A Master-Equation Reduction of the SGP4 Propagator from TLE to Topocentric Look Angle

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Abstract

I present a fully expanded derivation of the Simplified General Perturbations 4 (SGP4) propagator as it appears in a working surveillance pipeline that ingests Two-Line Element (TLE) sets, propagates each catalog object, and reduces the result to topocentric azimuth and elevation at a fixed ground station. The classical SGP4 model is well documented in *Spacetrack Report No. 3* and in Vallado's revisitation, but the published descriptions are organized around a Fortran reference implementation, which obscures the algebraic structure of the propagator. I rewrite the entire chain as a sequence of seven composable operators, each with an explicit input and output signature, and I collapse the chain into a single boxed master equation per satellite. I also derive a compact implicit-triplet form, a Lyddane Kepler equation followed by a perturbed radius and a three-axis rotation, that is more tractable for inspection and debugging than the operator chain. The topocentric reduction is treated as a separate composition layer: a sidereal rotation into the Earth-fixed frame, a difference against the ground-station vector, and a projection onto the local south-east-zenith basis. I conclude with a complete variable-accounting table classifying every named scalar in the pipeline as either an external driver (TLE-derived element, evaluation-time clock, or ground-station configuration), a hard-coded physical constant, or a derived intermediate. I validate the implementation against live observations from my own earth station in Austin, Texas, comparing predicted azimuth and elevation against observed pass geometry for a representative set of low-Earth-orbit satellites. The paper is intended as a self-contained reference for engineers implementing SGP4 from scratch and for analysts who need to audit an existing propagator at the equation level.

Keywords: SGP4, orbit propagation, two-line element set, Lyddane Kepler equation, J_2 secular perturbation, topocentric reduction, master equation, satellite tracking, low Earth orbit.

1. Introduction and scope

The Simplified General Perturbations 4 model, originally documented by Hoots and Roehrich in 1980 and later canonicalized by Vallado et al. in their revisitation of *Spacetrack Report No. 3*, remains the standard analytical propagator for low Earth orbit ephemeris distributed in TLE form. The model is widely implemented across the satellite-tracking community, but the published references are organized around the procedural structure of an Air Force Fortran code rather than around the underlying algebra. As a result, a practitioner who wants to audit a working implementation, port the model to a new language, or reason about its behavior under edge cases must reverse-engineer the algebraic structure from the code itself.

In this paper I take the opposite path. I write down the mathematics of SGP4 in the order that it executes, with every coefficient expanded, every conditional branch made explicit, and every shape constant identified by its standard name. I then collapse the entire propagation chain, from the seven mean elements parsed out of a TLE to the inertial position vector at an arbitrary UTC instant, into a single composite operator. I do this for two reasons. First, the operator-chain form is a clean specification of the contract between an SGP4 implementation and any downstream consumer, which is useful for testing and for hand-off between teams. Second, having the master equation in closed form makes it possible to talk about the model as a mathematical object rather than as a piece of code, which is a prerequisite for any analytical work on orbit determination, sensitivity analysis, or differentiable propagation.

The intended audience is an engineer or analyst who already knows Keplerian elements, the distinction between mean and osculating quantities, and the basic structure of the J_2 secular perturbation. I do not motivate the underlying physics; that material is well covered in standard celestial mechanics texts. Where the implementation cuts a corner, for instance by approximating geocentric latitude in place of geodetic latitude in the sub-satellite reduction, I say so plainly and quantify the cost.

2. Constants, conventions, and unit system

The propagator inherits a non-dimensional unit system from the original Air Force code. The unit of length is one Earth equatorial radius $a_E = 6378.137$ km, and the unit of time is chosen so that the Earth gravitational parameter $\mu_\oplus$ takes the dimensionless value

$$k_e = \sqrt{\frac{\mu_\oplus}{a_E^3}} = 0.0743669161 \text{ rad/min}.$$

The geopotential coefficients enter as scaled constants:

$$C_{K2} = \tfrac{1}{2} J_2 a_E^2 = 5.413079 \times 10^{-4},$$

$$C_{K4} = -\tfrac{3}{8} J_4 a_E^4 = 6.2098875 \times 10^{-7},$$

$$\frac{J_3}{J_2} = \frac{-2.53881 \times 10^{-6}}{2 C_{K2}}.$$

The dimensionless Earth radius is unity by convention. The atmospheric drag fit constants are

$$(q_0 - s)_{\text{ref}} = 1.01222928 \text{ ER}, \qquad (q_0 - s)_{\text{ref}}^4 = 1.88027916 \times 10^{-9} \text{ ER}^4.$$

Time inside the propagator is measured in minutes since TLE epoch, and angles are in radians. The dimensional bridge to kilometers happens once at the end of the chain, when the unit-radius position is multiplied by a_E.

The Earth shape model used in the topocentric reduction is WGS84, with flattening $f = 1/298.257223563$ and first-eccentricity-squared $e^2 = f(2 - f)$. SGP4 is internally a WGS72 model, so the inertial position is technically in a WGS72 frame and the topocentric reduction converts it as if it were WGS84. It seems, the position error from that mismatch is sub-kilometer for low Earth orbit, and is dominated by the TLE age error within a day or two of epoch. In my view, the discrepancy is invisible for visualization and pass alerting. For narrow-beam antenna pointing, it should be tightened.

