Regarding the Transcendental Equation for a 1-dimensional tube with a Hole in It

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Takuya Yabu(takuya.yabu@live.jp)

Abstract

While research on automobile whistling and aspiration noise has been conducted in the past, we believe that understanding the properties of whistling and aspiration sounds using a 1-dimensional tube as a simple model is necessary in fields such as automobiles. Therefore, we discussed transcendental equations for the case where a 1-dimensional tube has a hole. As a result, we found that the properties of automobile whistling and aspiration sounds can be simulated.

Nomenclature

S, S_H: Cross-sectional area of the 1-dimensional tube and area of the hole in the 1-dimensional tube

l: Position of the hole in the 1-dimensional tube

L: Length of the 1-dimensional tube

V: Volume of the 1-dimensional tube

p_1, v_1: Sound pressure and particle velocity for $0 \leq x \leq l$

A_1: Amplitude of sound pressure for $0 \leq x \leq l$

p_2, v_2: Sound pressure and particle velocity for $l \leq x \leq L$

A_2: Amplitude of sound pressure for $l \leq x \leq L$

p_H, v_H: Sound pressure and particle velocity at $x = l$

δ_H: Thickness of the 1-dimensional tube

ρ, c: Density of air and speed of sound in air

ω: Angular frequency

f: Frequency

k: Wavenumber

t: Time

j: Imaginary unit

1. Introduction

In the past, studies have been conducted on the whistling and suction sounds of automobiles (Calvo, Diaz, & San Roman, 2005) (Chien-Hsiung, Lung-Ming , Chang-Hsien , Yen-Loung , & Jik-Chang , 2009) (George, 1990) (Jagtiani, 1972) (Jung & Oh, 1995) (Münder & Carbon, 2022) (Oettle & Sims-Williams, 2017) (Qatu, Abdelhamid, Pang, & Sheng, 2009) (Wang, Chen, & Zhang, 2021) (Zhang, Meng, Li, & Zheng, 2022). However, to the best of the author's knowledge, there are no studies that discuss the properties of whistling and aspiration noise using a one-dimensional tube as a simple model. Understanding the properties of whistling and aspiration noise using a 1-dimensional tube is considered essential in the automotive field.

Therefore, we will consider the properties of whistling and aspiration noise using a 1-dimensional tube.

This will be discussed below.

2. Transcendental Equation for a 1-Dimensional Tube with Holes

2.1. Case with Open Ends on Both Sides

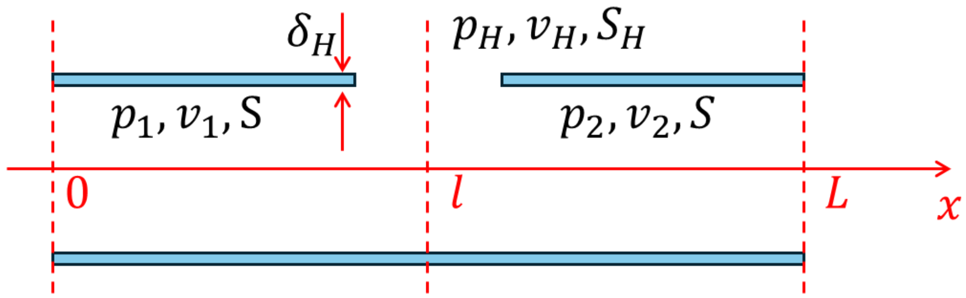

Fig. 1 1-Dimensional Tube which Both Sides Open

Fig. 1 shows a 1-dimensional tube with open ends on both sides. From the conditions for continuity of sound pressure and continuity of particle velocity, the following equations hold:

$$p_1(l,t) = p_2(l,t) \tag{1}$$

$$Sv_1(l,t) = Sv_2(l,t) + S_H v_H \tag{2}$$

The equation of motion that holds for the hole is as follows:

$$\rho \delta_H \frac{\partial v_H}{\partial t} = p_H(t) \tag{3}$$

The equation for continuity of sound pressure is as follows:

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} \tag{4}$$

Let the steady-state oscillation be $e^{j\omega t}$. From equation (3), the following equation is obtained:

$$\rho j\omega\delta_H v_H = p_H$$

$$\therefore v_H = \frac{p_H}{\rho j\omega\delta_H} \tag{5}$$

From equation (4), the following equation is obtained:

$$\rho j\omega v = -\frac{\partial p}{\partial x}$$

$$\therefore v = -\frac{1}{\rho j\omega}\frac{\partial p}{\partial x} \tag{6}$$

Substitute equations (5) and (6) into equation (2) and rearrange.

