Stochastic Dominance and Generalized Risk-Dominance for 2×2 Coordination Games

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Abstract

We provide a generalization of risk-dominance based on an elementary type of stochastic dominance vis-a-vis the uniform probability distribution function, to partially resolve the conflict between pay-off dominance and risk-dominance in coordination games. Such a generalization, unlike risk-dominance, is compatible with equilibrium action profile in coordination games like "battle of the sexes". Our generalization allows players to conjecture the choice of randomization by the other player, that includes but is not restricted to the uniform distribution function.

Keywords: two player, two actions, coordination games, pay-off dominance, risk-dominance with respect to, almost uniform probability distribution function, stochastic dominance

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1. Introduction: Coordination games, even in the case of simple 2×2 bi-matrix that we consider here, has been a major cause of concern, primarily because of the existence of multiple equilibrium action profiles. The formal analysis of such games can be traced back to the work of Harsanyi and Selten (1988), although it is quite possible that other scholars may have recognized the dilemma in informal discussions and correspondences among themselves.

Harsanyi and Selten (1988) postulated two criteria to distinguish between two different equilibrium action profiles: pay-off dominance and risk-dominance. If one equilibrium action profile pays more to both players than a second action profile does, then the first is said to pay-off dominate the second. On the other hand, if the probability with which an action yields greater expected pay-off than another action does is greater than $\frac{1}{2}$, *assuming the other player's probability of choosing each action* *is uniformly distributed over the interval* [0, 1], then the former action is said to risk dominate the latter action. A pair of actions, where each action is a risk dominant action for a player, is said to be a risk-dominant action profile. Harsanyi and Selten (1988) prioritizes pay-off dominance over risk-dominance, although in a later paper- Harsanyi (1995)- the priorities are reversed. In any case, it can be safely conjectured that in the absence of a pay-off dominant equilibrium action profile, the recommendation in Harsanyi and Selten (1988) would be to use the risk-dominance criterion. Some empirical analysis that shows both pay-off dominance and risk-dominance are possible is available in Jagau (2024).

If reducing the choice problem to a single criterion, resolved the problem of choice between multiple equilibrium action profiles, that would indeed be welcome. If however, instead of resolving the problem of choice between multiple equilibrium action profiles, it introduces the possibility of landing up at an action profile that is not an equilibrium, as for instance in the case of "battle of the sexes", then there is a need for rethinking about the decision making procedure. That is precisely what we do here.

Risk-dominance is based on the assumption, that each player believes that the other player follows a uniform probability distribution while choosing its probability of choosing any action. But, why must a player have such beliefs about the behavior of the other player? If based on familiarity with the tastes, preferences and temperament of the other player, there is reason to believe that the other player is "strongly biased" towards an action, then it should be more appropriate to choose a probability distribution about the choice probabilities of that action for the other player, that "*stochastically dominates*" the uniform probability distribution function. Stochastic dominance by one probability distribution function of another means that lower values get less weight in the first probability distribution function than in the second probability distribution function and higher values get more weight in the first probability distribution than in the second probability distribution function. Based on such an assumption, the best reply of the player who conjectures such a probability distribution function about the other player, may be compatible with an equilibrium action profile.

Thus, for instance with two actions {1, 2}, if a player believes that the other player strongly prefers action j over action 3-j, then it seems reasonable to assume that the former would conjecture a probability distribution function for the choice probability of action j by the other player, that stochastically dominates the uniform probability distribution function for the choice probability of action j. If the distribution function is chosen "appropriately" the best reply of the former to action j has greater probability of being chosen than best reply of the former to action 3-j. This kind of thinking resembles the kind of reasoning that takes place in the "ultimatum game" wherein a player (tacitly?) "conveys" to the other player a preference or a stance, and assumes that the latter's conjectures about the former are informed choices that are in the latter's best interests.

Allowing for probability distribution functions that exhibit the "simplest type of stochastic dominance" over the uniform probability distribution function, solves the problem that risk-dominance confronts, when applied to situations such as "battle of the sexes". Thus, the main contribution here, is a generalization of risk-dominance, that allows wider and intuitively greater applicability consistent with equilibrium, than what is possible with risk-dominance.

In order to allow the kind of stochastic dominance we require, we consider probability distribution functions which has a constant positive "probability density" on $[0, \frac{1}{2}]$ and a *possibly different* constant probability density on $(\frac{1}{2}, 1]$. We refer to such a probability distribution functions as an "almost uniform" probability distribution function. The set of such probability distribution functions include the uniform probability distribution function. An almost uniform probability distribution function stochastically dominates the uniform probability distribution function "if and only if" the former's probability density on $[0, \frac{1}{2}]$ is strictly less than 1 and is stochastically dominated by the uniform probability distribution function "if and only if" the former's probability density on $[0, \frac{1}{2}]$ is strictly greater than 1.