3. From TLE text to mean elements at epoch

A TLE is a fixed-column ASCII record with two 69-column lines of orbital data and an optional name line. The line-2 columns encode the inclination, right ascension of ascending node (RAAN), eccentricity, argument of perigee, mean anomaly, and mean motion. The line-1 columns encode the catalog number, the epoch, and the drag-like ballistic coefficient B^*. Conversion to SI-like units gives

$$i_0 = i_{\text{deg}} \cdot \pi/180, \qquad \Omega_0 = \Omega_{\text{deg}} \cdot \pi/180,$$

$$e_0 = 10^{-7} \cdot E_{\mathrm{TLE}}, \qquad \omega_0 = \omega_{\mathrm{deg}} \cdot \pi/180, \qquad M_0 = M_{\mathrm{deg}} \cdot \pi/180,$$

with the mean motion converted from revolutions per day to radians per minute,

$$n_0 = N_{\mathrm{rev/day}} \cdot \frac{2\pi}{1440}.$$

The drag coefficient is encoded in a packed exponential form. The mantissa M_{B^*} occupies columns 54 to 59 as an implicit-decimal five-digit field, and the signed power of ten E_{B^*} occupies columns 60 to 61. The recovered value is

$$B^* = 10^{-5} \cdot M_{B^*} \cdot 10^{E_{B^*}}.$$

The epoch is encoded as a two-digit year and a fractional day of year. The standard SGP4 pivot expands the year as

$$Y = \begin{cases} 2000 + Y_{\mathrm{TLE}}, & Y_{\mathrm{TLE}} < 57, \\ 1900 + Y_{\mathrm{TLE}}, & \text{otherwise}, \end{cases}$$

and the millisecond epoch is

$$t^{\mathrm{ms}}_{\mathrm{epoch}} = t_{\text{Jan 1 00:00 UTC}}(Y) + (D_{\mathrm{TLE}} - 1) \cdot 86400 \cdot 10^3.$$

Once the seven mean elements $(n_0, e_0, i_0, \Omega_0, \omega_0, M_0, B^*)$ are in hand, the propagator immediately performs the Brouwer reduction that absorbs the long-period J_2 effect baked into the TLE producer's mean motion. The Kozai semi-major axis is

$$a_1 = \left(\frac{k_e}{n_0}\right)^{2/3},$$

with two correction scalars

$$\delta_1 = \frac{3}{2} C_{K2} \cdot \frac{3\cos^2 i_0 - 1}{a_1^2 (1 - e_0^2)^{3/2}},$$

$$\delta_0 = \frac{3}{2} C_{K2} \cdot \frac{3\cos^2 i_0 - 1}{a_0^2 (1 - e_0^2)^{3/2}},$$

an intermediate axis

$$a_0 = a_1 \left[1 - \delta_1 \left(\tfrac{1}{3} + \delta_1 \left(1 + \tfrac{134}{81}\delta_1\right)\right)\right],$$

and the Brouwer mean values

$$n_0'' = \frac{n_0}{1 + \delta_0}, \qquad a_0'' = \frac{a_0}{1 - \delta_0}.$$

The doubled-prime quantities n_0'' and a_0'' are the mean elements that the rest of the propagator uses as the reference orbit.

Two flags are set at this stage. The deep-space switch fires when the orbital period reaches the lunar-solar resonance regime,

$$T_{\text{orbit}} = \frac{2\pi}{n_0''} \geq 225 \text{ min } \Rightarrow \text{ deep space},$$

and an SGP4-only kernel will refuse to propagate any object beyond that threshold because the SDP4 deep-space resonance machinery is not part of the model. The simple flag,

$$\text{simple} = (a_0''(1 - e_0) - 1) \cdot a_E \geq 220 \text{ km},$$

selects whether the higher-order drag terms are needed: when perigee is high enough, the cubic and quartic time corrections to the semi-major axis and mean longitude are dropped.

4. Atmospheric fit and the perigee-dependent reference altitude

SGP4 models drag with a power-law density that depends on a reference altitude s. The default value is $s = 1.01222928$ ER, corresponding to about 78 km above the surface. For low-perigee satellites this default is replaced by an altitude that brackets the perigee. The propagator computes the perigee in km,

$$h_p = (a_0''(1 - e_0) - 1) \cdot a_E,$$

and rewrites s and $(q_0 - s)^4$ conditionally:

$$s = \begin{cases} \frac{20}{a_E} + 1, & h_p \leq 98 \text{ km}, \\ \frac{h_p - 78}{a_E} + 1, & 98 < h_p < 156 \text{ km}, \\ 1.01222928, & h_p \geq 156 \text{ km}, \end{cases}$$

$$(q_0 - s)^4 = \left[\frac{120 - (s-1)a_E}{a_E}\right]^4 \quad \text{when } h_p < 156 \text{ km}.$$

The geometric meaning is that the drag fit is anchored at an altitude that always lies between perigee and 120 km, so the integrals that produce the C_2 and C_4 coefficients downstream stay well-conditioned even for decaying objects.

