted in place of spatial densities $s^{o,\alpha_k}_{l,k}$ and that the mGLMB constant $C^{o,\alpha_k}_{L,k}$ is more complex than the GLMB constant $C^{o,\alpha_k}_{L,k}$.

5.2.5 Factor Formulas

Inspection of Eqs (155,156) shows that Eq. (159) is incomplete in the absence of explicit formulas for the following factors:

1. $\mathring{F}^o_{l,k}[0,1]$ if $\alpha_k(l)=0$, and $l\in L^B_{k|k-1}$ or $l\in L_{k-1}$.

2. $\frac{\delta\mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0,1]$ if $\alpha_k(l)>0$, and $l\in L^B_{k|k-1}$ or $l\in L_{k-1}$

3. $\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k]$ if $l\in L^B_{k|k-1}$ or $l\in L_{k-1}$, and $\alpha_k(l)=0$ or $\alpha_k(l)>0$.

These eight possibilities are considered each in turn.

Case 1a: ($\mathring{F}^o_{l,k}[0,1]$ for $\alpha_k(l)=0$ and $l\in L^B_{k|k-1}$). From Eqs. (138,94,95) we have

$$\mathring{F}^o_{l,k}[0,1] = \mathring{G}^B_{l,k|k-1}[\mathring{T}_{0,1}] \tag{161}$$

$$= 1-q^B_{l,k|k-1}+q^B_{l,k|k-1}s^B_{l,k|k-1}[\mathring{T}_{0,1}] \tag{162}$$

$$= 1-q^B_{l,k|k-1}+q^B_{l,k|k-1}s^B_{l,k|k-1}[\mathring{p}^c_D] \tag{163}$$

$$= 1-q^B_{l,k|k-1}s^B_{l,k|k-1}[\mathring{p}_D]. \tag{164}$$

Case 1b: ($\mathring{F}^o_{l,k}[0,1]$ for $\alpha_k(l) = 0$ and $l \in L_{k-1}$). From Eqs. (138,84,36) we get

$$\begin{aligned} & \mathring{F}^o_{l,k}[0,1] && (165) \\ = \; & \sigma^o_{l,k-1}[\mathring{H}_{1,0,Z_{k-1}}] \\ = \; & 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \mathring{H}_{1,0,Z_{k-1}}(x_{k-1}, l)\, s^o_{l,k-1}(x_{k-1}) dx_{k-1} && (166) \\ = \; & 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S \mathring{p}^c_D}(x_{k-1}, l, Z_{k-1}) && (167) \\ & \cdot s^o_{l,k-1}(x_{k-1}) dx_{k-1}. \end{aligned}$$

Abbreviate

$$\begin{aligned} \mathring{P}(\mathring{x}_{k-1}, \mathring{x}_k) &= \mathring{p}^c_S(\mathring{x}_{k-1}) + \mathring{p}_S(\mathring{x}_{k-1})\, \mathring{p}^c_D(\mathring{x}_k) && (168) \\ &= 1 - \mathring{p}_S(\mathring{x}_{k-1})\, \mathring{p}_D(\mathring{x}_k). && (169) \end{aligned}$$

Then

$$\begin{aligned} \mathring{F}^o_{l,k}[0,1] &= 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \ddot{M}_{\mathring{P}}(x_{k-1}, l, Z_{k-1}) && (170) \\ & \quad \cdot s^o_{l,k-1}(x_{k-1}) dx_{k-1} \\ &= 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \mathring{P}(x_{k-1}, l, \mathring{x}_k) && (171) \\ & \quad \cdot \mathring{f}_{k|k-1}(\mathring{x}_k, y_k | x_{k-1}, l, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) d\mathring{x}_k dy_k dx_{k-1} \\ &= 1 - q^o_{l,k-1} + q^o_{l,k-1} \sum_{l_k \in L_k} \int \mathring{P}(x_{k-1}, l, x_k, l_k) && (172) \\ & \quad \cdot \mathring{f}_{k|k-1}(x_k, l_k, y_k | x_{k-1}, l, Z_{k-1}) \\ & \quad \cdot s^o_{l,k-1}(x_{k-1}) dx_k dy_k dx_{k-1} \\ &= 1 - q^o_{l,k-1} + q^o_{l,k-1} \sum_{l_k \in L_k} \int \mathring{P}(x_{k-1}, l, x_k, l_k) && (173) \\ & \quad \cdot \delta_{l_k,l}\, f_{k|k-1}(x_k, y_k | x_{k-1}, Z_{k-1}) \\ & \quad \cdot s^o_{l,k-1}(x_{k-1}) dx_k dy_k dx_{k-1} \\ &= 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \mathring{P}_l(x_{k-1}, x_k) && (174) \\ & \quad \cdot f_{k|k-1}(x_k, y_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dy_k dx_{k-1} \\ &= 1 - q^o_{l,k-1} + q^o_{l,k-1} \int \mathring{P}_l(x_{k-1}, x_k) && (175) \\ & \quad \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{aligned}$$

where from Eq. (168)

$$\mathring{P}_l(x_{k-1}, x_k) \stackrel{\text{def.}}{=} 1 - \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(x_k, l). \tag{176}$$

Case 2a ($\frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0,1]$ for $\alpha_k(l) > 0$ and $l \in L^B_{k|k-1}$). From Eqs. (138,94,95)

$$\frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0,1] = \left[\frac{\delta}{\delta z_{\alpha_k(l)}} \mathring{G}^B_{l,k|k-1}\left[\mathring{T}_{g_k,1}\right]\right]_{g_k=0} \tag{177}$$

$$= q^B_{l,k|k-1} s^B_{l,k|k-1} \left[\left[\frac{\delta}{\delta z_{\alpha_k(l)}} \mathring{T}_{g_k,1}\right]_{g_k=0}\right] \tag{178}$$

$$= q^B_{l,k|k-1} s^B_{l,k|k-1} \left[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}\right]. \tag{179}$$

Case 2b ($\frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0,1]$ for $\alpha_k(l) > 0$ and $l \in L_{k|-1}$). From Eqs. (138,84),

$$\frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0,1] \tag{180}$$

$$= \frac{\delta}{\delta z_{\alpha_k(l)}} \sigma^o_{l,k-1}\left[\mathring{H}_{1,g_k,Z_{k-1}}\right]$$

$$= q^o_{l,k-1} \int \ddot{M}_{\mathring{p}_S \mathring{p}_D \delta_{z_{\alpha_k(l)}}}(x_{k-1}, l, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_{k-1} \tag{181}$$

$$= q^o_{l,k-1} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(\mathring{x}_k)\, \delta_{z_{\alpha_k(l)}}(y_k) \tag{182}$$

$$\cdot \mathring{f}_{k|k-1}(\mathring{x}_k, y_k | x_{k-1}, l, Z_{k-1})\, s^o_{k-1}(x_{k-1}) d\mathring{x}_k\, dy_k dx_{k-1}$$

$$= q^o_{l,k-1} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(\mathring{x}_k) \tag{183}$$

$$\cdot \mathring{f}_{k|k-1}(\mathring{x}_k, z_{\alpha_k(l)} | x_{k-1}, l, Z_{k-1})\, s^o_{k-1}(x_{k-1}) d\mathring{x}_k dx_{k-1}.$$

$$= q^o_{l,k-1} \sum_{l_k \in L_k} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(x_k, l_k)\, \delta_{l_k,l} \tag{184}$$

$$\cdot \mathring{f}_{k|k-1}(x_k, l_k, z_{\alpha_k(l)} | x_{k-1}, l, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1}$$

$$= q^o_{l,k-1} \sum_{l_k \in L_k} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(x_k, l_k) \tag{185}$$

$$\cdot \delta_{l_k,l}\, f_{k|k-1}(x_k, z_{\alpha_k(l)} | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1}$$

$$= q^o_{l,k-1} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D(x_k, l) \tag{186}$$

$$\cdot f_{k|k-1}(x_k, z_{\alpha_k(l)} | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1}.$$