$$-S\frac{1}{\rho j\omega}\frac{\partial p_1}{\partial x}\bigg|_{x=l} = -S\frac{1}{\rho j\omega}\frac{\partial p_2}{\partial x}\bigg|_{x=l} + S_H\frac{1}{\rho j\omega}\frac{1}{\delta_H}p_H|_{x=l}$$

$$\therefore -S\frac{\partial p_1}{\partial x}\bigg|_{x=l} + S\frac{\partial p_2}{\partial x}\bigg|_{x=l} = +\frac{S_H}{\delta_H}p_H|_{x=l} \tag{7}$$

Now, from the open ends at both ends, p_1 and p_2 are determined as follows.

$$p_1(0,t) = 0 \text{ at } x = 0, \therefore p_1(x,t) = A_1\sin(kx) \tag{8}$$

$$p_2(L,t) = 0 \text{ at } x = L, \therefore p_2(x,t) = A_2\sin\big(k(L-x)\big) \tag{9}$$

Substitute into equations (1) and (7) and rearrange.

$$A_1\sin(kl) = A_2\sin(k(L-l)) \tag{10}$$

$$-SkA_1\cos(kl) - SkA_2\cos\big(k(L-l)\big) = \frac{S_H}{\delta_H}A_1\sin(kl) \tag{11}$$

$$\therefore \frac{1}{\tan(kl)} + \frac{1}{\tan\big(k(L-l)\big)} = -\frac{S_H}{S\delta_H}\frac{1}{k} \tag{12}$$

$$\therefore \cot(kl) + \cot(k(L-l)) = -\frac{S_H}{S\delta_H}\frac{1}{k} \tag{13}$$

Now, when $S_H = 0$, the resonance frequency of the 1-dimensional tube with both ends open, without holes, can be found from equation (12).

$$\tan\big(k(L-l)\big) = -\tan(kl)$$

$$k(L-l) = -kl + n\pi$$

$$\therefore kL = n\pi \tag{14}$$

$$\therefore f = n\frac{c}{2L} \tag{15}$$

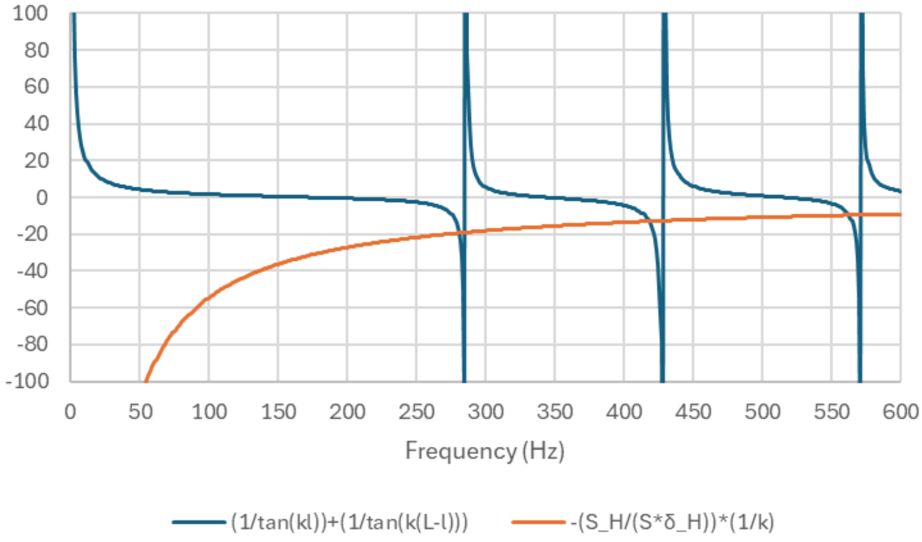

Fig. 2 Function Value of 1-Dimensional Tube which Both Sides Open

Fig. 2 shows the graphs of the functions on the left and right sides of equation (12). The first resonance frequency when $S_H = 0$ is 171.5 (Hz), and the first intersection point on the graph is at a higher frequency. That is, the pitch is higher. This can be considered equivalent to the tube being shortened by the length of the holes.