5. Secular and drag coefficients

The next block builds the dimensionless coefficients C_1 through C_5 that drive the secular evolution of the mean elements. The auxiliary quantities are

$$\beta_0 = \sqrt{1 - e_0^2}, \qquad \theta = \cos i_0, \qquad \theta^2 = \cos^2 i_0,$$

$$\xi = \frac{1}{a_0'' - s}, \qquad \eta = a_0'' e_0 \xi, \qquad \psi^2 = \left|1 - \eta^2\right|.$$

The ξ parameter is the inverse of the radial distance from the drag reference altitude to the mean orbital radius, and η is the dimensionless eccentric scale of the orbit relative to that reference. Then

$$\mathrm{coef} = (q_0 - s)^4 \cdot \xi^4, \qquad \mathrm{coef}_1 = \frac{\mathrm{coef}}{\psi^7},$$

$$C_2 = \mathrm{coef}_1 \cdot n_0'' \left[a_0'' \left(1 + \tfrac{3}{2}\eta^2 + e_0\eta(4+\eta^2)\right) + \tfrac{3}{4} C_{K2}\, \xi \,\frac{3\cos^2 i_0 - 1}{\psi^2} \left(8 + 3\eta^2(8+\eta^2)\right) \right],$$

$$C_1 = B^* C_2.$$

C_1 is the secular drag coefficient for the semi-major axis. With the inclination helpers $X_{1\mathrm{m}5\theta} = 1 - 5\theta^2$ and $X_{1\mathrm{m}\theta^2} = 1 - \theta^2$, the eccentricity drag coefficient is

$$C_4 = 2n_0'' \cdot \mathrm{coef}_1 \cdot a_0''\beta_0^2 \Bigg[\eta(2 + \tfrac{1}{2}\eta^2) + e_0(\tfrac{1}{2} + 2\eta^2)$$

$$- \frac{2C_{K2}\,\xi}{a_0''\psi^2} \Big(-3(3\cos^2 i_0 - 1)\big(1 - 2e_0\eta + \eta^2(\tfrac{3}{2} - \tfrac{1}{2}e_0\eta)\big) + \tfrac{3}{4} X_{1\mathrm{m}\theta^2} \big(2\eta^2 - e_0\eta(1+\eta^2)\big)\cos 2\omega_0 \Big) \Bigg],$$

and the long-period eccentricity coupling is

$$C_5 = 2 \cdot \mathrm{coef}_1 \cdot a_0''\beta_0^2 \left[1 + \tfrac{11}{4}(\eta^2 + e_0\eta) + e_0\eta^3 \right].$$

The factor $\cos 2\omega_0$ inside C_4 is the only place where the initial argument of perigee enters the secular drag, and it appears to me, to capture the dependence of the secular eccentricity decay rate on the orientation of the line of apsides relative to the equator at epoch.

6. Secular rates of the mean elements

With $p = a_0''\beta_0^2$ as the semi-latus rectum, the inverse-square auxiliary is

$$p^{-2} = \frac{1}{(a_0'')^2\beta_0^4},$$

and the geopotential time scales are

$$\tau_1 = 3C_{K2}\, p^{-2}\, n_0'', \qquad \tau_2 = \tau_1 \cdot C_{K2}\, p^{-2}, \qquad \tau_3 = \tfrac{5}{4} C_{K4}\, p^{-4}\, n_0''.$$

The mean-anomaly secular rate is

$$\dot{M} = n_0'' + \tfrac{1}{2}\tau_1\beta_0(3\cos^2 i_0 - 1) + \tfrac{1}{16}\tau_2\beta_0(13 - 78\cos^2 i_0 + 137\cos^4 i_0).$$

The argument-of-perigee rate is

$$\dot{\omega} = -\tfrac{1}{2}\tau_1(1 - 5\cos^2 i_0) + \tfrac{1}{16}\tau_2(7 - 114\cos^2 i_0 + 395\cos^4 i_0) + \tau_3(3 - 36\cos^2 i_0 + 49\cos^4 i_0).$$

The right-ascension-of-ascending-node rate is

$$\dot{\Omega} = -\tau_1 \cos i_0 + \cos i_0 \cdot \left[\tfrac{1}{2}\tau_2(4 - 19\cos^2 i_0) + 2\tau_3(3 - 7\cos^2 i_0)\right].$$

These are the classical Brouwer secular rates, truncated consistently with the rest of SGP4. The bare J_2 node piece $-\tau_1 \cos i_0$ is reused in the long-period node correction below.

The drag terms attached to ω and M are

$$\omega_{\text{cof}} = B^* C_2 \cos\omega_0,$$

$$M_{\text{cof}} = \begin{cases} -\frac{2}{3}\,(q_0 - s)^4\, \xi^4\, B^*\, \dfrac{a_E}{e_0 \eta}, & e_0 > 10^{-4}, \\ 0, & \text{otherwise}, \end{cases}$$

and the long-period node coupling is

$$\Omega_{\text{cof}} = \tfrac{7}{2}\,\beta_0^2\,(-\tau_1 \cos i_0)\, C_1.$$

The semi-major-axis time coefficient gets squared into $T_2 = \frac{3}{2}C_1$. When the orbit is non-simple, the higher-order drag terms are needed:

$$D_2 = 4\, a_0''\, \xi\, C_1^2,$$

$$D_3 = (17a_0'' + s) \cdot \frac{D_2 \xi C_1}{3},$$

$$D_4 = \tfrac{1}{2}\, \frac{D_2 \xi C_1}{3} \cdot a_0''\, \xi\, (221a_0'' + 31s) \cdot C_1,$$

$$T_3 = D_2 + 2C_1^2,$$

$$T_4 = \tfrac{1}{4}(3D_3 + C_1(12D_2 + 10C_1^2)),$$

$$T_5 = \tfrac{1}{5}(3D_4 + 12C_1 D_3 + 6D_2^2 + 15C_1^2(2D_2 + C_1^2)).$$

These coefficients capture the cubic and quartic drag corrections to the mean longitude and the semi-major axis, and they are what differentiate SGP4 from the cheaper SGP that ignores them.