Case 3a: $q^o_{l,k}$ and $s^o_{l,k}$ for an undetected newly-appearing target l ($\alpha_k(l) = 0$ and $l \in L^B_{kk-1}$). From Eqs. (156,138,94,95),

$$\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k] = \frac{\mathring{F}^o_{l,k}[0, \mathring{h}_k]}{\mathring{F}^o_{l,k}[0, 1]} = \frac{\mathring{G}^B_{l,k|k-1}[\mathring{T}_{0,\mathring{h}_k}]}{\mathring{G}^B_{l,k|k-1}[\mathring{T}_{0,1}]} \tag{187}$$

$$= \frac{1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]}{1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}^c_D]} \tag{188}$$

$$= \frac{1 - q^B_{l,k|k-1}}{1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}^c_D]} + \frac{q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}^c_D]}{1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}^c_D]} \frac{s^B_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]}{s^B_{l,k|k-1}[\mathring{p}^c_D]} \tag{189}$$

$$= 1 - q^o_{l,k} + q^o_{l,k} s^o_{l,k}[\mathring{h}_k] \tag{190}$$

where

$$q^o_{l,k} \overset{\text{def.}}{=} \frac{q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}^c_D]}{1 - q^B_{l,k|k-1} + q^B_{l,k|k-1} s^B_{l,k|k-1}[\mathring{p}^c_D]} \tag{191}$$

$$s^o_{l,k}(x_{k-1}) \overset{\text{def.}}{=} \frac{\mathring{p}^c_D(x_{k-1}, l)\ s^B_{l,k|k-1}(x_{k-1})}{\int \mathring{p}^c_D(x_{k-1}, l)\, s^B_{l,k|k-1}(x_{k-1}) dx_{k-1}} \tag{192}$$

and where Eq. (192) follows from

$$\frac{s^B_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]}{s^B_{l,k|k-1}[\mathring{p}^c_D]} \tag{193}$$

$$= \frac{\int \mathring{h}_k(x_{k-1}, l)\, \mathring{p}^c_D(x_{k-1}, l)\, s^B_{l,k|k-1}(x_{k-1}) dx_{k-1}}{\int \mathring{p}^c_D(x_{k-1}, l)\, s^B_{l,k|k-1}(x_{k-1}) dx_{k-1}}$$

$$= \int \mathring{h}_k(x_{k-1}, l) \frac{\mathring{p}^c_D(x_{k-1}, l)\ s^B_{l,k|k-1}(x_{k-1})}{\int \mathring{p}^c_D(x_{k-1}, l)\, s^B_{l,k|k-1}(x_{k-1}) dx_{k-1}} dx_{k-1} \tag{194}$$

$$= \int \mathring{h}_k(x_{k-1}, l)\, s^{o,\alpha_k}_{l,k}(x_{k-1}) dx_{k-1} \tag{195}$$

where

$$s^{o,\alpha_k}_{l,k}(x_{k-1}) = \frac{\mathring{p}^c_D(x_{k-1}, l)\ s^B_{l,k|k-1}(x_{k-1})}{s^B_{l,k|k-1}[\mathring{p}^c_D]}. \tag{196}$$

Case 3b: $q^o_{l,k}$ and $s^o_{l,k}$ for an undetected surviving target l ($\alpha_k(l) = 0$ and $l \in L_{k-1}$). From Eqs. (156,138,84,36),

$$\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k] = \frac{\mathring{F}^o_{l,k}[0,\mathring{h}_k]}{\mathring{F}^o_{l,k}[0,1]} = \frac{\sigma^o_{l,k-1}[\mathring{H}_{\mathring{h}_k,0,Z_{k-1}}]}{\sigma^o_{l,k-1}[\mathring{H}_{1,0,Z_{k-1}}]} \tag{197}$$

$$= \frac{\begin{pmatrix} 1 - q^o_{l,k-1} + q^o_{l,k-1}\int \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S\mathring{h}_k\mathring{p}^c_D}(x_{k-1},l,Z_{k-1}) \\ \cdot s^o_{l,k-1}(x_{k-1})dx_{k-1}\end{pmatrix}}{\begin{pmatrix} 1 - q^o_{l,k-1} + q^o_{l,k-1}\int \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1}) \\ \cdot s^o_{l,k-1}(x_{k-1})dx_{k-1}\end{pmatrix}} \tag{198}$$

$$= \frac{1 - q^o_{l,k-1} + q^o_{l,k-1}\int \ddot{M}_{\mathring{p}^c_S}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}}{1 - q^o_{l,k-1} + q^o_{l,k-1}\int \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}} \tag{199}$$
$$+ \frac{q^o_{l,k-1}\int \ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}}{1 - q^o_{l,k-1} + q^o_{l,k-1}\int \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}}$$

$$= 1 - q^o_{l,k} + q^o_{l,k} s^k_l[\mathring{h}_k] \tag{200}$$

where

$$q^o_{l,k} \overset{\text{def.}}{=} 1 - \frac{\begin{pmatrix} 1 - q^o_{l,k-1} + q^o_{l,k-1} \\ \cdot\int \ddot{M}_{\mathring{p}^c_S}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}\end{pmatrix}}{\begin{pmatrix} 1 - q^o_{l,k-1} + q^o_{l,k-1} \\ \cdot\int \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}\end{pmatrix}} \tag{201}$$

$$= \frac{q^o_{l,k-1}\int \ddot{M}_{\mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1}}{\begin{pmatrix} 1 - q^o_{l,k-1} + q^o_{l,k-1} \\ \cdot\int \ddot{M}_{\mathring{p}^c_S + \mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l\sigma,k-1}(x_{k-1})dx_{k-1}\end{pmatrix}} \tag{202}$$

and where the numerator of Eq. (202) is

$$q^o_{l,k-1}\int \ddot{M}_{\mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_{k-1} \tag{203}$$
$$= q^o_{l,k-1}\int \mathring{p}_S(x_{k-1},l)\,\mathring{p}^c_D(\mathring{x}_k)\,\mathring{f}_{k|k-1}(\mathring{x}_k|x_{k-1},l,Z_{k-1})\, s^o_{l,k-1}(x_{k-1})d\mathring{x}_k dx_{k-1}$$

$$= q^o_{l,k-1}\sum_{l_k\in L_k}\int \mathring{p}_S(x_{k-1},l)\,\mathring{p}^c_D(x_k,l_k)\,\delta_{l_k,l}\, f_{k|k-1}(x_k|x_{k-1},Z_{k-1}) \tag{204}$$
$$\cdot s^o_{l,k-1}(x_{k-1})dx_k dx_{k-1}$$

$$= q^o_{l,k-1}\int \mathring{p}_S(x_{k-1},l)\,\mathring{p}^c_D(x_k,l)\, f_{k|k-1}(x_k|x_{k-1},Z_{k-1})\, s^o_{l,k-1}(x_{k-1})dx_k dx_{k-1}. \tag{205}$$