2.2. Case with One Open End and One Closed End

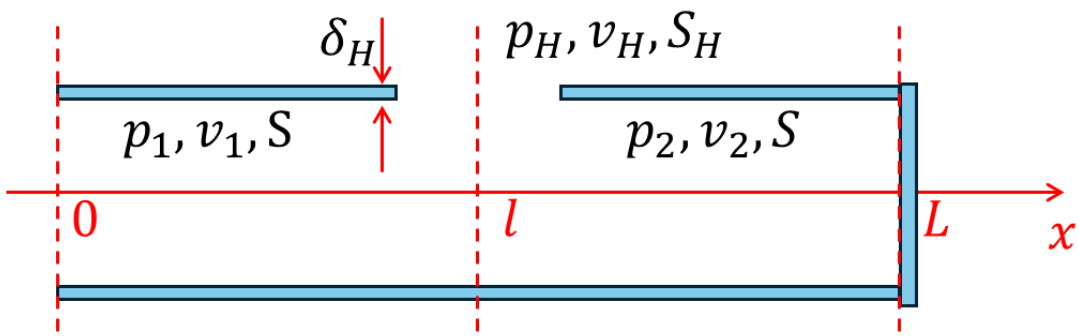

Fig. 3 1-Dimensional Tube which Left Side Opens and Right Side Closes

Fig. 3 shows a 1-dimensional tube with one open end and one closed end. From one open end and one closed end, p_1 and p_2 are determined as follows:

$$p_1(0,t) = 0 \text{ at } x = 0, \therefore p_1(x,t) = A_1 \sin(kx) \tag{16}$$

$$v_2(L,t) = 0 \text{ at } x = L, \therefore p_2(x,t) = A_2 \cos\big(k(L-x)\big) \tag{17}$$

Substitute into equations (1) and (7) and rearrange.

$$A_1 \sin(kl) = A_2 \cos(k(L-l)) \tag{18}$$

$$-SkA_1 \cos(kl) + SkA_2 \sin\big(k(L-l)\big) = \frac{S_H}{\delta_H} A_1 \sin(kl) \tag{19}$$

$$\tan(k(L-l)) - \cot(kl) = \frac{S_H}{S\delta_H}\frac{1}{k} \tag{20}$$

Now, when $S_H = 0$, the resonance frequency of a 1-dimensional tube with one open end and one closed end can be found from equation (20) in the case of no holes.

$$\tan\big(k(L-l)\big) = \cot(kl)$$

$$k(L-l) = \frac{\pi}{2} - kl + n\pi$$

$$\therefore kL = \frac{\pi}{2} + n\pi \tag{21}$$

$$\therefore f = (2n+1)\frac{c}{4L} \tag{22}$$

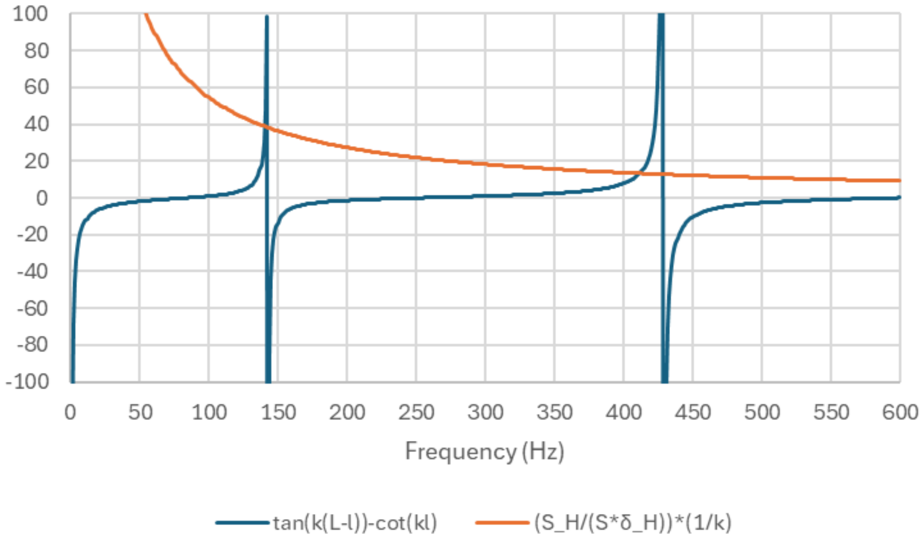

Fig. 4 Function Value of 1-Dimensional Tube which Left Side Opens and Right Side Closes

Fig. 4 shows the graphs of the functions on the left and right sides of equation (20). The first resonant frequency when $S_H = 0$ is 85.75 (Hz), and the first intersection point on the graph is at a higher frequency. That is, the pitch is higher. This can be considered equivalent to the tube becoming shorter by the length of the hole.