7. Long-period and inclination-coupled corrections

The long-period J_3/J_2 coupling produces three small corrections that sit on top of the secular drift,

$$L_{\text{cof}} = \tfrac{1}{8}\,\frac{J_3}{J_2}\,\sin i_0 \cdot \frac{3 + 5\cos i_0}{1 + \cos i_0},$$

$$A_{y,\mathrm{cof}} = \tfrac{1}{4}\,\frac{J_3}{J_2}\,\sin i_0,$$

$$\delta_{M,0} = (1+\eta\cos M_0)^3.$$

L_{cof} is the longitude correction coefficient, $A_{y,\mathrm{cof}}$ is the eccentricity-vector y-component correction coefficient, and $\delta_{M,0}$ stores the cube that lets SGP4 compute the long-period mean-anomaly increment as a difference of cubes rather than as a direct integral. The denominator $(1+\cos i_0)$ is singular for retrograde polar orbits ($\cos i_0 = -1$), and a numerically careful implementation guards against this by replacing the denominator with 1.5×10^{-12} when it falls below that threshold. A retrograde polar orbit will not propagate cleanly in any case.

8. Time evolution of the mean elements

Let time advance by $\Delta t = t - t_{\mathrm{epoch}}$ minutes. The first-pass mean elements are

$$M_{\mathrm{df}} = M_0 + \dot{M}\,\Delta t, \qquad \omega_{\mathrm{adf}} = \omega_0 + \dot{\omega}\,\Delta t, \qquad \Omega_{\mathrm{ndf}} = \Omega_0 + \dot{\Omega}\,\Delta t,$$

with the second-order node correction

$$\Omega = \Omega_{\mathrm{ndf}} + \Omega_{\mathrm{cof}}\,\Delta t^2.$$

The semi-major axis decay factor, the eccentricity decay, and the mean-longitude polynomial accumulator are

$$T_a = 1 - C_1\Delta t, \qquad T_e = B^*\,C_4\,\Delta t, \qquad T_\ell = T_2\,\Delta t^2.$$

For the non-simple branch the drag-induced libration of the argument of perigee and mean anomaly enters as

$$\Delta\omega = \omega_{\mathrm{cof}}\,\Delta t,$$

$$\Delta M = \begin{cases} M_{\mathrm{cof}}\left[(1+\eta\cos M_{\mathrm{df}})^3 - \delta_{M,0}\right], & e_0 > 10^{-4}, \\ 0, & \text{otherwise}, \end{cases}$$

$$M = M_{\mathrm{df}} + \Delta\omega + \Delta M, \qquad \omega = \omega_{\mathrm{adf}} - (\Delta\omega + \Delta M),$$

and the polynomial accumulators pick up the higher-order drag contributions,

$$T_a \ -= D_2\Delta t^2 + D_3\Delta t^3 + D_4\Delta t^4,$$

$$T_e \ += B^*C_5(\sin M - \sin M_0),$$

$$T_\ell \ += T_3\Delta t^3 + T_4\Delta t^4 + T_5\Delta t^5.$$

The propagated mean elements are then

$$a = a_0'' \cdot T_a^2, \qquad e = e_0 - T_e, \qquad L = M + \omega + \Omega + n_0'' T_\ell,$$

$$\beta = \sqrt{1 - e^2}, \qquad n = \frac{k_e}{a^{3/2}}.$$

If $a < 0.5$ ER or $e \notin [0, 1)$, the satellite is decayed or unphysical and the propagator returns a null result. Any consumer downstream of the propagator is expected to handle this gracefully.

9. The Lyddane long-period correction

SGP4 uses the Lyddane formulation to avoid the small-eccentricity singularity in the classical Kepler equation. Instead of (e, ω, M) it works with the eccentricity vector (a_{xn}, a_{yn}) and the mean longitude L:

$$a_{xn} = e \cos \omega, \qquad a_{yn,L} = \frac{1}{a\beta^2} A_{y,\mathrm{cof}}, \qquad L_L = \frac{1}{a\beta^2} L_{\mathrm{cof}}\, a_{xn},$$

$$L_T = L + L_L, \qquad a_{yn} = e \sin \omega + a_{yn,L}.$$

The longitude that Kepler's equation will be solved against is

$$U = L_T - \Omega \pmod{2\pi}.$$

10. Solving Kepler's equation for the eccentric longitude

The eccentric longitude E_w is the root of the Lyddane Kepler equation

$$U = E_w - a_{yn} \cos E_w + a_{xn} \sin E_w.$$

Newton iteration starting from $E_w^{(0)} = U$ produces the increment

$$\Delta E_w = \frac{U - a_{yn} \cos E_w + a_{xn} \sin E_w - E_w}{1 - a_{xn} \cos E_w - a_{yn} \sin E_w},$$

with update $E_w \leftarrow E_w + \Delta E_w$, terminating when $|\Delta E_w| < 10^{-12}$ or after a fixed maximum (typically 10 iterations). For non-pathological orbits the iteration converges in three or four steps. The denominator is the Jacobian $1 - \mathbf{e} \cdot \hat{\mathbf{r}}$ in the rotating frame, which vanishes only for rectilinear orbits, a regime the deep-space gate already excludes.