Similarly, as in Eq. (168) define $\mathring{P}(x_{k-1},l,\mathring{x}_k,l_k) = 1-\mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(x_k,l_k)$. Then the denominator of Eq. (202) is

$$1-q^o_{l,k-1}+q^o_{l,k-1}\int \ddot{M}_{\mathring{p}^c_S+\mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1}) \cdot s^o_{l\sigma,k-1}(x_{k-1})dx_{k-1} \tag{206}$$

$$= 1-q^o_{l,k-1}+q^o_{l,k-1}\int \ddot{M}_{\mathring{P}}(x_{k-1},Z_{k-1})\,s^o_{l\sigma,k-1}(x_{k-1})dx_{k-1} \tag{207}$$

$$= 1-q^o_{l,k-1}+q^o_{l,k-1}\int \mathring{P}(x_{k-1},l,l_k,\mathring{x}_k) \cdot \mathring{f}_{k|k-1}(\mathring{x}_k|x_{k-1},l,Z_{k-1})\ s^o_{l,k-1}(x_{k-1})d\mathring{x}_kdx_{k-1} \tag{208}$$

$$= 1-q^o_{l,k-1}+q^o_{l,k-1}\sum_{l_k}\int \mathring{P}(x_{k-1},l,x_k,l_k) \cdot\delta_{l_k,l}\,f_{k|k-1}(x_k|x_{k-1},Z_{k-1})\ s^o_{l,k-1}(x_{k-1})dx_kdx_{k-1} \tag{209}$$

$$= 1-q^o_{l,k-1}+q^o_{l,k-1}\int \mathring{P}_l(x_{k-1},x_k) \cdot f_{k|k-1}(x_k|x_{k-1},Z_{k-1})\ s^o_{l,k-1}(x_{k-1})dx_kdx_{k-1} \tag{210}$$

where as in Eq. (168), $\mathring{P}_l(x_{k-1},\mathring{x}_k) = 1-\mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(x_k,l)$. So we get

$$q^o_{l,k} = \frac{\left(\begin{array}{c} q^o_{l,k-1}\int \mathring{p}_S(x_{k-1},l)\,\mathring{p}^c_D(x_k,l) \\ \cdot f_{k|k-1}(x_k|x_{k-1},Z_{k-1})\,s^o_{l,k-1}(x_{k-1})dx_kdx_{k-1} \end{array}\right)}{\left(\begin{array}{c} 1-q^o_{l,k-1}+q^o_{l,k-1}\int \mathring{P}_l(x_{k-1},\mathring{x}_k) \\ \cdot f_{k|k-1}(x_k|x_{k-1},Z_{k-1})\ s^o_{l,k-1}(x_{k-1})dx_kdx_{k-1} \end{array}\right)}. \tag{211}$$

Now from Eqs. (199,200) we must have, for some $s^o_{l,k}[\mathring{h}_k]$ (the formula for which is to be derived),

$$\frac{q^o_{l,k-1}\int \ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\,s^o_{l,k-1}(x_{k-1})dx_{k-1}}{1-q^o_{l,k-1}+q^o_{l,k-1}\int \ddot{M}_{\mathring{p}^c_S+\mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\,s^o_{l,k-1}(x_{k-1})dx_{k-1}} = q^o_{l,k}s^o_{l,k}[\mathring{h}_k]. \tag{212}$$

From this follows

$$s^o_{l,k}[\mathring{h}_k] = \frac{\int \ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\,s^o_{l,k-1}(x_{k-1})dx_{k-1}}{\int \ddot{M}_{\mathring{p}_S\mathring{p}^c_D}(x_{k-1},l,Z_{k-1})\,s^o_{l,k-1}(x_{k-1})dx_{k-1}} \tag{213}$$

$$= \frac{\left(\begin{array}{c} \int \mathring{p}_S(x_{k-1},l)\,\mathring{h}_k(\mathring{x}_k)\,\mathring{p}^c_D(\mathring{x}_k) \\ \cdot\mathring{f}_{k|k-1}(\mathring{x}_k,y_k|x_{k-1},l,Z_{k-1})\,s^o_{l,k-1}(x_{k-1})d\mathring{x}_kdy_kdx_{k-1} \end{array}\right)}{\left(\begin{array}{c} \int \mathring{p}_S(x_{k-1},l)\,\mathring{p}^c_D(\mathring{x}_k) \\ \cdot\mathring{f}_{k|k-1}(\mathring{x}_k,y_k|x_{k-1},l,Z_{k-1})\,s^o_{l,k-1}(x_{k-1})d\mathring{x}_kdy_kdx_{k-1} \end{array}\right)} \tag{214}$$

$$= \frac{\begin{pmatrix} \sum_{l_k \in L_k} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{h}_k(x_k, l_k)\, \mathring{p}_D^c(x_k, l_k) \\ \cdot \mathring{f}_{k|k-1}(x_k, l_k, y_k | x_{k-1}, l, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dy_k dx_{k-1} \end{pmatrix}}{\begin{pmatrix} \sum_{l_k \in L_k} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D^c(x_k, l_k) \\ \cdot \mathring{f}_{k|k-1}(x_k, l_k, y_k | x_{k-1}, l, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dy_k dx_{k-1} \end{pmatrix}} \tag{215}$$

$$= \frac{\begin{pmatrix} \sum_{l_k \in L_k} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{h}_k(x_k, l_k)\, \mathring{p}_D^c(x_k, l_k) \\ \cdot \delta_{l_k, l}\, f_{k|k-1}(x_k, y_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dy_k dx_{k-1} \end{pmatrix}}{\begin{pmatrix} \sum_{l_k \in L_k} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D^c(x_k, l_k) \\ \cdot \delta_{l_k, l}\, f_{k|k-1}(x_k, y_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dy_k dx_{k-1} \end{pmatrix}} \tag{216}$$

$$= \frac{\begin{pmatrix} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{h}_k(x_k, l)\, \mathring{p}_D^c(x_k, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{pmatrix}}{\begin{pmatrix} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D^c(x_k, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{pmatrix}} \tag{217}$$

$$= \int \mathring{h}_k(x_k, l) \tag{218}$$

$$\cdot \frac{\begin{pmatrix} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D^c(x_k, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_{k-1} \end{pmatrix}}{\begin{pmatrix} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D^c(x_k, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{pmatrix}} dx_k$$

$$= \int \mathring{h}_k(x_k, l)\, s^o_{l,k}(x_k) dx_k \tag{219}$$

where

$$s^o_{l,k}(x_k) = \frac{\begin{pmatrix} \mathring{p}_D^c(x_k, l) \int \mathring{p}_S(x_{k-1}, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_{k-1} \end{pmatrix}}{\begin{pmatrix} \int \mathring{p}_S(x_{k-1}, l)\, \mathring{p}_D^c(x_k, l) \\ \cdot f_{k|k-1}(x_k | x_{k-1}, Z_{k-1})\, s^o_{l,k-1}(x_{k-1}) dx_k dx_{k-1} \end{pmatrix}}. \tag{220}$$

