A powerful series solution of some Eshelby's problems of a single smooth inclusion

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Chengpeng Wei[1,*]
1. Institute of Engineering Mechanics
Nanchang University
Nanchang 330031, China
jizangcheng@dingtalk.com

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ABSTRACT

Except for elliptical inclusions, the solution of the anisotropic Eshelby's problem has been limited to polygonal inclusion for a long time. In this paper, the anisotropic Eshelby's problem of a smooth inclusion characterized by Laurent polynomials and undergoing nonuniform eigenstrain in a magneto-electro-elastic (MEE) plane is investigated. Under the framework of Stroh formalism, the multiphysical perturbance can be described by a set of eigenfunctions, which are analytical functions expressed through the Cauchy integrations along the inclusion's boundaries in the transformed complex planes. A powerful series solution is provided in order to solve the Cauchy integration without material parameter constrains. And the method can be used to solve a smooth inclusion undergoing nonuniform eigenstrain in an anisotropic bimaterial plane or an isotropic disk. (This version is a preprint. Some example figures will be supplemented in a future version).

Keywords inclusion · nonuniform eigenstrain · smooth shape

1 Introduction

Eshelby's problem is concerned with the elastic perturbance in an infinite homogeneous medium due to an embedded inclusion undergoing uniform eigenstrain. More than half a century ago, Eshelby's seminal works on the ellipsoidal inclusion opened a fertile realm in solid mechanics [Eshelby, 1957, 1959], where the uniformity of elastic field in the inclusion is widely used to transform the problem of analyzing the stress field in matrix-inclusion solids into an algebraic operation problem. This concept has become the cornerstone of composite mechanics and material science [Mura, 1982, Nemat-Nasser and Hori, 1993], even been extensively applied in nanoscience and nanotechnologies [Li and Wang, 2008]. During this period, analytical solutions for the Eshelby's problems of a non-elliptic inclusion are very rare. Taking the case of isotropic materials as an example, analytical solutions for polygonal inclusion problems were still cumbersome [Nozaki and Taya, 1997, Kawashita and Nozaki, 2001] until the succinct solution in a complex representation by Zou et al. [2010]. Analytical exterior solutions for smooth inclusion problems, say with the shapes characterized by the Laurent polynomials, were obtained by Lee and Zou [2016a,b].

Kinoshita and Mura [1971] derived the integration formulae of the anisotropic Eshelby's problem for an ellipsoidal inclusion (also see Mura [1982]), and Wang [1992] extended the problem to the piezoelectric composite materials. But except for these, Ting [2000] particularly emphasized the importance of Stroh formalism [Stroh, 1958, Ting, 1996] in the study of plane anisotropic linear elasticity problems, and pointed out that analytically solving the anisotropic Eshelby problem for a non-elliptic inclusion is still a challenging problem. Through conformal mapping method and technique of analytic continuation, Ru attempted to construct some auxiliary functions to express the analytical solutions of the piezoelectric inclusions of arbitrary shapes, but available solutions are still limited to ellipse [Ru, 2000, 2003]. In the meanwhile, Pan and coworkers made use of the Green's function method [Pan, 2004] to express the solutions of piezoelectric Eshelby problem of polygonal inclusions with a finite number of straight-line integrals, which can be easily extended to magneto-electro-elastic materials [Jiang and Pan, 2004]. Later, Zou and coworkers proposed a unified approach [Zou et al., 2011] with generic and compact integral formulae for the eigenfunctions and their averages in the Stroh formalism, and all results reported by Ru [2000] and Pan [2004] can be derived as the particular ones. Furthermore, Zou and Pan [2012] extended analytical solutions from arbitrary polygon to those characterized by Jordan curves.

In resent years, the Eshelby's problem has continued to attract significant attention from the academic comminuity [Hsieh and Hwu, 2023, Xie and Linder, 2024, Wu and Yin, 2024]. Two elegant but imperfect methods are provided to solve the smooth cavity in an anisotropic plane [Hsieh and Hwu, 2025]. By exploiting the computational efficiency of Fourier space, a universal approach for solving Eshelby's problems (both homogeneous and inhomogeneous) is proposed [Zhu et al., 2025], however, nonuniform eigenstrains are not considered. In contrast, the study focuses on the analytical solutions of smooth inclusions, say those shaped by the Laurent polynomials, can clearly indicate the shape effects on the physical fields, and so are of great interest. The present study will provide a more comprehensive and succinct solution of kind of smooth inclusions undergoing uniform eigenstrain in the MEE full plane, bimaterial plane or isotropic disk.

The structure of this paper is as follows. In Section 2, an Eshelby's problem is proposed, where a smooth inclusion characterized by Laurent polynomials and disturbed by nonuniform eigenstrain in a MEE plane. And the integral solution of it is derived, and some numerical examples are given. The powerful method can also use to solve a smooth inclusion undergoing extended nonuniform eigenstrain in an anisotropic bimaterial plane in Section 3 or an isotropic circular disk in Section 4. And concluding remarks are provided in Section 5. The processing of derivation of the series solution to the integral solution is detailed in the Appendix A.

2 A smooth inclusion in an anisotropic full plane

2.1 Description of the smooth inclusion problem

In the infinite homogeneous plane made of an anisotropic medium, consider a smooth inclusion ω embedded into a matrix Ω, as shown in Fig. 1, with the boundary indicated by $\Gamma = \partial\omega$. As is shown in Fig. 1, the boundary of a smooth

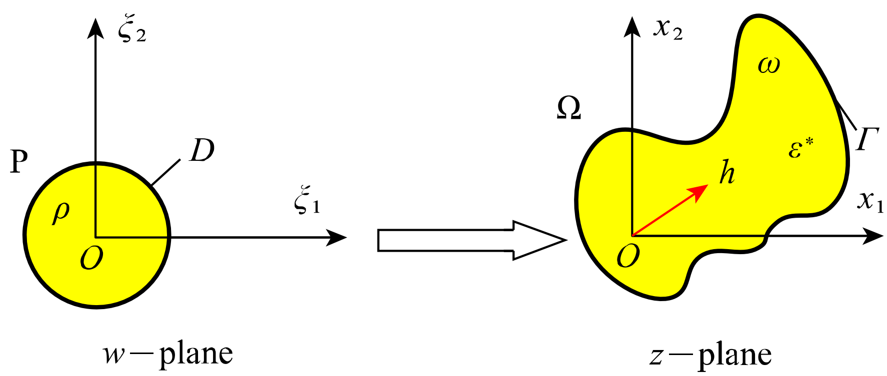

Figure 1: Riemann mapping for the smooth inclusion.

inclusion ω can be approximately expressed by the Laurent polynomial

$$t = h + R\phi(\eta) = h + R\left(\eta + \sum_{k=1}^{N} b_k \eta^{-k}\right), \quad |\eta| = 1, \tag{1}$$

within any accuracy [Zou et al., 2011, England, 1971], where t and η are the boundary points of inclusion ω and the unit disk ρ, respectively, h indicates the center of the inclusion, the positive real parameter R indicates the size of the inclusion, the complex coefficients b_k satisfy $|b_k| \leq \frac{1}{k}$. In the meanwhile, the Riemann mapping

$$z = h + R\phi(w), \; \phi(w) = w + \sum_{k=1}^{N} b_k w^{-k}, \quad |w| \geq 1, \tag{2}$$

maps the domain Ω outside the inclusion ω to the domain P outside of the unit disk ρ (as shown in Fig. 1), where w and z are the points of P and Ω, respectively.