11. Position in the orbital plane

With E_w in hand, the orbital-plane scalars fall out. The eccentric-longitude trig combinations are

$$\epsilon_c = a_{xn} \cos E_w + a_{yn} \sin E_w, \qquad \epsilon_s = a_{xn} \sin E_w - a_{yn} \cos E_w,$$

representing the $e \cos E$ and $e \sin E$ scalars respectively. The in-plane eccentricity squared, the semi-latus rectum, and the mean orbital radius are

$$e_L^2 = a_{xn}^2 + a_{yn}^2, \qquad p_L = a(1 - e_L^2), \qquad r = a(1 - \epsilon_c).$$

The instantaneous orbital-plane direction cosines, written in a form that stays smooth as $e \to 0$, are

$$\cos u = \frac{a}{r}\left[\cos E_w - a_{xn} + \frac{a_{yn}\,\epsilon_s}{1+\sqrt{1-e_L^2}}\right],$$

$$\sin u = \frac{a}{r}\left[\sin E_w - a_{yn} - \frac{a_{xn}\,\epsilon_s}{1+\sqrt{1-e_L^2}}\right],$$

$$u = \operatorname{atan2}(\sin u, \cos u),$$

with double-angle helpers $\sin 2u = 2\sin u\cos u$ and $\cos 2u = 2\cos^2 u - 1$. A negative value of p_L signals a degenerate orbit and the propagator should return a null result.

12. Short-period J_2 corrections

The osculating geometry differs from the mean-element geometry by the short-period J_2 terms,

$$r_k = r\left(1 - \tfrac{3}{2}\,C_{K2}\,\sqrt{1/p_L}\,(3\cos^2 i_0 - 1)\right) + \tfrac{1}{2}C_{K2}\,\frac{1}{a\beta^2}\,(1-\cos^2 i_0)\,\cos 2u,$$

$$u_k = u - \tfrac{1}{2}\,C_{K2}\,\frac{(7\cos^2 i_0 - 1)\,\sin 2u}{p_L^2},$$

$$\Omega_k = \Omega + \tfrac{3}{2}\,C_{K2}\,\frac{\cos i_0\,\sin 2u}{p_L^2},$$

$$i_k = i_0 + \tfrac{3}{2}\,C_{K2}\,\frac{\cos i_0 \sin i_0\,\cos 2u}{p_L^2}.$$

The inclination short-period uses the original i_0 rather than a propagated i, because SGP4 does not propagate the inclination secularly to first order in J_2. This is consistent with the published model.

13. Rotation into the inertial frame

The unit vector to the satellite is built from the orientation triad

$$M_x = -\sin\Omega_k \cos i_k, \qquad M_y = \cos\Omega_k \cos i_k,$$

so that

$$\hat{u}_x = M_x \sin u_k + \cos\Omega_k \cos u_k,$$

$$\hat{u}_y = M_y \sin u_k + \sin\Omega_k \cos u_k,$$

$$\hat{u}_z = \sin i_k \sin u_k.$$

These are the components of the rotation $R_z(-\Omega_k)R_x(-i_k)R_z(-u_k)$ applied to $\hat{e}_x$, written in closed form. The dimensional inertial position is

$$\mathbf{r}_{\text{ECI}} = a_E \cdot r_k \cdot (\hat{u}_x, \hat{u}_y, \hat{u}_z) \quad [\text{km}].$$

14. The master equation per satellite

Sections 3 through 13 collapse into a single composite operator. Let

$$\boldsymbol{\theta}_s = (n_0, e_0, i_0, \Omega_0, \omega_0, M_0, B^*, t_{\text{epoch}})_s$$

be the TLE-derived parameter vector for satellite s, and let $\Delta t = t - t_{\text{epoch}}$ be the time since epoch in minutes. Define the seven sub-operators

$$\Phi_{\text{Brouwer}}(\boldsymbol{\theta}_s) \mapsto (n_0'', a_0'', \beta_0, \theta, s, q_{04}, \xi, \eta, \psi),$$

$$\Phi_{\text{coef}}(\Phi_{\text{Brouwer}}) \mapsto (C_1, C_2, C_4, C_5, D_2, D_3, D_4, T_3, T_4, T_5, \dot{M}, \dot{\omega}, \dot{\Omega}, \omega_{\text{cof}}, M_{\text{cof}}, \Omega_{\text{cof}}, L_{\text{cof}}, A_{y,\text{cof}}, \delta_{M,0}),$$

$$\Phi_{\text{sec}}(\boldsymbol{\theta}_s, \Delta t) \mapsto (a, e, \omega, M, \Omega, L, n, \beta),$$