Case 3c: $q^o_{l,k}$ and $s^o_{l,k}$ for a detected newly-appearing target l ($\alpha_k(l) > 0$ and $l \in L^B_{k|k-1}$). From Eqs. (156,138,94,95),

$$\sigma^{o,\alpha_k}_{l,k}[\mathring{h}_k] = \frac{\frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0, \mathring{h}_k]}{\frac{\delta \mathring{F}^o_{l,k}}{\delta z_{\alpha_k(l)}}[0, 1]} = \frac{q^B_{l,k-1} \frac{\delta}{\delta z_{\alpha_k(l)}} s^B_{l,k|k-1}\left[\mathring{T}_{g_k, \mathring{h}_k}\right]}{q^B_{l,k-1} \frac{\delta}{\delta z_{\alpha_k(l)}} s^B_{l,k|k-1}\left[\mathring{T}_{g_k, 1}\right]} \tag{221}$$

$$= \frac{q^B_{l,k-1} s^B_{l,k|k-1}\left[\mathring{h}_k \mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}\right]}{q^B_{l,k-1} s^B_{l,k|k-1}\left[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}\right]} \tag{222}$$

$$= \int \mathring{h}_k(x_k, l) \frac{\mathring{p}_D(x_k, l)\, \mathring{L}_{z_{\alpha_k(l)}}(x_k, l)\, s^B_{l,k|k-1}(x_k)}{s^B_{l,k|k-1}\left[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}\right]} dx_k \tag{223}$$

$$= s^{o,\alpha_k}_{l,k}[\mathring{h}_k] \tag{224}$$

where

$$s_{l,k}^{o,\alpha_k}(x_k) = \frac{\mathring{p}_D(x_k,l)\,\mathring{L}_{z_{\alpha_k(l)}}(x_k,l)\,s_{l,k|k-1}^B(x_k)}{s_{l,k|k-1}^B[\mathring{p}_D\mathring{L}_{z_{\alpha_k(l)}}]}. \tag{225}$$

Case 3d: $q_{l,k}^o$ and $s_{l,k}^o$ for a detected surviving target l ($\alpha_k(l) > 0$ and $l \in L_{k-1}$). From (156,138,84,36)

$$\sigma_{l,k}^{o,\alpha_k}[\mathring{h}_k] = \frac{\frac{\delta\mathring{F}_{l,k}^o}{\delta z_{\alpha_k(l)}}[0,\mathring{h}_k]}{\frac{\delta\mathring{F}_{l,k}^o}{\delta z_{\alpha_k(l)}}[0,1]} = \frac{q_{l,k-1}^o\frac{\delta}{\delta z_{\alpha_k(l)}}s_{l,k-1}^o\left[\mathring{H}_{\mathring{h}_k,g_k,Z_{k-1}}\right]}{q_{l,k-1}^o\frac{\delta}{\delta z_{\alpha_k(l)}}s_{l,k-1}^o\left[\mathring{H}_{1,\delta_{g_k},Z_{k-1}}\right]} \tag{226}$$

$$= \frac{\int\ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D\delta_{z_{\alpha_k(l)}}}(\mathring{x}_{k-1},x_{k-1},l,Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_{k-1}}{\int\ddot{M}_{\mathring{p}_S\mathring{p}_D\delta_{z_{\alpha_k(l)}}}(\mathring{x}_{k-1},x_{k-1},l,Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_{k-1}} \tag{227}$$

$$= \frac{\left(\begin{array}{c}\int\mathring{p}_S(x_{k-1},l)\,\mathring{h}_k(\mathring{x}_k)\,\mathring{p}_D(\mathring{x}_k)\,\delta_{z_{\alpha_k(l)}}(y_k)\\ \cdot\mathring{f}_{k|k-1}(\mathring{x}_k,y_k|x_{k-1},l,Z_{k-1})\,s_{l,k-1}^o(x_{k-1})d\mathring{x}_kdy_kdx_{k-1}\end{array}\right)}{\left(\begin{array}{c}\int\mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(\mathring{x}_k)\,\delta_{z_{\alpha_k(l)}}(y_k)\\ \cdot\mathring{f}_{k|k-1}(\mathring{x}_k,y_k|x_{k-1},l,Z_{k-1})\,s_{l,k-1}^o(x_{k-1})d\mathring{x}_kdy_kdx_{k-1}\end{array}\right)} \tag{228}$$

$$= \frac{\left(\begin{array}{c}\sum_{l_k\in L_k}\int\mathring{p}_S(x_{k-1},l)\,\mathring{h}_k(x_k,l_k)\,\mathring{p}_D(x_k,l_k)\\ \cdot\delta_{l_k,l}\,f_{k|k-1}(x_k,l_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_kdx_{k-1}\end{array}\right)}{\left(\begin{array}{c}\sum_{l_k\in L_k}\int\mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(x_k,l_k)\\ \cdot f_{k|k-1}(x_k,l_k,z_{\alpha_k(l)}|x_{k-1},l,Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_kdx_{k-1}\end{array}\right)} \tag{229}$$

$$= \frac{\left(\begin{array}{c}\int\mathring{p}_S(x_{k-1},l)\,\mathring{h}_k(x_k,l)\,\mathring{p}_D(x_k,l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_kdx_{k-1}\end{array}\right)}{\left(\begin{array}{c}\int\mathring{p}_S(x_{k-1},l)\,\mathring{h}_k(x_k,l)\,\mathring{p}_D(x_k,l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_kdx_{k-1}\end{array}\right)} \tag{230}$$

$$= \int\mathring{h}_k(x_k,l)\frac{\left(\begin{array}{c}\int\mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(x_k,l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_{k-1}\end{array}\right)}{\left(\begin{array}{c}\int\mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(x_k,l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_kdx_{k-1}\end{array}\right)}dx_k \tag{231}$$

$$= \int\mathring{h}_k(x_k,l)\,s_{l,k}^{o,\alpha_k}(x_k)dx_k \tag{232}$$

where

$$s_{l,k}^{o,\alpha_k}(x_k) = \frac{\left(\begin{array}{c}\mathring{p}_D(x_k,l)\int\mathring{p}_S(x_{k-1},l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_{k-1}\end{array}\right)}{\left(\begin{array}{c}\int\mathring{p}_S(x_{k-1},l)\,\mathring{p}_D(x_k,l)\\ \cdot f_{k|k-1}(x_k,z_{\alpha_k(l)}|x_{k-1},Z_{k-1})\,s_{l,k-1}^o(x_{k-1})dx_kdx_{k-1}\end{array}\right)}. \tag{233}$$