What we concern with is to find the elastic perturbance due to the eigenstrain when the sub-domain ω undergoes an eigenstrain ε^*. Since the elastic fields of the Eshelby's problem in an infinite medium are independent of the size and position, in the following we will take $h = 0$ and $R = 1$ if there is no special explanation.

2.2 Governing equations of anisotropic MEE media

The constitutive laws for a linear MEE material without body force, electric charge density, and electric current density are given as

$$\left.\begin{array}{l}\sigma_{ij} = C_{ijkl}u_{k,l} + e_{kij}\varphi_{,k} + q_{kij}\phi_{,k},\\ D_k = e_{kij}u_{i,j} - m_{kl}\varphi_{,l} - d_{kl}\phi_{,l},\\ B_k = q_{kij}u_{i,j} - d_{lk}\varphi_{,l} - \mu_{kl}\phi_{,l},\\ \sigma_{ij,j} = 0,\ \ D_{k,k} = 0,\ \ B_{k,k} = 0,\end{array}\right\} \tag{3}$$

repeated indices in (3) mean summation,a comma followed by i $(= 1, 2, 3)$ denotes the partial derivative with respect to the ith spatial coordinate; u_i, φ and ϕ are the elastic displacements, electric potential, and magnetic potential; σ_{ij}, D_i and B_i are the stress, electric displacement, and magnetic induction (i.e.,magnetic flux); C_{ijkl}, m_{ij} and μ_{ij} are the elastic, dielectric and magnetic permeability constants; e_{ijk}, q_{ijk} and d_{ij} are the piezoelectric, piezomagnetic, and magnetoelectric constants.

The the extended stiffness notation can be introduced as

$$C_{IjKl} = \begin{cases} C_{ijkl}, & I, K = i,\ k = 1, 2, 3,\\ e_{ijl}, & K = 4,\ I = i = 1, 2, 3,\\ e_{jkl}, & I = 4,\ K = k = 1, 2, 3,\\ q_{ijl}, & K = 5,\ I = i = 1, 2, 3,\\ q_{jkl}, & I = 5,\ K = k = 1, 2, 3,\\ -m_{jl}, & I = K = 4,\\ -\mu_{jl}, & I = K = 5,\\ -d_{ij} & I = 4,\ K = 5 \text{ or } I = 5,\ K = 4,\end{cases} \tag{4}$$

then extended displacement and stress components are defined as below

$$u_I = \begin{cases} u_i, & I = i = 1, 2, 3,\\ \varphi, & I = 4,\\ \phi, & I = 5,\end{cases} \qquad \sigma_{ij} = \begin{cases} \sigma_{ij}, & I = i = 1, 2, 3,\\ D_j, & I = 4,\\ B_j, & I = 5,\end{cases} \tag{5}$$

Eq.(3) can be expressed in a more compact form

$$\sigma_{Ij} = C_{IjKl}u_{K,l},\ \sigma_{Ij,j} = 0. \tag{6}$$

The 2D problem is described by the Cartesian coordinates x_1 and x_2. According to the extended Stroh formalism, the solution to (6) is in the form as

$$\mathbf{u} = (u_1, u_2, u_3, \varphi, \phi)^{\mathrm{T}} = \mathbf{a}f(x_1 + px_2), \tag{7}$$

where $\mathbf{a}$ is a five-dimensional vector; p is a complex number; $f(*)$ is an analytic function of its variable '*' and the superscript 'T' denotes the transpose of a matrix or vector. Substituting (7) into (6) results in the characteristic problem

$$[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}]\mathbf{a} = \mathbf{0}, \tag{8}$$

where the 5×5 matrix $\mathbf{R}$ and the 5×5 symmetric matrices $\mathbf{Q}$ and $\mathbf{T}$ are defined by

$$R_{IK} = C_{I1K2}, \quad Q_{IK} = C_{I1K1}, \quad T_{IK} = C_{I2K2}, \tag{9}$$

due to the vector $\mathbf{a}$ in (8) is non-zero, we have

$$\det\left[\mathbf{Q} + p\left(\mathbf{R} + \mathbf{R}^{\mathrm{T}}\right) + p^2\mathbf{T}\right] = 0. \tag{10}$$

As [Ting, 2000] shows the extended displacement $\mathbf{u}$ and the extended stress potential function $\boldsymbol{\psi}$ can be expressed as

$$\begin{aligned}\mathbf{u} &= (u_1, u_2, u_3, \varphi, \phi)^{\mathrm{T}} = 2\mathrm{Re}[\mathbf{A}\mathbf{f}(z)],\\ \boldsymbol{\psi} &= (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^{\mathrm{T}} = 2\mathrm{Re}[\mathbf{B}\mathbf{f}(z)],\end{aligned} \tag{11}$$

where 'Re' stands for the real part, and the constant matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\begin{aligned}&\mathbf{b}_I=(\mathbf{R}^{\mathrm{T}}+p_I\mathbf{T})\mathbf{a}_I=-p_I^{-1}(\mathbf{Q}+p_I\mathbf{R})\mathbf{a}_I,\\&\mathbf{A}=(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4,\mathbf{a}_5),\ \ \mathbf{B}=(\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3,\mathbf{b}_4,\mathbf{b}_5).\end{aligned}\tag{12}$$