$$\Phi_{\text{Kep}}(L, \Omega, e, \omega, a, \beta) \mapsto E_w \text{ such that } (L - \Omega + L_L) = E_w - a_{yn}\cos E_w + a_{xn}\sin E_w,$$

$$\Phi_{\text{plane}}(E_w, a, e, \omega) \mapsto (r, u),$$

$$\Phi_{\text{sp}}(r, u, i_0, \Omega, p_L) \mapsto (r_k, u_k, i_k, \Omega_k),$$

$$\Phi_{\text{rot}}(r_k, u_k, i_k, \Omega_k) \mapsto \hat{\mathbf{u}} \in S^2.$$

The master per-satellite equation that maps a TLE plus a UTC time to an inertial position is

$$\boxed{\mathbf{r}^{(s)}_{\text{ECI}}(t) \;=\; a_E \cdot r_k(\Phi_{\text{sec}}(\boldsymbol{\theta}_s, \Delta t),\ \Phi_{\text{Kep}},\ \Phi_{\text{sp}}) \;\cdot\; \hat{\mathbf{u}}(\Phi_{\text{rot}}(\Phi_{\text{sp}}(\Phi_{\text{plane}}(\Phi_{\text{Kep}}(\Phi_{\text{sec}}(\boldsymbol{\theta}_s, \Delta t)))\,,\ \Phi_{\text{sec}})))\ [\text{km}].}$$

Every closed-form substitution in the operator chain is one of the equations in sections 5 through 13. The two parts that have no closed form are the Newton iteration in Φ_{Kep} and the conditional branches inside Φ_{sec} for the simple flag and the small-eccentricity guard.

A more compact form is the implicit triplet

$$
\begin{aligned}
U(\Delta t;\boldsymbol{\theta}_s) &= E_w - a_{yn}(\Delta t)\cos E_w + a_{xn}(\Delta t)\sin E_w,\\
r_k(\Delta t;\boldsymbol{\theta}_s) &= a(\Delta t)\,(1-\epsilon_c(E_w))\,\big[1+\delta_{r,J_2}(E_w,\Delta t)\big],\\
\mathbf{r}^{(s)}_{\mathrm{ECI}}(t) &= a_E\, r_k\, R_z(-\Omega_k(\Delta t))\, R_x(-i_k(\Delta t))\, R_z(-u_k(\Delta t))\,\hat{e}_x,
\end{aligned}
$$

where δ_{r,J_2} collects the short-period radial perturbations from section 12, and the angles Ω_k, i_k, u_k already include both the secular drift and the short-period J_2 contribution. The first line is the Lyddane Kepler equation, solved by Newton iteration on E_w. The second line is the perturbed radius. The third line is the three-axis rotation through the propagated node, inclination, and argument of latitude. This form is more tractable for inspection and for reasoning about which stage of the chain produces a given numerical artifact.

15. Inertial to Earth-fixed to topocentric reduction

A topocentric az-el at a fixed ground station extends the master equation by a sidereal rotation, a difference, and a projection. The Greenwich mean sidereal time at the requested instant is

$$
\theta_g = \mathrm{GMST}(JD), \qquad JD = \mathrm{toJulianDate}(t),
$$

with the Julian date implementing the Meeus algorithm,

$$
JD = \lfloor 365.25(Y+4716)\rfloor + \lfloor 30.6001(m+1)\rfloor + d + f_{\mathrm{day}} + B - 1524.5,
$$

$$
B = 2 - \lfloor Y/100\rfloor + \lfloor Y/400\rfloor, \qquad m \to m+12 \text{ if } m \le 2,
$$

and the GMST in radians comes from the IAU 1982 polynomial,

$$
\mathrm{GMST}(JD) = 24110.54841 + T(8640184.812866 + T(0.093104 - 6.2\times 10^{-6}T)) + 86400\cdot 1.00273790935\cdot \mathrm{UT},
$$

$$
T = (JD_0 - 2451545.0)/36525, \qquad \mathrm{UT} = (JD+0.5) \bmod 1, \qquad JD_0 = JD - \mathrm{UT},
$$

reduced modulo 86400, and rescaled to $[0, 2\pi)$. It seems, the polynomial-based GMST agrees with the IAU 2006 sidereal-time expression to better than 0.1 arcsecond across the 20th and 21st centuries, which is well below the SGP4 position error floor.

The ECI-to-ECEF rotation is the z-axis rotation by $-\theta_g$,

$$
\begin{pmatrix} x_E \\ y_E \\ z_E \end{pmatrix} = \begin{pmatrix} \cos\theta_g & \sin\theta_g & 0 \\ -\sin\theta_g & \cos\theta_g & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_I \\ y_I \\ z_I \end{pmatrix},
$$

ignoring polar motion and equation-of-the-equinoxes corrections, which are below the precision floor of SGP4 itself. The ground-station position in ECEF, with WGS84 prime-vertical radius of curvature

$$
N = \frac{a_E}{\sqrt{1 - e^2 \sin^2 \phi_g}},
$$

is

$$\mathbf{r}_{gs} = \big((N + h_g)\cos\phi_g \cos\lambda_g,\ (N + h_g)\cos\phi_g \sin\lambda_g,\ (N(1 - e^2) + h_g)\sin\phi_g\big),$$

where ϕ_g, λ_g, h_g are the geodetic latitude, longitude, and altitude of the station. The relative vector is $\boldsymbol{\rho} = \mathbf{r}_{\text{ECEF}} - \mathbf{r}_{gs}$, and its projection onto the local south-east-zenith (SEZ) basis is

$$\rho_S = \sin\phi_g \cos\lambda_g\, \rho_x + \sin\phi_g \sin\lambda_g\, \rho_y - \cos\phi_g\, \rho_z,$$

$$\rho_E = -\sin\lambda_g\, \rho_x + \cos\lambda_g\, \rho_y,$$

$$\rho_Z = \cos\phi_g \cos\lambda_g\, \rho_x + \cos\phi_g \sin\lambda_g\, \rho_y + \sin\phi_g\, \rho_z.$$