6 Mathematical Derivations

6.1 Proof that mGLMB = GLMB

It is trivially true that a GLMB PGFL is an mGLMB PGFL. Conversely, let us be given an mGLMB PGFL:

$$\mathring{G}[\mathring{h}] = \sum_{L \in \mathbb{L}} \sum_{o \in O} \omega^o(L) \prod_{l \in L} \left(1 - q_l^o + q_l^o s_l^o[\mathring{h}]\right). \tag{234}$$

If $L, J \subseteq \mathbb{L}$ are finite define

$$\delta_J^L = \begin{cases} 1 & \text{if} \quad J \subseteq L \\ 0 & \text{if} \quad \text{otherwise} \end{cases}. \tag{235}$$

Then from the generalized binomial theorem for finite sets L,

$$\prod_{l \in L} (a_l + b_l) = \sum_{J \subseteq I} \left(\prod_{l \in J} a_i\right) \left(\prod_{l \in I-J} b_l\right), \tag{236}$$

we get

$$\begin{aligned}
& \mathring{G}[\mathring{h}] && (237)\\
= \; & \sum_{L \subseteq \mathbb{L}} \sum_{o \in O} \omega^o(L) \prod_{l \in L} \left(1 - q_l^o + q_l^o s_l^o[\mathring{h}]\right) \\
= \; & \sum_{L \in \mathbb{L}} \sum_{o \in O} \omega^o(L) \sum_{J \subseteq L} \left(\prod_{l \in L-J} (1 - q_l^o)\right) \left(\prod_{l \in J} q_l^o\right) \left(\prod_{l \in J} s_l^o[\mathring{h}]\right) && (238)\\
= \; & \sum_{J \subseteq \mathbb{L}} \sum_{o \in O} \sum_{L \subseteq \mathbb{L}} \delta_J^L \, \omega^o(L) \left(\prod_{l \in L-J} (1 - q_l^o)\right) \left(\prod_{l \in J} q_l^o\right) \prod_{l \in J} s_l^o[\mathring{h}] && (239)\\
= \; & \sum_{J \subseteq \mathbb{L}} \sum_{o \in O} \tilde{\omega}^o(J) \prod_{l \in J} s_l^o[\mathring{h}] && (240)
\end{aligned}$$

where

$$\tilde{\omega}^o(J) = \sum_{L \subseteq \mathbb{L}} \delta_J^L \, \omega^o(L) \left(\prod_{l \in L-J} (1 - q_l^o)\right) \left(\prod_{l \in J} q_l^o\right). \tag{241}$$

Note that $\mathring{G}[\mathring{h}]$ is GLMB since

$$\begin{aligned}
\sum_{J \subseteq \mathbb{L}} \sum_{o \in O} \tilde{\omega}^o(J) &= \sum_{o \in O} \sum_{J \subseteq \mathbb{L}} \sum_{L \subseteq \mathbb{L}} \delta_J^L \, \omega^o(L) \left(\prod_{l \in L-J} (1 - q_l^o)\right) \left(\prod_{l \in J} q_l^o\right) && (242)\\
&= \sum_{o \in O} \sum_{L \subseteq \mathbb{L}} \sum_{J \subseteq L} \omega^o(L) \left(\prod_{l \in L-J} (1 - q_l^o)\right) \left(\prod_{l \in J} q_l^o\right) && (243)
\end{aligned}$$

$$= \sum_{o\in O}\sum_{L\subseteq \mathbb{L}} \omega^o(L) \sum_{J\subseteq L} \left(\prod_{l\in L-J}(1-q_l^o)\right)\left(\prod_{l\in J} q_l^o\right) \quad (244)$$

$$= \sum_{o\in O}\sum_{L\subseteq \mathbb{L}} \omega^o(L) \prod_{l\in L}(1-q_l^o+q_l^o) \quad (245)$$

$$= \sum_{o\in O}\sum_{L\subseteq \mathbb{L}} \omega^o(L) = 1. \quad (246)$$

6.2 Proofs of Eqs. (41-45)

This section provides a more polished version of the PGFL-based derivation of the measurement-update step of the GLMB filter, which originally appeared in [6, Sect. III-C]. The results are summarized in Section 2.10.

The derivation consists of five steps: set-theoretic (Section 6.2.1), product-rule (Section 6.2.2); first LMA (Section 6.2.3); second LMA (Section 6.2.4); and GLMB (Section 6.2.5).

6.2.1 Set-Theoretic Step

From [13, Eqs. (14.280-14.281)], [7, Eqs. (5.58-5.59)] the posterior PGFL is

$$\mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] = \frac{\frac{\delta F_k}{\delta Z_k}[0,\mathring{h}_k]}{\frac{\delta F_k}{\delta Z_k}[0,1]} \quad (247)$$

where

$$F_k[g_k,\mathring{h}_k] = \int \mathring{h}_k^{\mathring{X}}\, G_k[g_k|\mathring{X}]\, \mathring{f}_{k|k-1}(\mathring{X}|Z_{1:k-1})\delta\mathring{X}. \quad (248)$$

From the PGFL form of the standard multitarget measurement model, Eq. (39), we have

$$G_k[g_k|\mathring{X}] = e^{\kappa_k[g_k-1]}\,(\mathring{p}_D^c + \mathring{p}_D\mathring{L}_{g_k})^{\mathring{X}}. \quad (249)$$

From this we get

$$\mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] \propto \frac{\delta F_k}{\delta Z_k}[0,\mathring{h}_k] \quad (250)$$

$$= \int \mathring{h}_k^{\mathring{X}} \left[\frac{\delta \mathring{G}_k}{\delta Z_k}[g_k|\mathring{X}_k]\right]_{g_k=0} \mathring{f}_{k|k-1}(\mathring{X}_k|Z_{1:k-1})\delta\mathring{X}_k$$

$$= \int \left[\frac{\delta}{\delta Z_k}\left(e^{\kappa_k[g_k-1]}\,(\mathring{h}^{\mathring{X}}(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k}))^{\mathring{X}}\right)\right]_{g_k=0} \quad (251)$$

$$\cdot \mathring{f}_{k|k-1}(\mathring{X}_k|Z_{1:k-1})\delta\mathring{X}_k$$

$$= \int \left[\frac{\delta}{\delta Z_k}\left(e^{\kappa_k[g_k-1]}\,\mathring{G}_{k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})|Z_{1:k-1}]\right)\right]_{g_k=0} \quad (252)$$

where Eq. (252) follows from the definition of a PGFL, Eq. (16).