In Eq.(11), the five-dimensional vector $\mathbf{f}(z)$ is formed by five arbitrary analytic functions $f_I(z_I)$ (I=1,2,3,4,5) as

$$\begin{aligned}&\mathbf{f}(z)=[f_1(z_1),f_2(z_2),f_3(z_3),f_4(z_4),f_5(z_5)]^{\mathrm{T}},\\&z_I=x_1+p_Ix_2,\ p_I=\alpha_I+\iota\beta_I,\ \beta_I>0,\end{aligned}\tag{13}$$

where imaginary unit $\iota=\sqrt{-1}$, furthermore, the extended stress function vector $\boldsymbol{\psi}$ is related to the extended stress components by

$$\sigma_{I1}=-\psi_{I,2},\quad \sigma_{I2}=\psi_{I,1}.\tag{14}$$

For convenient, we can use following eigenrelations to calculate the eigenvalue p_I, the constant matrix $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{N}\boldsymbol{\xi}=p\boldsymbol{\xi},\tag{15}$$

$$\begin{cases}\mathbf{N}=\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}}\end{bmatrix},\quad \boldsymbol{\xi}_I=\begin{pmatrix}\mathbf{a}_I\\ \mathbf{b}_I\end{pmatrix},\\ \mathbf{N}_1=-\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}},\quad \mathbf{N}_2=\mathbf{T}^{-1},\\ \mathbf{N}_3=\mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}-\mathbf{Q}.\end{cases}\tag{16}$$

Further consider the possible rigid-body displacement of the inclusion relative to the matrix to be $\boldsymbol{u}_{\text{rigid}}=\boldsymbol{u}_0+[\gamma x_1,\ -\gamma x_2,\ 0]^{\mathrm{T}}$, we have the continuity conditions of displacement and traction at the interface of the inclusion as

$$u_i^-=u_i^++u_i^*+u_i^{\text{rigid}},\quad n_i\sigma_{ij}^-=n_i\sigma_{ij}^+,\tag{17}$$

where the superscripts '+' and '–' denote quantities in and out of the inclusion, and n_α is the component of the outward normal $\mathbf{n}$ of the interface. Denote by s the arc coordinate of the boundary, there are relations

$$n_1\mathrm{d}s=\mathrm{d}x_2,\ n_2\mathrm{d}s=-\mathrm{d}x_1,\tag{18}$$

Furthermore, electromagnetic continuity conditions can be list as

$$\begin{cases}\mathbf{n}\cdot\mathbf{D}^-=\mathbf{n}\cdot\mathbf{D}^+,\\ \mathbf{n}\times\mathbf{E}^-=\mathbf{n}\times(\mathbf{E}^++\mathbf{E}^*),\\ \mathbf{n}\cdot\mathbf{B}^-=\mathbf{n}\cdot\mathbf{B}^+,\\ \mathbf{n}\times\mathbf{H}^-=\mathbf{n}\times(\mathbf{H}^++\mathbf{H}^*),\end{cases}\tag{19}$$

where, $E_i=-\varphi_{,i}$, $H_i=-\phi_{,i}$ are the components of electric and magnetic fields respectively, then following equations can be derived

$$\begin{cases}\mathrm{d}(\psi_I^--\psi_I^+)/\mathrm{d}s=0,\\ \mathrm{d}(\varphi^--\varphi^+-\varphi^*)/\mathrm{d}s=0,\\ \mathrm{d}(\phi^--\phi^+-\phi^*)/\mathrm{d}s=0,\end{cases}\tag{20}$$

thus we can obtain

$$\psi_I^-=\psi_I^++f_I^0;\ \varphi^-=\varphi^++\varphi^*+u_4^{\text{rigid}},\ \phi^-=\phi^++\phi^*+u_5^{\text{rigid}},\tag{21}$$

the rigid-body displacement $\mathbf{u}_{\text{rigid}}$, rotation angle γ and the integral constant $\mathbf{f}_0$ in (17) and (21) directly by confining the fields at the origin, say

$$\mathbf{u}_0=\mathbf{u}\left(0\right),\quad \mathbf{f}_0=\boldsymbol{\psi}\left(0\right),\quad \gamma=\frac{1}{2}\left[u_{1,2}\left(0\right)-u_{2,1}\left(0\right)\right],\tag{22}$$

which means if the inclusion is allowed to deform, the rigid-displacement and integration constant can be regarded as part of the total extended displacement and stress potential. It need not to be described separately, except in special cases such as rigid inclusion or voids.

In some practical problems, non-uniform eigenstrains must be taken into account in order to properly describe the effects of disturbances, see [Sharma and Sharma, 2003]. Without loss of generality, we assume that the non-uniform eigendisplacements take the following form

$$\mathbf{u}^*(x_1,x_2)=\mathbf{u}_{MN}^*\left(\frac{x_1}{a}\right)^M\left(\frac{x_2}{a}\right)^N,\tag{23}$$

where $\mathbf{u}_{MN}^{*}$ is a real constant vector with the dimension of generalized displacement, and a is the characteristic length of the inclusion. Combined (11) with (17) and (21), we have

$$\begin{aligned} \mathbf{Af}^{-}(\mathbf{t}) + \overline{\mathbf{Af}^{-}(\mathbf{t})} &= \mathbf{Af}^{+}(\mathbf{t}) + \overline{\mathbf{Af}^{+}(\mathbf{t})} + \mathbf{u}_{*}, \\ \mathbf{Bf}^{-}(\mathbf{t}) + \overline{\mathbf{Bf}^{-}(\mathbf{t})} &= \mathbf{Bf}^{+}(\mathbf{t}) + \overline{\mathbf{Bf}^{+}(\mathbf{t})}, \end{aligned} \tag{24}$$

where '$\overline{*}$' is the conjugate of '$*$'. By using the Stroh orthogonality relations [Chung and Ting, 1996]

$$\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \tag{25}$$

the decoupled Stroh eigenfunction relations can be obtained, where $\mathbf{1}$ is the 5×5 identity matrix.

$$\mathbf{f}^{-}(\mathbf{t}) = \mathbf{f}^{+}(\mathbf{t}) + \mathbf{B}^{\mathrm{T}}\mathbf{u}_{*}. \tag{26}$$

Eq.(26) is the degenerated Privalov problem [Muskhelishvili, 1963], which has the solution as

$$f_I(z_I) = \frac{1}{2\pi\imath} \sum_{p=0}^{M} \sum_{q=0}^{N} C_{MN}^{I,pq} \int_{\partial\omega_I} \frac{t^{p+q}\bar{t}^{M+N-p-q}}{t_I - z_I} \mathrm{d}t_I, \tag{27}$$

where t_I be the value of $z_I = x_1 + p_I x_2$ on the boundary $\partial\omega_I$ of transformed domain $\omega_I \doteq \{z_I = x_1 + p_I x_2 | z = x_1 + \imath x_2 \in \omega\}$, and

$$C_{MN}^{I,pq} = \sum_{J=0}^{5} B_{IJ} \mathbf{u}_{MN}^{*,J} a^{-(N+M)} 2^{-M} (2\imath)^{-N} \binom{M}{p} \binom{N}{q} (-1)^{N-q}. \tag{28}$$

Up to now, the integral solution is derived, which can be expressed in the form of a powerful series solution, and the detailed procedure is given in the Appendix.