The topocentric look angles are

$$\text{el} = \arcsin\left(\frac{\rho_Z}{\|\boldsymbol{\rho}\|}\right), \qquad \text{az} = \operatorname{atan2}(\rho_E, -\rho_S),$$

with the azimuth shifted into $[0, 360^\circ)$. The negative sign on ρ_S converts the SEZ azimuth (measured from south) to the standard north-referenced compass azimuth.

16. Sub-satellite latitude, longitude, and altitude

For ground-track plotting and footprint visualization, the sub-satellite point is derived directly from the ECEF position. Given $\mathbf{r}_{\text{ECEF}} = (x_E, y_E, z_E)$,

$$r = \|\mathbf{r}_{\text{ECEF}}\|, \qquad \phi = \arcsin(z_E/r), \qquad \lambda = \operatorname{atan2}(y_E, x_E),$$

$$h = r - a_E \quad [\text{km, geocentric}].$$

The geocentric latitude differs from the geodetic latitude by up to about 11 arcminutes at mid-latitudes, which, it seems, is invisible at typical ground-track display scales. For applications that drive antenna pointing or precise geolocation, the geodetic latitude should be recovered by Bowring iteration, or by Heiskanen's closed-form expansion.

17. Pass prediction by coarse-then-fine bracketing

Pass prediction over a fixed interval reduces to bracketing zero crossings of the elevation. A coarse scan steps through time at Δt_{scan} (typically 30 s) and tags transitions in the sign of elevation. Each acquisition-of-signal (AOS) and loss-of-signal (LOS) time is then refined by bisection between the bracketing samples,

$$t_{\text{AOS/LOS}} = \text{bisect}_N\{t : \text{el}(t) = 0\},$$

with $N \approx 20$ iterations giving sub-second precision. A pass is reported only if the maximum elevation in the bracket exceeds a user-configured threshold (commonly 5 degrees), which suppresses very low passes that would be clipped by terrain or local horizon obstructions.

The geometric quantities that characterize a pass are the AOS time and azimuth, the time of closest approach (TCA), the maximum elevation at TCA, the TCA azimuth, the LOS time and azimuth, and the duration. Each is a direct evaluation of the master equation at the relevant sample instant.

18. End-to-end pipeline

The end-to-end workflow for one satellite, from TLE source to topocentric output, has the following structure:

1. Retrieve the TLE for the satellite, either from a cached file or from a public catalog source (typical cache lifetimes are 12 to 24 hours).
2. Parse the TLE to extract $\boldsymbol{\theta}_s$ and pre-compute the Brouwer reduction and the secular and drag coefficients of sections 3 through 7.
3. At evaluation time, compute Δt in minutes from the requested UTC instant to the TLE epoch.
4. Evaluate the master equation of section 14 to produce $\mathbf{r}_{\text{ECI}}$ in km.
5. Rotate to ECEF using the GMST from section 15 and reduce to the sub-satellite footprint (ϕ, λ, h) via section 16.
6. Reduce to topocentric SEZ via section 15 to produce azimuth and elevation at the ground station.

For a real-time display, the per-satellite position is interpolated between propagation updates with a one-pole infinite-impulse-response filter

$$x_{n+1} = x_n + \alpha(x_n^* - x_n),$$

with cut-off frequency

$$f_c \approx -\frac{f_{\text{frame}}}{2\pi} \ln(1 - \alpha).$$

A blend fraction of $\alpha = 0.08$ at 60 frames per second produces a visual time constant of about 200 milliseconds, which feels responsive without strobing. Longitude wraparound across the date line should be handled by adjusting $\Delta\lambda$ into $[-180°, +180°)$ before the lerp, so a satellite crossing the antimeridian does not skate across an entire ocean in one frame.

19. Variable accounting

Every named scalar in the propagator and the topocentric reduction falls into one of four classes by source. The external drivers, EXT-TLE and EXT-RUN, are the only quantities that must be supplied from outside the kernel. Hard-coded physical constants, CONST, are baked into the model. Everything else, DERIV, is bookkeeping the kernel performs in service of the master equation.

The eight EXT-TLE rows are the seven mean elements plus the epoch parsed from the TLE: $n_0, e_0, i_0, \Omega_0, \omega_0, M_0, B^*, t_{\text{epoch}}$. These describe one specific satellite at one epoch and they are the only quantities that change when the kernel is called against a different object.

The five EXT-RUN rows are the wall-clock UTC instant t and the four ground-station configuration values ϕ_g, λ_g, h_g plus the minimum elevation threshold for pass reporting. These describe when the question is being asked and from where, and they are independent of any satellite.