Now note that

$$
\begin{aligned}
& e^{\kappa_k[g_k-1]}\,\mathring{G}_{k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})|Z_{1:k-1}] && (253)\\
= \;& e^{\kappa_k[g_k-1]}\sum_{o\in O_{k-1}}\sum_{L\subseteq L_{k|k-1}}\omega^o_{k|k-1}(L)\prod_{l\in L}s^o_{l,k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})]
\end{aligned}
$$

and so

$$
\begin{aligned}
& \mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] \\
\propto \;& \frac{\delta}{\delta Z_k}\begin{pmatrix} e^{\kappa_k[g_k-1]}\sum_{o\in O_{k-1}}\sum_{L\subseteq L_{k|k-1}}\omega^o_{k|k-1}(L) \\ \cdot\prod_{l\in L}s^o_{l,k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})]\end{pmatrix} && (254)\\
= \;& \sum_{o\in O_{k-1}}\sum_{L\subseteq L_{k|k-1}}\omega^o_{k|k-1}(L) && (255)\\
& \cdot\frac{\delta}{\delta Z_k}\begin{pmatrix} e^{\kappa_k[g_k-1]} \\ \cdot\prod_{l\in L}s^o_{l,k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})]\end{pmatrix}.
\end{aligned}
$$

6.2.2 Product-Rule Step

For a fixed L, let $L=\{l_1,...,n\}$ with $|L|=n$. From the general product rule for functional derivatives [13, Eq. (11.274)],

$$
\begin{aligned}
& \frac{\delta}{\delta Z_k}\left(e^{\kappa_k[g_k-1]}\prod_{l\in L}s^o_{l,k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})]\right) && (256)\\
= \;& \sum_{W_0\uplus W_1\uplus...\uplus W_n=Z_k}\frac{\delta}{\delta W_0}e^{\kappa_k[g_k-1]} \\
& \cdot\prod_{i=1}^n\frac{\delta}{\delta W_i}s^o_{l_i,k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})] \\
= \;& e^{\kappa_k[g_k-1]}\sum_{W_0\uplus W_1\uplus...\uplus W_n=Z_k}\kappa_k^{W_0} && (257)\\
& \cdot\prod_{i=1}^n\frac{\delta}{\delta W_i}s^o_{l_i,k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})]
\end{aligned}
$$

where the summation is taken over all possibly empty and mutually disjoint subsets $W_0,W_1,...,W_n\subseteq Z_k$ such that $W_0\uplus W_1\uplus...\uplus W_n=Z_k$. Thus:

$$
\begin{aligned}
= \;& e^{\kappa_k[g_k-1]}\sum_{W_0\uplus W_1\uplus...\uplus W_n=Z_k}\kappa_k^{W_0}\left(\prod_{i=1}^n s^o_{l_i,k|k-1}[\mathring{p}_D^c]\right) && (258)\\
& \cdot\left(\prod_{i=1}^n\frac{\delta}{\delta W_i}\frac{s^o_{l_i,k|k-1}[\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})])}{s^o_{l,k|k-1}[\mathring{p}_D^c]}\right)
\end{aligned}
$$

$$= \kappa_k^{Z_k} e^{\kappa_k[g_k-1]} \left(\prod_{i=1}^{n} s^o_{l_i,k|k-1}[\mathring{h}_k \mathring{p}^c_D] \right) \tag{259}$$

$$\cdot \sum_{W_1,...,W_n \subseteq Z_k} \prod_{i=1}^{n} \frac{1}{\kappa_k^{W_i}} \frac{\delta}{\delta W_i} \frac{s^o_{l_i,k|k-1}[\mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D \mathring{L}_{g_k})]}{s^o_{l_i,k|k-1}[\mathring{h}_k \mathring{p}^c_D]}$$

where for any $l \in L_{k|k-1}$,

$$\left[\frac{\delta}{\delta W} \frac{s^o_{l,k|k-1}[\mathring{h}_k(\mathring{p}^c_D + \mathring{p}_D \mathring{L}_{g_k})])}{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]} \right]_{g_k=0} = \begin{cases} 1 & \text{if} \quad W = \emptyset \\ \frac{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D \mathring{L}_z]}{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]} & \text{if} \quad W = \{z\} \\ 0 & \text{if} \quad |W| \geq 2 \end{cases} . \tag{260}$$

6.2.3 First LMA Step

Recall the procedure used to construct the multitarget likelihood function for the standard measurement model (originally presented in [13, pp. 742-745] and later summarized in [10, p. 6, Sec. 9]). This procedure will be emulated with the aim of transforming Eq. (259) into a conceptually more useful form.

Given each W_i, define a label-to-measurement association (LMA) $\alpha_{L,k} : L \to \{0, 1, ..., |Z_k|\}$ by $\alpha_{L,k}(l_i) = 0$ if $W_i = \emptyset$ (target undetected) and $\alpha_{L,k}(l_i) = j$ if $W_i = \{z_j\}$ (target detected). Conversely, given an LMA $\alpha_{L,k}$, if $\alpha_{L,k}(l_i) > 0$ define $W_i = \{z_{\alpha_{L,k}(l_i)}\}$ if $\alpha_{L,k}(l_i) > 0$ and $W_i = \emptyset$ otherwise. It follows that there is a one-to-one correspondence between LMAs on L and lists of $n = |L|$ mutually disjoint subsets $W_1, ..., W_n$ of Z_k such that $|W_i| \leq 1$ for $i = 1, ..., n$.

Thus after setting $g_k = 0$ and using Eqs. (259,260), Eq. (255) can be recast into LMA form:

$$\mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] \tag{261}$$

$$\propto e^{-\kappa_k[1]} \sum_{o \in O_{k-1}} \sum_{Ls \subseteq L_{k|k-1}} \omega^o_{k|k-1}(L)$$

$$\cdot \kappa_k^{Z_k} \left(\prod_{l \in L} s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D] \right)$$

$$\cdot \sum_{\alpha_{L,k}} \prod_{l:\alpha_{L,k}(l)>0} \frac{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D \mathring{L}_{z_{\alpha_{L,k}(l)}}]}{\kappa_k(z_{\alpha_{L,k}(l)})\, s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]}.$$

6.2.4 Second LMA Step

Eq. (261) is problematic because it involves LMAs $\alpha_{L,k} : L \to \{0, 1, ..., |Z_k|\}$ rather than, as should be the case, LMAs $\alpha_k : L_{k|k} \to \{0, 1, ..., |Z_k|\}$.