2.3 Numerical examples and discussion

The extended stiffness matrix of the MEE material is taken from [Buroni and Sáez, 2010], setting $M = N = 1$, $\mathbf{u}_{MN}^{*} = [1, 0, 0, 0, 0]^{\mathrm{T}}$, $a = 1\mu\mathrm{m}$ in (23). The smooth pentagram is selected as the example shape, setting $R = 0.786$ in (2) to ensure that the area of smooth pentagram precisely equals to the area of a polygonal pentagram inscribed in the unit circle, and the boundary equation [Zou and He, 2018] of the pentagram inclusion is shown as below

$$t = 0.786 \begin{pmatrix} \eta + \frac{3\imath}{10}\eta^{-4} + \frac{13}{225}\eta^{-9} - \frac{4\imath}{125}\eta^{-14} - \frac{214}{11875}\eta^{-19} \\ + \frac{1231\imath}{93750}\eta^{-24} + \frac{20974}{2265625}\eta^{-29} - \frac{2908\imath}{390625}\eta^{-34} \\ - \frac{441199}{76171875}\eta^{-39} + \frac{95763\imath}{19531250}\eta^{-44} + \frac{6890609}{1708984375}\eta^{-49} \end{pmatrix}, \; |\eta| = 1. \tag{29}$$

By analyse the Fig.2, it can be observed that the displacement components, electric and magnetic potentials are all continuous along the inclusion boundary, and the results satisfy the interface conditions. Compared with [Shen et al., 2026], this approach is more efficient and has a broader range of applicability, as it avoids Taylor expansion and the differentiation of higher-order composite function.

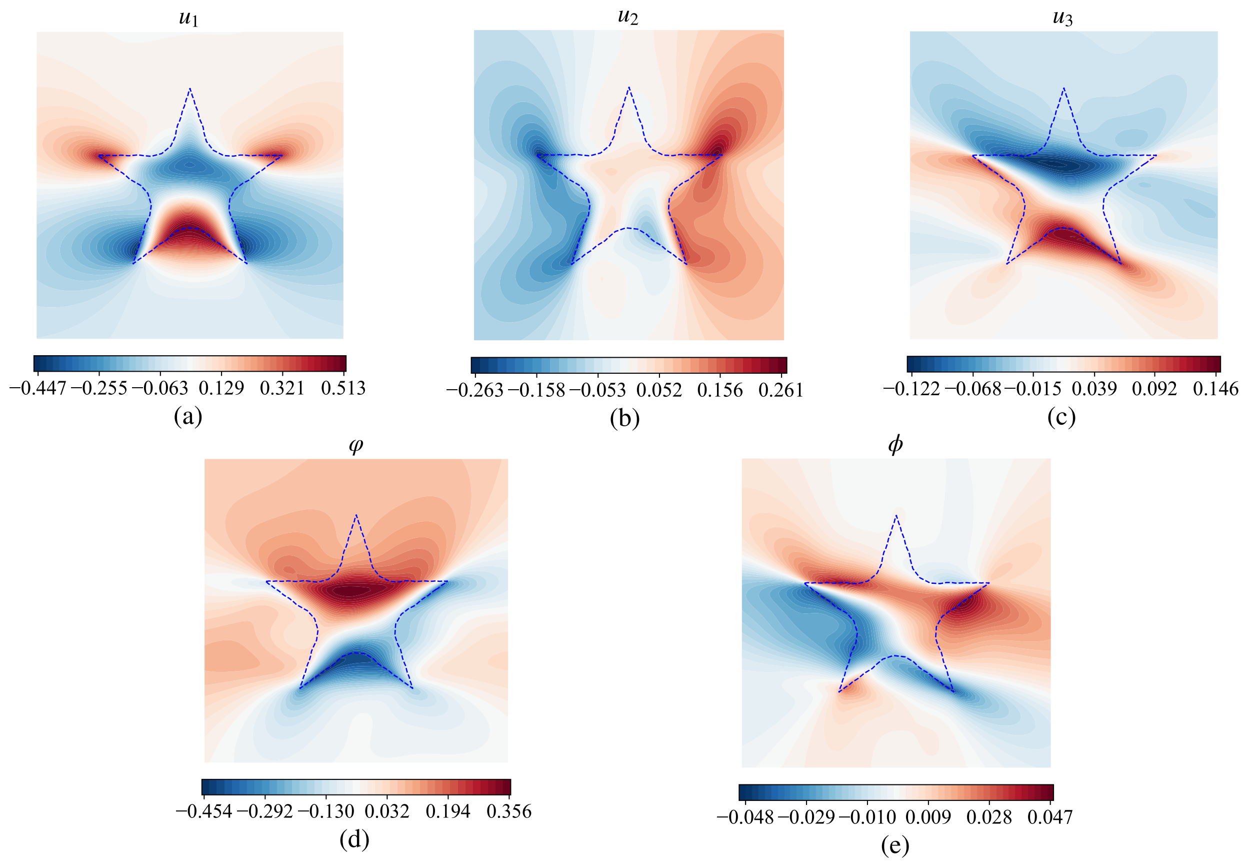

Figure 2: Displacement, electric and magnetic potentials of the smooth pentagram described by (29) disturbed by the extended nonuniform eigenstrain $\varepsilon_{11}^* = x_2,\ \varepsilon_{12}^* = 0.5x_1$ (a) u_1 (10 μm); (b) u_2 (10 μm); (c) u_3 (10 μm); (d) φ (10^5 V); (e) ϕ (10^3 A).