The ten CONST rows are the geometric and gravitational constants $a_E, k_e, C_{K2}, C_{K4}, J_3/J_2$, the two drag-fit references $(q_0 - s)_{\text{ref}}$ and $(q_0 - s)^4_{\text{ref}}$, the dimensionless Earth radius, the WGS84 flattening f, and the first-eccentricity-squared e^2. These describe the planet and the propagator and they are not user-tunable in any reasonable sense.

The remaining variables, on the order of 130 named scalars for a non-simple satellite with full topocentric output, are derived inside the kernel. They subdivide naturally into the Brouwer reduction (about 15), the secular and drag coefficient block (about 38), the time-evolved mean elements (about 16), the Lyddane Kepler iteration (about 8), the orbital-plane geometry (about 10), the short-period J_2 corrections (4), the rotation triad (about 11), the topocentric reduction (about 17), and the sub-satellite footprint (about 4). A complete table of these variables, with code identifiers and stage labels, is straightforward to compile from the equations of sections 3 through 16 and is omitted here for compactness.

The simple-flag branch suppresses 8 derived variables ($D_2, D_3, D_4, T_3, T_4, T_5, \Delta\omega, \Delta M$) plus the cubic and quartic contributions to the semi-major axis and mean longitude polynomials. This is the dominant computational savings for high-perigee satellites.

20. Known limitations

Three limitations are inherent to an SGP4-only implementation and should be kept in mind when interpreting its output.

The propagator silently refuses any deep-space object with orbital period ≥ 225 minutes. Catalogs that include geosynchronous, highly elliptical, or molniya orbits will return null for those objects, and any visible-count or summary statistic must account for the gap. Extending the kernel beyond LEO requires the SDP4 deep-space resonance terms (lunar and solar third-body and the 12- and 24-hour resonance buckets), which are documented in Vallado's revisitation of *Spacetrack Report No. 3* and in the corresponding Fortran reference.

The internal frame is WGS72, and the topocentric reduction uses WGS84. It seems, the position discrepancy is sub-kilometer for low Earth orbit, and is dominated by the TLE age error within a day or two of epoch. In my view, the discrepancy is invisible for ground-track plotting and pass alerting. For narrow-beam antenna pointing, or for laser ranging, the mismatch should be removed by using a consistent shape model end-to-end, or by upgrading to SGP4-XP, which uses a higher-fidelity geopotential and an EOP-aware ECI-to-ECEF rotation.

The Julian-date routine uses the Gregorian leap rule via $B = 2 - \lfloor Y/100 \rfloor + \lfloor Y/400 \rfloor$, which is correct for all dates after the 1582 Gregorian reform and silently wrong before. The constraint is irrelevant for any practical satellite-tracking work.

21. Validation against a working earth station

The implementation behind this paper is in continuous use our lab's earth station in Austin, Texas, at near sea-level altitude. The station serves as the topocentric reference for the EXT-RUN block of section 19, and it provides the ground truth I use to confirm that the operator chain of section 14 and the SEZ projection of section 15 are producing the correct azimuth and elevation.

The validation procedure is direct. For a given catalog object, I pull the most recent TLE from CelesTrak, propagate through the master equation at one-second resolution across a predicted pass window, and compare the resulting azimuth and elevation track against the observed track from the station. For low-Earth-orbit objects within a day or two of TLE epoch, it appears to me, that the residuals in azimuth and elevation are consistent with the published SGP4 error budget for fresh TLEs, of order one to three kilometers in along-track position, which projects to a few hundredths of a degree, at typical pass ranges of several hundred to a couple of thousand kilometers. Residuals grow predictably with TLE age, in the manner described by Vallado's revisitation, and the growth rate is the standard diagnostic I use, to decide when to refresh the catalog.

The Austin station is not an absolute reference, and the validation is certainly not a formal accuracy assessment. However, it is a convenient continuity check: the same TLE, propagated through the same master equation, must produce the same azimuth and elevation track that the station antenna actually points at when the satellite is overhead. I contend, that this check is sufficient to certify that the operator chain in section 14 has been transcribed correctly, and that the topocentric reduction in section 15 is using the right rotation conventions. It seems, any structural error in the propagator, a swapped sign in the Lyddane long-period correction, a missing J_3 term in the inclination-coupled block, or a confused sidereal-time polynomial, surfaces as a systematic offset in either azimuth or elevation, and in my view, the implicit-triplet form of section 14 is the diagnostic tool that localizes the failure.

22. Conclusion

In my view, the operator-chain master equation in section 14 is the contract between an SGP4 propagator and every consumer in the pipeline: pass prediction, ground-track plotting, az-el computation, footprint visualization, and any analytical work that needs to differentiate position with respect to TLE elements, or evaluation time. The implicit-triplet form is the version that survives at the whiteboard, and the form to reach for when a propagator implementation is producing the wrong number, and the engineer needs to localize the failure to one of three stages.

The variable-accounting structure of section 19, with its sharp distinction between external drivers and derived intermediates, is the structural map that separates the inputs to the model from its internal bookkeeping. It is my goal to implement the EXT-TLE, EXT-RUN, and CONST blocks as named structures, and to treat everything else as opaque kernel state, making the process more straightforward to test, port, and audit.

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