To remedy this, given $\alpha_{L,k} : L \to \{0, 1, ..., |Z_k|\}$ define [6, Eq. (69)]

$$\alpha_k(l) = \begin{cases} \alpha_{L,k}(l) & \text{if} \quad l \in L,\, \alpha_{L,k}(l) > 0 \\ 0 & \text{if} \quad \text{otherwise} \end{cases} \tag{262}$$

The choice $\alpha = \alpha_k$ is distinguished from all other $\alpha : L_{k|k} \to \{0, 1, ..., |Z_k|\}$ by the fact that $\alpha_k(l) > 0$ if and only if $\alpha_{L,k}(l) > 0$. Specifically, α must satisfy the identity[11]

$$L_\alpha \stackrel{\text{def.}}{=} \{l \in L_{k|k} : \alpha(l) > 0\} = \{l \in L : \alpha_{L,k}(l) > 0\} \stackrel{\text{def.}}{=} \alpha_{L,k}(L) \tag{263}$$

This is equivalent to $L_\alpha = L \cap L_\alpha$, which is equivalent to $L_\alpha \subseteq L$, which is equivalent to

$$\mathbf{1}_L^{L_\alpha} \stackrel{\text{def.}}{=} \prod_{l \in L_\alpha} \mathbf{1}_L(l) = 1, \tag{264}$$

which, finally, is equivalent to the factor [7, Eq. (15.185)]

$$\lambda_k^\alpha(L) \stackrel{\text{def.}}{=} \prod_{l \in L_k - L} \delta_{\alpha(l),0} = 1 \tag{265}$$

in the original GLMB filter's measurement update step—i.e., $\lambda_k^\alpha(L) = \mathbf{1}_L^{L_\alpha}$.

Let $\mathcal{A}_k$ denote the set of all LMAs $\alpha_k : L_{k|k} \to \{0, 1, ..., |Z_k|\}$. Then

$$\begin{aligned} & \mathring{G}_{k|k}[\mathring{h}_k|Z_{1:k}] \\ \propto \quad & e^{-\kappa_k[1]} \sum_{o \in O_{k-1}} \sum_{L \subseteq L_{k|k-1}} \omega^o_{k|k-1}(L) \\ & \cdot \kappa_k^{Z_k} \left(\prod_{l \in L} s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D^c] \right) \\ & \cdot \sum_{\alpha_k \in \mathcal{A}_k} \mathbf{1}^L_{L_{a_k}} \prod_{l \in L : \alpha_k(l) > 0} \frac{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}]}{\kappa_k(z_{\alpha_k(l)}) \, s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D^c]} \end{aligned} \tag{266}$$

$$\begin{aligned} = \quad & e^{-\kappa_k[1]} \sum_{o \in O_{k-1}} \sum_{\alpha_k \in \mathcal{A}_k} \sum_{L \subseteq L_{k|k-1}} \omega^o_{k|k-1}(L) \; \mathbf{1}^L_{L_{a_k}} \\ & \cdot \left(\prod_{l \in L : \alpha_k(l) > 0} \kappa_k(z_{\alpha_k(l)}) \right) \left(\prod_{l \in L : \alpha_k(l) = 0} s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D^c] \right) \\ & \cdot \left(\prod_{l \in L : \alpha_k(l) > 0} s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}] \right) \end{aligned} \tag{267}$$

[11] *Errata*: Eqs. (70,71) in [6] have typos. The correct definition is in the line immediately following Eq. (69).

$$= \quad e^{-\kappa_k[1]} \sum_{o \in O_{k-1}} \sum_{\alpha_k \in \mathcal{A}_k} \sum_{L \subseteq L_{k|k-1}} \omega^o_{k|k-1}(L)\ \mathbf{1}^L_{L_{a_k}} \tag{268}$$

$$\cdot \left(\prod_{l \in L : \alpha_k(l) > 0} \kappa_k(z_{\alpha_k(l)}) \right) \left(\prod_{l \in L : \alpha_k(l) = 0} s^o_{l,k|k-1}[\mathring{p}^c_D] \right)$$

$$\cdot \left(\prod_{l \in L : \alpha_k(l) > 0} s^o_{l,k|k-1}[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}] \right)$$

$$\cdot \left(\prod_{l \in L : \alpha_k(l) = 0} \frac{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]}{s^o_{l,k|k-1}[\mathring{p}^c_D]} \right) \left(\prod_{l \in L : \alpha_k(l) > 0} \frac{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}]}{s^o_{l,k|k-1}[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}]} \right)$$

$$= e^{-\kappa_k[1]} \sum_{o \in O_{k-1}} \sum_{\alpha_k \in \mathcal{A}_k} \sum_{L \subseteq L_{k|k-1}} \omega^o_{k|k-1}(L)\ C^{o,\alpha_k}_{L,k} \prod_{l \in L} s^{o,\alpha_k}_{l,k|k}[\mathring{h}_k] \tag{269}$$

where

$$C^{o,\alpha_k}_{L,k} \quad = \quad \mathbf{1}^L_{L_{a_k}} \left(\prod_{l \in L : \alpha_k(l) > 0} \kappa_k(z_{\alpha_k(l)}) \right) \left(\prod_{l \in L : \alpha_k(l) = 0} s^o_{l,k|k-1}[\mathring{p}^c_D] \right) \tag{270}$$

$$\cdot \left(\prod_{l \in L : \alpha_k(l) > 0} s^o_{l,k|k-1}[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}] \right)$$

and

$$s^{o,\alpha_k}_{l,k|k}[\mathring{h}_k] = \begin{cases} \frac{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}^c_D]}{s^o_{l,k|k-1}[\mathring{p}^c_D]} & \text{if} \quad \alpha_k(l) = 0 \\ \frac{s^o_{l,k|k-1}[\mathring{h}_k \mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}]}{s^o_{l,k|k-1}[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}]} & \text{if} \quad \alpha_k(l) > 0 \end{cases} . \tag{271}$$

From Eq. (271) and Eq. (8) it is easily shown that

$$s^{o,\alpha_k}_{l,k|k}(x_k) = \begin{cases} \frac{\mathring{p}^c_D(x_k,l)\, s^o_{l,k|k-1}(x_k)}{s^o_{l,k|k-1}[\mathring{p}^c_D]} & \text{if} \quad \alpha_k(l) = 0 \\ \frac{\mathring{p}_D(x_k,l)\, \mathring{L}_{z_{\alpha_k(l)}}(x_k,l)\, s^o_{l,k|k-1}(x_k)}{s^o_{l,k|k-1}[\mathring{p}_D \mathring{L}_{z_{\alpha_k(l)}}]} & \text{if} \quad \alpha_k(l) > 0 \end{cases} . \tag{272}$$