3 A smooth inclusion in an anisotropic bimaterial plane

For buried strained semiconductor devices, the thickness of the top barrier layer is typically much thinner than that of the underlying substrate. Therefore, modeling such buried devices as inclusions in a half-plane (rather than in a full plane) provides a more accurate representation of their actual operating conditions [Zou and Pan, 2012]. As is shown in Fig.3 we consider the more general case where two anisotropic MEE media $m2$, $m1$ occupying the lower half-plane S_2 ($x_2 < 0$) and upper half-plane S_1 ($x_2 > 0$), respectively. Furthermore, the lower half-plane S_2 contains an internal subdomain ω which undergoes the nonuniform eigenstrains. The quantities in S_1 and S_2 are denoted by the superscripts (or subscripts) '$m1$' and '$m2$', respectively. According the superposition model [Zou and Pan, 2012], the Stroh eigenfunction of the Eshelby's inclusion problem in the bimaterial plane can be expressed as

$$\mathbf{f}(z) = \mathbf{f}^{\infty}(z) + \mathbf{f}^{b}(z), \tag{30}$$

with $\mathbf{f}^{\infty}(z)$ denotes the corresponding Eshelby solution of inclusion in the full plane,as given in Section 2, and $\mathbf{f}^{b}(z)$ is a analytic function that needs to be solved. Across the interface $x_2 = 0$ the continuities of the extended displacement and stress potential are

$$\mathbf{u}_{m1} = \mathbf{u}_{m2}, \quad \boldsymbol{\psi}_{m1} = \boldsymbol{\psi}_{m2}, \quad \text{on} \quad x_2 = 0. \tag{31}$$

The interface conditions along the curve $\partial\omega$ in the lower half-plane are similar to (26), and due to the analyticity of $\mathbf{f}^{b}(z)$ we have

$$\mathbf{f}_{-}^{b}(t) = \mathbf{f}_{+}^{b}(t), \quad t \in \partial\omega. \tag{32}$$

Substitution of (11) and (32) into (31) yields

$$\begin{cases} \mathbf{A}_{m1}\mathbf{f}_{m1}^{b}(x_1) + \bar{\mathbf{A}}_{m1}\overline{\mathbf{f}_{m1}^{b}(x_1)} = \mathbf{A}_{m2}\mathbf{f}_{m2}^{b}(x_1) + \bar{\mathbf{A}}_{m2}\overline{\mathbf{f}_{m2}^{b}(x_1)} + \mathbf{F}_{A}^{\infty}(x_1), \\ \mathbf{B}_{m1}\mathbf{f}_{m1}^{b}(x_1) + \bar{\mathbf{B}}_{m1}\overline{\mathbf{f}_{m1}^{b}(x_1)} = \mathbf{B}_{m2}\mathbf{f}_{m2}^{b}(x_1) + \bar{\mathbf{B}}_{m2}\overline{\mathbf{f}_{m2}^{b}(x_1)} + \mathbf{F}_{B}^{\infty}(x_1); \end{cases} \tag{33}$$

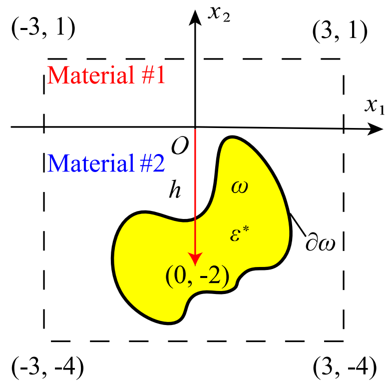

Figure 3: A smooth inclusion in material #2 ($x_2 < 0$) of the bimaterial plane. The center of the inclusion is at (x_1, x_2) = (0, 2), and the domain for the numerical calculation is $-3 < x_1 < 3$, $-4 < x_2 < 1$.

where

$$\begin{cases} \mathbf{F}_A^\infty(x_1) = (\mathbf{A}_{m2} - \mathbf{A}_{m1})\mathbf{f}^\infty(x_1) + (\bar{\mathbf{A}}_{m2} - \bar{\mathbf{A}}_{m1})\overline{\mathbf{f}^\infty(x_1)}, \\ \mathbf{F}_B^\infty(x_1) = (\mathbf{B}_{m2} - \mathbf{B}_{m1})\mathbf{f}^\infty(x_1) + (\bar{\mathbf{B}}_{m2} - \bar{\mathbf{B}}_{m1})\overline{\mathbf{f}^\infty(x_1)}. \end{cases} \tag{34}$$

Then, by virtue of the Stroh orthogonality relations (25), it results in

$$\begin{aligned} &\mathbf{C}_2\mathbf{f}_{m1}^b(x_1) + \mathbf{D}_2\overline{\mathbf{f}_{m1}^b(x_1)} = \mathbf{f}_{m2}^b(x_1) + (\mathbf{1} - \mathbf{C}_2)\mathbf{f}^\infty(x_1) - \mathbf{D}_2\overline{\mathbf{f}^\infty(x_1)}, \\ &\mathbf{f}_{m1}^b(x_1) = \mathbf{C}_1\mathbf{f}_{m2}^b(x_1) + \mathbf{D}_1\overline{\mathbf{f}_{m2}^b(x_1)} + (\mathbf{C}_1 - 1)\mathbf{f}^\infty(x_1) + \mathbf{D}_1\overline{\mathbf{f}^\infty(x_1)}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \mathbf{C}_1^{\mathrm{T}} &= \mathbf{C}_2 = \mathbf{B}_{m2}^{\mathrm{T}}\mathbf{A}_{m1} + \mathbf{A}_{m2}^{\mathrm{T}}\mathbf{B}_{m1}, \\ \overline{\mathbf{D}}_1^{\mathrm{T}} &= \mathbf{D}_2 = \mathbf{B}_{m2}^{\mathrm{T}}\overline{\mathbf{A}}_{m1} + \mathbf{A}_{m2}^{\mathrm{T}}\overline{\mathbf{B}}_{m1}, \end{aligned} \tag{36}$$

Using the Cauchy formulae and the properties $\mathbf{f}_{m1}^b(\infty) = \mathbf{f}_{m2}^b(\infty) = 0$, we can derive that

$$\left.\begin{aligned} \mathbf{f}_+^b(z) &= \tfrac{C_2^{-1}-1}{2\pi\iota}\int_{-\infty}^{\infty}\tfrac{\mathbf{f}^\infty(t)}{t-z}\mathrm{d}t - \tfrac{C_2^{-1}D_2}{2\pi\iota}\int_{-\infty}^{\infty}\tfrac{\overline{\mathbf{f}^\infty(t)}}{t-z}\mathrm{d}t,\ z \in S_1, \\ \mathbf{f}_-^b(z) &= \tfrac{1-C_1^{-1}}{2\pi\iota}\int_{-\infty}^{\infty}\tfrac{\mathbf{f}^\infty(t)}{t-z}\mathrm{d}t + \tfrac{C_1^{-1}D_1}{2\pi\iota}\int_{-\infty}^{\infty}\tfrac{\overline{\mathbf{f}^\infty(t)}}{t-z}\mathrm{d}t,\ z \in S_2, \end{aligned}\right\} \tag{37}$$

with

$$\begin{aligned} &\frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{f_I^\infty(t)}{t-z}\mathrm{d}t = \begin{cases} 0,\ z \in S_2, \\ \frac{1}{2\pi\iota}\sum_{p=0}^{M}\sum_{q=0}^{N}C_{MN}^{I,pq}\oint_{\partial\omega_I}\frac{t^{p+q}\bar{t}^{M+N-p-q}}{t_I-z}\mathrm{d}t_I,\ z \in S_1, \end{cases} \\ &\frac{1}{2\pi\iota}\int_{-\infty}^{\infty}\frac{\overline{f_I^\infty(t)}}{t-z}\mathrm{d}t = \begin{cases} \frac{-1}{2\pi\iota}\sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{I,pq}}\oint_{\partial\omega_I}\frac{\bar{t}^{p+q}t^{M+N-p-q}}{\overline{t_I}-z}\mathrm{d}\overline{t_I},\ z \in S_2, \\ 0,\ z \in S_1. \end{cases} \end{aligned} \tag{38}$$