2.3. Case with One Closed End and One Open End

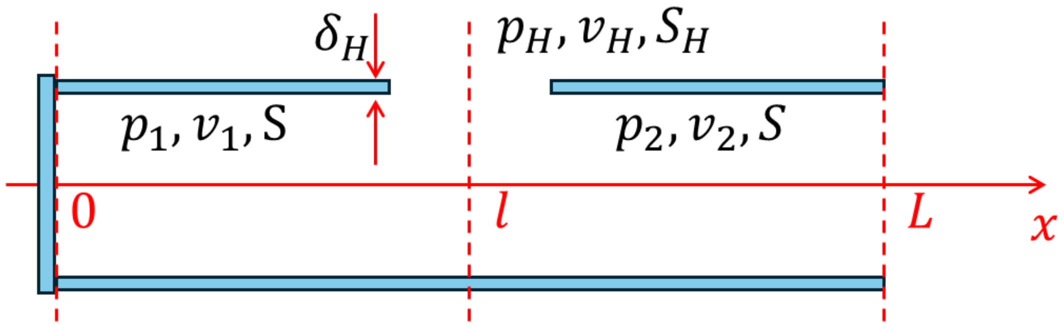

Fig. 5 1-Dimensional Tube which Left Side Closes and Right Side Opens

Fig. 5 shows a 1-dimensional tube with one closed end and one open end. From one closed end and one open end, p_1 and p_2 are determined as follows:

$$v_1(0,t) = 0 \text{ at } x = 0, \therefore p_1(x,t) = A_1 \cos(kx) \tag{23}$$

$$p_2(L,t) = 0 \text{ at } x = L, \therefore p_2(x,t) = A_2 \sin\big(k(L-x)\big) \tag{24}$$

Substitute into equations (1) and (7) and rearrange.

$$A_1 \cos(kl) = A_2 \sin(k(L-l)) \tag{25}$$

$$SkA_1 \sin(kl) - SkA_2 \cos\big(k(L-l)\big) = \frac{S_H}{\delta_H} A_1 \cos(kl) \tag{26}$$

$$\tan(kl) - \cot(k(L-l)) = \frac{S_H}{S\delta_H} \frac{1}{k} \tag{27}$$

Now, when $S_H = 0$, the resonance frequency of a 1-dimensional tube with one open end and one closed end can be found from equation (27) in the case of no holes.