6.2.5 GLMB Step

From Eq. (269) the posterior PGFL is

$$\mathring{G}_{k|k}[\mathring{h}_k | Z_{1:k}] \tag{273}$$

$$= \quad \frac{\sum_{o \in O_{k-1}} \sum_{L \subseteq L_k} \omega^o_{k|k-1}(L) \sum_{\alpha_k \in \mathcal{A}_k} C^{o,\alpha_k}_{L,k} \prod_{l \in L_k} s^{o,\alpha_k}_{l,k|k}[\mathring{h}_k]}{\sum_{o \in O_{k-1}} \sum_{L \subseteq L_k} \omega^o_{k|k-1}(L) \sum_{\alpha_k \in \mathcal{A}_k} C^{o,\alpha_k}_{L,k}}$$

$$= \sum_{o\in O_{k-1}} \sum_{L\subseteq L_k} \sum_{\alpha_k\in\mathcal{A}_k} \frac{\omega^o_{k|k-1}(L)\, C^{o,\alpha_k}_{L,k}}{\sum_{o\in O_{k-1}} \sum_{L\subseteq L_k} \sum_{\alpha_k\in\mathcal{A}_k} \omega^o_{k|k-1}(L)\, C^{o,\alpha_k}_{L,k}} \cdot \prod_{l\in L} s^{o,\alpha_k}_{l,k|k}[\mathring{h}_k] \tag{274}$$

$$= \sum_{L\subseteq L_k} \sum_{(o,\alpha_k)\in O_k} \omega^{o,\alpha_k}_k(L) \prod_{l\in L} s^{o,\alpha_k}_{l,k|k}[\mathring{h}_k] \tag{275}$$

where $O_k = O_{k-1}\times\mathcal{A}_k$ and

$$\omega^{o,\alpha_k}_k(L) = \frac{\omega^o_{k|k-1}(L)\, C^{o,\alpha_k}_{L,k}}{\sum_{(o,\alpha_k)\in O_k} \sum_{L\subseteq L_k} \omega^o_{k|k-1}(L)\, C^{o,\alpha_k}_{L,k}}. \tag{276}$$

6.3 Proofs of Eqs. (49,50)

These are as follows:

$$\mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1},y_{k-1}) = \int \mathring{f}_{k|k-1}(x_k,l_k,y_k|x_{k-1},l_{k-1},y_{k-1})dy_k \tag{277}$$

$$= \int \delta_{l_k,l_{k-1}}\, f_{k|k-1}(x_k,y_k|x_{k-1},y_{k-1})dy_k \tag{278}$$

$$= \delta_{l_k,l_{k-1}}\, f_{k|k-1}(x_k|x_{k-1},y_{k-1}). \tag{279}$$

$$f_{k|k-1}(y_k|\mathring{x}_{k-1},y_{k-1}) \tag{280}$$

$$= \sum_{l\in L_k} \int \mathring{f}_{k|k-1}(x_k,l_k,y_k|x_{k-1},l_{k-1},y_{k-1})dx_k$$

$$= \sum_{l\in L_k} \int \delta_{l_k,l_{k-1}}\, f_{k|k-1}(x_k,y_k|x_{k-1},y_{k-1})dx_k \tag{281}$$

$$= f_{k|k-1}(y_k|x_{k-1},y_{k-1}). \tag{282}$$

6.4 Proof of Eq. (81)

This result originally appeared in [15, Eq. (56)]. The following is a cleaner derivation. We are to show that

$$\ddot{M}_{\mathring{p}^c_S+\mathring{p}_S\mathring{h}_k(\mathring{p}^c_D+\mathring{p}_D g_k)}(\mathring{x}_{k-1},y_{k-1}) = \mathring{M}_{\mathring{p}^c_S+\mathring{p}_S\mathring{h}_k(\mathring{p}^c_D+\mathring{p}_D \mathring{L}_{g_k})}(\mathring{x}_{k-1}). \tag{283}$$

Since

$$\ddot{M}_{\mathring{p}^c_S+\mathring{p}_S\mathring{h}_k(\mathring{p}^c_D+\mathring{p}_D g_k)}(\mathring{x}_{k-1},y_{k-1}) = \ddot{M}_{\mathring{p}^c_S}(\mathring{x}_{k-1},y_{k-1}) + \ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}^c_D}(\mathring{x}_{k-1},y_{k-1}) + \ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D g_k}(\mathring{x}_{k-1},y_{k-1}) \tag{284}$$

and

$$\mathring{M}_{\mathring{p}_S^c+\mathring{p}_S\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})}(\mathring{x}_{k-1}) = \mathring{M}_{\mathring{p}_S^c}(\mathring{x}_{k-1}) + \mathring{M}_{\mathring{p}_S\mathring{h}_k(\mathring{p}_D^c+\mathring{p}_D\mathring{L}_{g_k})}(\mathring{x}_{k-1}) \tag{285}$$
$$+\mathring{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D^c}(\mathring{x}_{k-1}) + \mathring{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D\mathring{L}_{g_k}}(\mathring{x}_{k-1})$$

it is enough to show that

$$\ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D g_k}(\mathring{x}_{k-1}, y_{k-1}) = \mathring{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D\mathring{L}_{g_k}}(\mathring{x}_{k-1}). \tag{286}$$

From Eq. (79) we have, assuming that the PMM is HMM,

$$\ddot{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D g_k}(\mathring{x}_{k-1}, y_{k-1}) = \int \mathring{p}_S(\mathring{x}_{k-1})\, \mathring{h}_k(\mathring{x}_k)\, \mathring{p}_D(\mathring{x}_k)\, g_k(y_k) \tag{287}$$
$$\cdot \mathring{f}_{k|k-1}(\mathring{x}_k, y_k|\mathring{x}_{k-1}, y_{k-1}) d\mathring{x}_k dy_k$$
$$= \int \mathring{p}_S(\mathring{x}_{k-1})\, \mathring{h}_k(\mathring{x}_k)\, \mathring{p}_D(\mathring{x}_k)\, g_k(y_k) \tag{288}$$
$$\cdot f_k(y_k|\mathring{x}_k)\, \mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1}) d\mathring{x}_k dy_k$$

$$= \mathring{p}_S(\mathring{x}_{k-1}) \int \mathring{h}_k(\mathring{x}_k)\, \mathring{p}_D(\mathring{x}_k)\, \mathring{L}_{g_k}(\mathring{x}_k)\, \mathring{f}_{k|k-1}(\mathring{x}_k|\mathring{x}_{k-1}) d\mathring{x}_k \tag{289}$$
$$= \mathring{p}_S(\mathring{x}_{k-1})\, \mathring{M}_{\mathring{h}_k\mathring{p}_D\mathring{L}_{g_k}}(\mathring{x}_{k-1}) \tag{290}$$
$$= \mathring{M}_{\mathring{p}_S\mathring{h}_k\mathring{p}_D\mathring{L}_{g_k}}(\mathring{x}_{k-1}) \tag{291}$$

where Eq. (289) follows from Eq. (38) and Eq. (290) from Eq. (36).

7 Conclusions

This paper has demonstrated:

1. a more polished probability generating functional (PGFL)-based derivation of the measurement-update step of the generalized labeled multi-Bernoulli (GLMB) filter;
2. a general multitarget pairwise Markov model (MPMM) recursive Bayes filter, formulated as a direct generalization of Pieczynski's original single-target PMM recursive Bayes filter;
3. an MPMM transition function that is consistent with the "standard" labeled random finite set (LRFS) multitarget motion and measurement models; and
4. a PMM generalization of the GLMB filter, the "PMM-mGLMB filter," which is similar in form to the GLMB filter's measurement-update step, and the derivation of which is closedly patterned after the new PGFL derivation of the latter.

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