4 A smooth inclusion in an isotropic disk

4.1 Description of the problem and the superposition principle

As is shown in Fig.4, symbol Ω represents a finite domain in a two dimensional Euclidean space, its boundary is denoted by $\partial\Omega$, and its elastic moduli is C_{ijkl}. A subdomain ω of Ω is now subjected to a nonuniform eigenstrain field ε_{ij}^*. Assuming the boundary $\partial\Omega$ of Ω is free of displacements or tractions, the governing equations and boundary conditions can be expressed as

$$C_{ijkl}u_{k,lj} = C_{ijkl}\varepsilon_{kl,j}^p \quad \text{in } \Omega, \quad M_{ij}u_j = 0 \quad \text{on } \partial\Omega, \tag{39}$$

with

$$\varepsilon_{kl}^p = \varepsilon_{kl}^*\chi^\omega, \quad M_{ij} = \begin{cases} \delta_{ij} & \text{if } \partial\Omega \text{ is free of displacements,} \\ C_{ilkj}n_l\partial_k & \text{if } \partial\Omega \text{ is free of tractions.} \end{cases} \tag{40}$$

Above, χ^ω is the characteristic function of ω, namely, $\chi^\omega(\mathbf{x}) = 1$ for $\mathbf{x} \in \omega$ and $\chi^\omega(\mathbf{x}) = 0$ for $\mathbf{x} \notin \omega$. Similar to Section 3, based on the superposition principle [Zou et al., 2012], the displacement of this problem can be decomposed into

$$u_i = u_i^a + u_i^b, \tag{41}$$

here the displacement fields u_i^a and u_i^b satisfy the following respective equations

$$C_{ijkl}u^a_{k,lj} = C_{ijkl}\varepsilon^p_{kl,j} \quad \text{in } \Omega, \quad M_{ij}u^a_j = g_i \quad \text{on } \partial\Omega, \tag{42}$$

$$C_{ijkl}u^b_{k,lj} = 0 \quad \text{in } \Omega, \quad M_{ij}u^b_j = -g_i \quad \text{on } \partial\Omega, \tag{43}$$

where g_i is an arbitrary function defined on $\partial\Omega$.

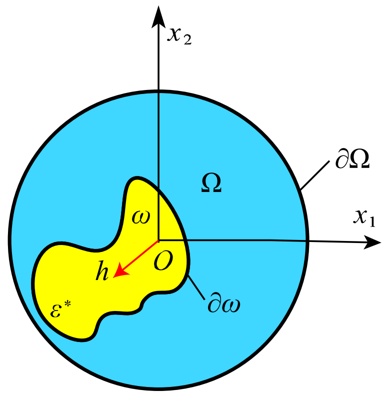

Figure 4: Illustration of an arbitrary shaped and located inclusion ω in a disk domain Ω.

In fact, to solve this problem, we can set $u_i^a = u_i^\infty$ where u_i^∞ is the solution of the corresponding full plane Eshelby's problem under the same conditions.

$$C_{ijkl}u^b_{k,lj} = 0 \quad \text{in } \Omega, \quad M_{ij}u^b_j = -M_{ij}u^\infty_j \quad \text{on } \partial\Omega, \tag{44}$$

then, combined with (42), (43) and (44), u_i^b can be solved. The schematic diagram of superposition principle is given in Fig.5

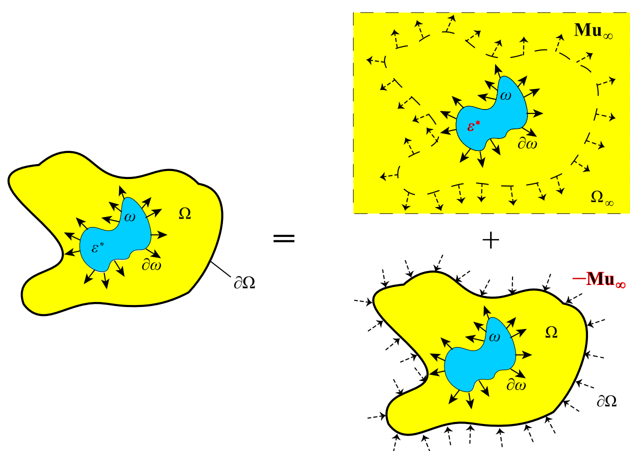

Figure 5: The decomposition of the Eshelby's problem that a smooth inclusion is embedded in an isotropic finite domain.

4.2 Derivation of the integral solution for this problem

Due to the fact that material of Ω is linearly elastic and isotropic, the stiffness matrix of Ω has only two independent parameters, namely, Young's modulus Y and Poisson's ratio ν. The Kolosov-Muskhelishvili potentials γ and ψ is used to represent the solution to the problem [Muskhelishvili, 1963], which are arbitrary analytic functions. Then, the displacement, stress and strain components can be expressed as following equations.

$$u_1 + \imath u_2 = \frac{1}{2\mu}\left[\kappa\gamma(z) - z\overline{\gamma'(z)} - \overline{\psi(z)}\right], \tag{45}$$

$$\begin{cases}\sigma_{11}+\sigma_{22}=2\left[\gamma'(z)+\overline{\gamma'(z)}\right],\\ \sigma_{22}-\sigma_{11}+2\iota\sigma_{12}=2[\bar{z}\gamma''(z)+\psi'(z)],\end{cases} \tag{46}$$

$$\begin{cases}\varepsilon_{11}+\varepsilon_{22}=\frac{\kappa-1}{4\mu}\left[\gamma'(z)+\overline{\gamma'(z)}\right],\\ \varepsilon_{22}-\varepsilon_{11}+2\iota\varepsilon_{12}=\frac{1}{\mu}[\bar{z}\gamma''(z)+\psi'(z)],\end{cases} \tag{47}$$

where

$$\mu=\frac{Y}{2(1+\nu)},\quad \kappa=\begin{cases}\frac{3-\nu}{1+\nu}, & \text{plane stress;}\\ 3-4\nu, & \text{plane strain.}\end{cases} \tag{48}$$