$$\tan(kl) = \cot\big(k(L-l)\big)$$

$$kl = \frac{\pi}{2} - k(L-l) + n\pi$$

$$\therefore kL = \frac{\pi}{2} + n\pi \tag{28}$$

$$\therefore f = (2n+1)\frac{c}{4L} \tag{29}$$

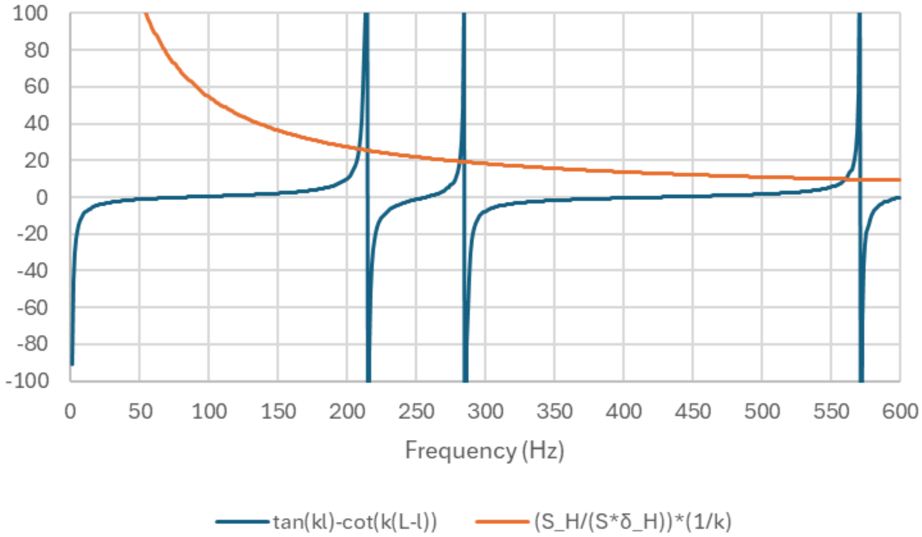

Fig. 6 Function Value of 1-Dimensional Tube which Left Side Closes and Right Side Open

Fig. 6 shows the graphs of the functions on the left and right sides of equation (27). The first resonant frequency when $S_H = 0$ is 85.75 (Hz), and the first intersection point of the graph is at a higher frequency. That is, the pitch is higher. This can be considered equivalent to the tube becoming shorter by the length of the hole.

2.4. Case with Both Sides Closed

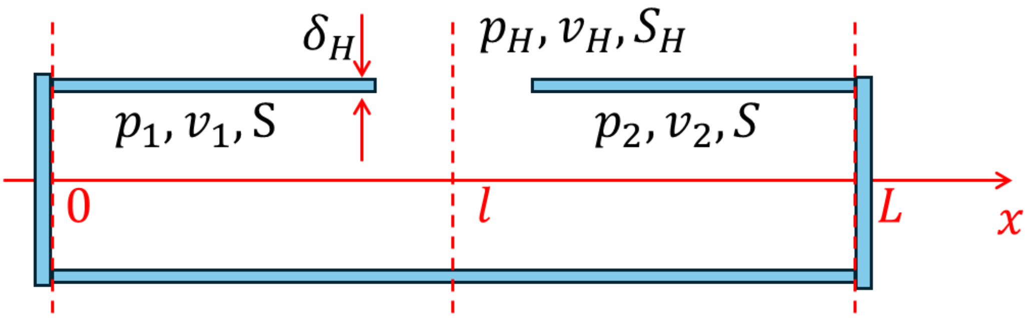

Fig. 7 1-Dimensional Tube which Both Sides Close

Fig. 7 shows a 1-dimensional tube with both sides closed. From the closed ends on both sides, p_1 and p_2 are determined as follows:

$$v_1(0,t) = 0 \text{ at } x = 0, \therefore p_1(x,t) = A_1\cos(kx) \tag{30}$$

$$v_2(L,t) = 0 \text{ at } x = L, \therefore p_2(x,t) = A_2\cos\big(k(L-x)\big) \tag{31}$$

Substitute into equations (1) and (7) and rearrange.

$$A_1\cos(kl) = A_2\cos(k(L-l)) \tag{32}$$

$$SkA_1\sin(kl) + SkA_2\sin\big(k(L-l)\big) = \frac{S_H}{\delta_H}A_1\cos(kl) \tag{33}$$

$$\tan(kl) + \tan(k(L-l)) = \frac{S_H}{S\delta_H}\frac{1}{k} \tag{34}$$

Now, when $S_H = 0$, the resonance frequency of a 1-dimensional tube with closed ends on both sides in the case without holes can be found from equation (34).