The traction vector $f=f_1+\iota f_2$ on the boundary $\partial\Omega$ is given by

$$f=-\iota\frac{\mathrm{d}}{\mathrm{d}s}\left[\gamma(z)+z\overline{\gamma'(z)}+\overline{\psi(z)}\right]. \tag{49}$$

By virtue of the superposition principle, the potential functions can be decomposed as

$$\gamma=\gamma_\infty+\gamma_b,\quad \psi=\psi_\infty+\psi_b, \tag{50}$$

where γ_∞ and ψ_∞ are the potentials of the corresponding full plane Eshelby's problem under the same conditions, while γ_b and ψ_b are the potentials to be solved for the auxiliary boundary value problem formulated by (43). The following boundary equations are obtained depending on whether the boundary constraints are Dirichlet or Neumann type.

$$\kappa\gamma_b(\tau)-\tau\overline{\gamma_b'(\tau)}-\overline{\psi_b(\tau)}=-\kappa\gamma_\infty(\tau)+\tau\overline{\gamma_\infty'(\tau)}+\overline{\psi_\infty(\tau)}, \tag{51}$$

$$-\iota\frac{\mathrm{d}}{\mathrm{d}s}\left[\gamma_b(\tau)+\tau\overline{\gamma_b'(\tau)}+\overline{\psi_b(\tau)}\right]=\iota\frac{\mathrm{d}}{\mathrm{d}s}\left[\gamma_\infty(\tau)+\tau\overline{\gamma_\infty'(\tau)}+\overline{\psi_\infty(\tau)}\right]. \tag{52}$$

An auxiliary function can be introduced

$$g(\tau;\eta_0)=-\eta_0\gamma_\infty(\tau)+\tau\overline{\gamma_\infty'(\tau)}+\overline{\psi_\infty(\tau)}=\eta_0\gamma_b(\tau)-\tau\overline{\gamma_b'(\tau)}-\overline{\psi_b(\tau)},\quad \tau\in\partial\Omega, \tag{53}$$

with $\eta_0=\kappa$ for the Dirichlet case and $\eta_0=1$ for the Neumann case. Since the potentials γ_b and ψ_b are analytical, there exists a analytic function ϑ on $\partial\Omega$, and the potentials can be expressed as

$$\begin{aligned}\gamma_b(z)&=\frac{1}{2\pi\iota}\oint_{\partial\Omega}\frac{\vartheta(t)}{t-z}\mathrm{d}t,\\ \psi_b(z)&=-\frac{\eta_0}{2\pi\iota}\oint_{\partial\Omega}\frac{\overline{\vartheta(t)}}{t-z}\mathrm{d}t-\frac{1}{2\pi\iota}\oint_{\partial\Omega}\frac{\bar{t}\vartheta'(t)}{t-z}\mathrm{d}t.\end{aligned} \tag{54}$$

Then, (53) can rewritten as

$$\eta_0\vartheta(t)+\frac{\eta_0}{2\pi\iota}\oint_{\partial\Omega}\vartheta(\tau)\mathrm{d}\left(\ln\frac{t-\tau}{\bar{t}-\bar{\tau}}\right)+\frac{1}{2\pi\iota}\oint_{\partial\Omega}\overline{\vartheta(\tau)}\mathrm{d}\left(\frac{t-\tau}{\bar{t}-\bar{\tau}}\right)=g(\tau;\eta_0). \tag{55}$$

As stated in [Muskhelishvili, 1963], the solution to (43) can be expressed as

$$\begin{aligned}\gamma_b(z)&=\frac{1}{2\pi\iota\eta_0}\oint_{\partial\Omega}\frac{g(\tau;\eta_0)\mathrm{d}\tau}{\tau-z}+a_1z,\\ \psi_b(z)&=-\frac{1}{2\pi\iota}\int_{\partial\Omega}\frac{\overline{g(\tau;\eta_0)\mathrm{d}\tau}}{\tau-z}-\frac{1}{2\pi\iota\eta_0}\int_{\partial\Omega}\frac{\tau g'(\tau;\eta_0)\mathrm{d}\tau}{\tau-z},\\ a_1&=-\frac{1}{\pi\eta_0(\eta_0-1)}\mathrm{Im}\int_{\partial\Omega}\frac{g(\tau;\eta_0)\mathrm{d}\tau}{\tau^2},\end{aligned} \tag{56}$$

where a_1 is a real constant. And according to [Lee et al., 2020], the solution of a smooth inclusion undergoing a nonuniform eigenstarin in a full isotropic plane is

$$\begin{aligned}\gamma_\infty(z)&=-\frac{1}{2\pi\iota}\sum_{p=0}^{M}\sum_{q=0}^{N}C_{MN}^{pq}\oint_{\partial\omega}\frac{t^{p+q}\bar{t}^{M+N-p-q}}{t-z}\mathrm{d}t,\\ \psi_\infty(z)&=\frac{1}{2\pi\iota}[\sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{\bar{t}^{p+q}t^{M+N-p-q}}{t-z}\mathrm{d}t\\ &\quad+\sum_{p=0}^{M}\sum_{q=0}^{N}C_{MN}^{pq}\oint_{\partial\omega}\frac{\bar{t}\mathrm{d}(t^{p+q}\bar{t}^{M+N-p-q})}{t-z}],\end{aligned} \tag{57}$$

where the nonuniform eigendisplacement and coefficient C_{MN}^{pq} are given as

$$
\begin{aligned}
u_1^p + \iota u_2^p &= \frac{\kappa+1}{2\mu}\frac{\alpha_{MN}+\iota\beta_{MN}}{M!N!}\left(\frac{x_1}{a}\right)^M\left(\frac{x_2}{a}\right)^N \\
&= (\kappa+1)\sum_{p=0}^{M}\sum_{q=0}^{N} C_{MN}^{pq} z^{p+q}\bar{z}^{M+N-p-q}, \\
C_{MN}^{pq} &= \frac{1}{2\mu}\frac{\alpha_{MN}+\iota\beta_{MN}}{M!N!}2^{-M}(2\iota)^{-N}\binom{M}{p}\binom{N}{q}(-1)^{N-q}.
\end{aligned} \tag{58}
$$