$$\tan(kl) = -\tan\big(k(L-l)\big)$$

$$kl = -k(L-l) + n\pi$$

$$\therefore kL = n\pi \tag{35}$$

$$\therefore f = n\frac{c}{2L} \tag{36}$$

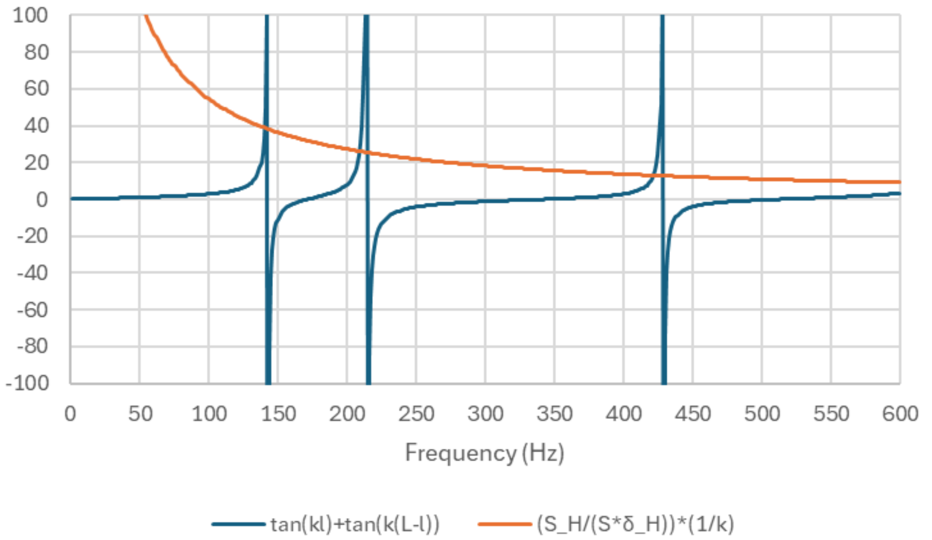

Fig. 8 Function Value of 1-Dimensional Tube which Both Sides Close

Fig. 8 shows the graphs of the functions on the left and right sides of equation (34). The first resonant frequency when $S_H = 0$ is 171.5 (Hz), and the first intersection point of the graph is at a lower frequency. That is, the pitch is lower. The resonant frequency decreases by the amount of the hole.

Now, consider the case where the "tube is short (L is small)" and the "pitch is low ($k =$

$\frac{\omega}{c}$ is small)". Since kl and $k(L-l)$ become very small values, from $\tan\theta \approx \theta$, equation (34) becomes as follows.

$$kl + k(L-l) \approx \frac{S_H}{S\delta_H}\frac{1}{k}$$

$$kL = \frac{S_H}{S\delta_H}\frac{1}{k}$$

$$\therefore k = \sqrt{\frac{S_H}{(S\cdot L)\delta_H}} \tag{37}$$

The volume V of the tube is obtained from $V = S\cdot L$, as shown in the following equation:

$$k = \sqrt{\frac{S_H}{V\delta_H}}$$

$$\therefore f = \frac{c}{2\pi}\sqrt{\frac{S_H}{V\delta_H}} \tag{38}$$

Equation (38) is the resonance frequency of a Helmholtz resonator. In other words, a 1-dimensional tube with closed ends on both sides can be considered a Helmholtz resonator.

When both ends are closed, if a hole is made, that hole becomes the only outlet. Having an open end simply adds "one more outlet" to a large number of potential exits. However, with closed ends on both sides, the trapped energy erupts from a single point of opening. This causes a rapid drop in sound pressure, making the resonance extremely unstable.

Furthermore, in a 1-dimensional tube with open ends at both ends, similar to a Helmholtz resonator, the air throughout the tube can be considered a "spring," and the air in the opening can be considered "mass." At frequencies lower than the resonant frequency, the "spring" is primarily at work, causing sound to be emitted. At the resonant frequency, the movement of the air in the opening is amplified, resulting in the strongest sound. At frequencies higher than the resonant frequency, the "mass" takes over, the air in the opening acts as a lid, and no sound is emitted. In vibration engineering, this is sometimes referred to as the spring line and mass line.

3. Characteristics of Whistling and Aspiration Noise

Automobiles often have spaces with one open end and one closed end, such as those with grooves, as well as spaces with both closed ends. From spaces with one open end and one closed end, a sound with a higher frequency than the resonant frequency of a space without holes is generated. This is a whistling noise. Similarly, from spaces with both closed ends, a sound with a lower frequency than the resonant frequency of a space without holes is generated. This is an aspiration noise. Air in the space is drawn out through the holes due to negative pressure, and air is drawn back in as the pressure decreases. This continuous process generates sound. This vibration of air entering and leaving the space is called self-excited vibration in vibration engineering.

Therefore, to find the source of whistling and aspiration noise, it is important to find spaces with holes. In other words, leak analysis can be used to discover the source of whistling and aspiration noise.

4. Conclusion

While research on automobile whistling and aspiration noise has been conducted in the past, we believe that understanding the properties of whistling and aspiration sounds using a 1-dimensional tube as a simple model is necessary in fields such as automobiles. Therefore, we discussed transcendental equations for the case where a 1-dimensional tube has a hole. As a result, we found that the properties of automobile whistling and aspiration sounds can be simulated.

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