Substituting (57) into (53) yields

$$
\begin{aligned}
g(\tau;\eta_0) = &\frac{\eta_0}{2\pi\iota}\sum_{p=0}^{M}\sum_{q=0}^{N} C_{MN}^{pq}\oint_{\partial\omega}\frac{t^{p+q}\bar{t}^{M+N-p-q}}{t-\tau}\mathrm{d}t - \frac{\tau}{2\pi\iota}\sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{1}{\bar{t}-\bar{\tau}}\mathrm{d}(\bar{t}^{p+q}t^{M+N-p-q}) \\
&- \frac{1}{2\pi\iota}[\sum_{p=0}^{M}\sum_{q=0}^{N} C_{MN}^{pq}\oint_{\partial\omega}\frac{t^{p+q}\bar{t}^{M+N-p-q}}{\bar{t}-\bar{\tau}}\mathrm{d}\bar{t} + \sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{t\mathrm{d}(\bar{t}^{p+q}t^{M+N-p-q})}{\bar{t}-\bar{\tau}}],
\end{aligned} \tag{59}
$$

let the radius of the disk be R, then $\bar{\tau} = \frac{R^2}{\tau}$. Furthermore, the auxiliary potential functions can be derived as

$$
\begin{aligned}
\gamma_b(z) = &-\frac{\tau}{2\pi\iota}\sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{1}{\bar{t}-R^2/z}\mathrm{d}(\bar{t}^{p+q}t^{M+N-p-q}) - \frac{1}{2\pi\iota}[\sum_{p=0}^{M}\sum_{q=0}^{N} C_{MN}^{pq}\oint_{\partial\omega}\frac{t^{p+q}\bar{t}^{M+N-p-q}}{\bar{t}-R^2/z}\mathrm{d}\bar{t} \\
&+ \sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{t\mathrm{d}(\bar{t}^{p+q}t^{M+N-p-q})}{\bar{t}-R^2/z}] + a_1 z, \\
\psi_b(z) = &\frac{\eta_0}{2\pi\iota}\sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{\bar{t}^{p+q}t^{M+N-p-q}}{\bar{t}-R^2/z}\mathrm{d}\bar{t} + \frac{1}{2\pi\iota\eta_0}\sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{z^3\bar{t}-2R^2z^2}{(z\bar{t}-R^2)^2}\mathrm{d}(\bar{t}^{p+q}t^{M+N-p-q}) \\
&+ \frac{\eta_0}{2\pi\iota}[\sum_{p=0}^{M}\sum_{q=0}^{N} C_{MN}^{pq}\oint_{\partial\omega}\frac{-zR^2t^{p+q}\bar{t}^{M+N-p-q}}{(z\bar{t}-R^2)^2}\mathrm{d}\bar{t} + \sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\frac{-zR^2t\mathrm{d}(\bar{t}^{p+q}t^{M+N-p-q})}{(z\bar{t}-R^2)^2}], \\
a_1 = &\frac{1}{\pi\eta_0(\eta_0-1)} \\
&\times \mathrm{Im}\frac{[\sum_{p=0}^{M}\sum_{q=0}^{N}\overline{C_{MN}^{pq}}\oint_{\partial\omega}\bar{t}^{p+q}t^{M+N-p-q}\mathrm{d}t + \sum_{p=0}^{M}\sum_{q=0}^{N} C_{MN}^{pq}\oint_{\partial\omega}\bar{t}\mathrm{d}(t^{p+q}\bar{t}^{M+N-p-q})]}{2\pi\iota},
\end{aligned} \tag{60}
$$

5 Conclusion

Based on [Shen et al., 2026, Zou and Pan, 2012, Zou et al., 2012], this paper presents a simple yet efficient method to calculate the solutions for the physical fields of a smooth inclusion undergoing a nonuniform eigenstrain perturbation in a full MEE plane, an anisotropic bimaterial plane or an isotropic disk. This method has the following features:

1. The method presented in the Appendix A is very concise and effective. Compared to the method in [Shen et al., 2026], it avoids Taylor expansion and the cumbersome differentiation of composite functions, and expresses the integral solution using a finite-term series. This makes the computational process highly efficient and independent of material parameter constraints.

2. Similarly, in comparison with [Zou et al., 2012], it still avoids the Taylor expansion process, which keeps the calculation simple. Moreover, when the inclusion is too close to the boundary of the disk, it can still perform rapid calculations without the issue of slow convergence.

3. Of course, compared with these references, since it does not use Faber series and its associated series, it cannot be expressed as a function of z or w. Therefore, it cannot iteratively handle inhomogeneity problems as in [Lee and Zou, 2025].

For problems solvable without iterative calculations, the series solution proposed herein demonstrates remarkable precision and simplicity.

Appendix A The powerful series solution

For the integral solutions (27), (38) and (60) to the three problems listed above, a powerful series solution is presented in this Appendix A, which is more efficient and has a wider ranger of application than the method in [Zou et al., 2012, Shen et al., 2026]. Since the inclusion boundary is described by (1), the contour integral in these integral solutions (27), (38) and (60) can ultimately be reduced to the following form

$$\frac{1}{2\pi\iota}\oint_{\partial\omega}\frac{P(t,\bar{t})}{t-z}\mathrm{d}t=\frac{C}{2\pi\iota}\oint_{\partial\omega_0}\frac{\sum_{i=-n_0}^{n_1}a_i\eta^i\mathrm{d}\eta}{\prod_{j=0}^{n_2}(\eta-c_j)^{b_j}},\ |\eta|=1, \tag{A1}$$

where $P(t,\bar{t})$, is a polynomial of t and $\bar{t}$, C is introduced to make the denominator a monic polynomial ($b_j>0$), the set of coefficients a^i can be obtained recursively by computer program, and b_j, c_j can be obtained by the same way. In general, we assume no repeated roots. When it do occur, a small perturbation of z can be applied to split the repeated roots into simple ones, except for the integral in Section 4, where the dominator is already a power of a polynomial like $\frac{1}{(z\bar{t}-R^2)^2}$.

To facilitate the caculation, (A1) can be further decomposed as

$$\frac{1}{\prod_{j=0}^{n_2}(\eta-c_j)^{b_j}}=\sum_{j=0}^{n_2}\sum_{l=1}^{a_j}\frac{d_{jl}}{(\eta-c_j)^l},\ d_{jl}=\frac{1}{(b_j-l)!}\lim_{\eta\to c_j}\frac{\mathrm{d}^{b_j-l}}{\mathrm{d}\eta^{b_j-l}}\frac{(\eta-c_j)^{b_j}}{\prod_{j=0}^{n_2}(\eta-c_j)^{b_j}}. \tag{A2}$$

Subsequently, (A1) can be expressed as a series with a finite number terms as

$$\frac{C}{2\pi\iota}\oint_{\partial\omega_0}\sum_{i=-n_0}^{n_1}\sum_{j=0}^{n_2}\sum_{l=1}^{a_j}\frac{a_i d_{jl}\eta^i}{(\eta-c_j)^l}\mathrm{d}\eta, \tag{A3}$$

this integral can be evaluated quite straightforwardly via the residue theorem.

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