It ought to be noted, that the magnitude of the probability density function on $[0, \frac{1}{2}]$ needs to be suitably chosen, in order to arrive at the desired equilibrium action profile. Thus, the stochastic dominance that leads to the desired solution is not unconditional. There is a non-degenerate open interval in (0, 2) with either 1 as an upper bound or with 1 as a lower bound in which the value of the conjectured density function on $[0, \frac{1}{2}]$ needs to belong so that an equilibrium action profile is realized as "best reply" for each player in response to the choice of strategy by the other player. Hence, the concept of generalized risk-dominance introduced here is a "justification based on stochastic dominance" for the desired equilibrium action profile. Stochastic dominance is not an unrestricted criterion for decision making in our context. It requires restrictions on the conjectures, much the same way that "rational expectations" requires of agents in a market. However, in our context there is considerably greater flexibility for compatible conjectures than what is required for the parameters used in models based on "rational expectations". Thus, the restriction we require on conjectures is considerably milder than the restrictions required for rational expectations.

2. The Framework of 2×2 Bi-matrix Games: Let the row player be denoted by 1, the column player by 2 and the pay-off matrix of player $h \in \{1,2\}$ be G_h =

$$\begin{bmatrix} x_h(1,1) & x_h(1,2) \\ x_h(2,1) & x_h(2,2) \end{bmatrix}.$$

The action set of the each player is {1, 2} and a pair $(i, j) \in \{1, 2\} \times \{1, 2\}$,with 'i' denoting the action (row) chosen by the row player (1) and 'j' denoting the action (column) chosen by the column player (2), is referred to as an **action profile**.

The pair (G_1, G_2) is referred to as a **2×2 bi-matrix game**.

Some times it may be convenient to represent (G_1, G_2) as

$$\begin{bmatrix} (x_1(1,1), x_2(1,1)) & (x_1(1,2), x_2(1,2)) \\ (x_1(2,1), x_2(2,1)) & (x_1(2,2), x_2(2,2)) \end{bmatrix}.$$

A **strategy for the row player** is a real number $p_1 \in [0, 1]$ such p_1 is the probability with which the row player chooses action 1 and 1- p_1 is the probability with which the row player chooses action 2.

A **strategy for the column player** is a real number $q_1 \in [0, 1]$ such q_1 is the probability with which the column player chooses action 1 and 1- q_1 is the probability with which the row player chooses action 2.

Note 1: If $p_1 = 1$, then the strategy p_1 is "action 1" and if $p_1 = 0$, then the strategy "p_1 is action 2". If $q_1 = 1$, then the strategy q_1 is action 1 and if $q_1 = 0$, then the strategy q_1 is action 2. ▌

A pair (p_1, q_1) where p_1 is a strategy for the row player and q_1 is a strategy for the column player is a **strategy profile**.

Note 2: If $(p_1, q_1) \in \{0, 1\} \times \{0, 1\}$ then it is an action profile (i, j), where i is chosen with probability 1 by the row player and column j is chosen with probability 1 by the column player. In such a situation (p_1, q_1) is often represented by the pair (i, j). ∎

For $h \in \{1, 2\}$ and $j \in \{1, 2)$, let $G_{h(j)}$ denote the j^{th} row of G_h and $G_h^{(j)}$ denote the j^{th} column of G_h.

A strategy profile (p_1^*, q_1^*) is said to be an **equilibrium strategy profile** for (G_1, G_2) if : $(p_1^*, 1- p_1^*)\, G_1 \begin{pmatrix} q_1^* \\ 1 - q_1^* \end{pmatrix} \geq G_{1(j)} \begin{pmatrix} q_1^* \\ 1 - q_1^* \end{pmatrix}$ for $j \in \{1, 2\}$ and $(p_1^*, 1- p_1^*)\, G_2 \begin{pmatrix} q_1^* \\ 1 - q_1^* \end{pmatrix} \geq (p_1^*, 1- p_1^*) G_2^{(j)}$ for $j \in \{1, 2\}$.

If an equilibrium strategy profile $(p_1^*, q_1^*) \in \{0, 1\} \times \{0, 1\}$ then it is said to be an **equilibrium action profile**.

Consider the 2×2 bi-matrix game (G_1, G_2) =

$$\begin{bmatrix} (x_1(1,1), x_2(1,1)) & (x_1(1,2), x_2(1,2)) \\ (x_1(2,1), x_2(2,1)) & (x_1(2,2), x_2(2,2)) \end{bmatrix}.$$

It is well known that the set of equilibrium strategy profiles of (G_1, G_2) is equal to the set of equilibrium strategy profiles of $(D(G_1), D(G_2))$ =

$$\begin{bmatrix} (x_1(1,1) - x_1(2,1), x_2(1,1) - x_2(1,2)) & (0, 0) \\ (0, 0) & (x_1(2,2) - x_1(1,2), x_2(2,2) - x_2(2,1)) \end{bmatrix},$$

i.e., (G_1, G_2) is *strategically equivalent* to $(D(G_1), D(G_2))$.

For $h \in \{1, 2\}$ and $j \in \{1, 2)$, let $D_{(j)}(G_h)$ denote the j^{th} row of $D(G_h)$ and $D^{(j)}(G_h)$ denote the j^{th} column of $D(G_h)$.

Important observation: Note that for $\alpha \in \{1, 2\}$, $\{1, 2\} \setminus \{\alpha\} = \{3 - \alpha\}$. ∎

It is easy to see that $\{q_1 \in [0, 1] \mid G_{1(j)} \begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix} > G_{1(3-j)} \begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix}\} = \{q_1 \in [0, 1] \mid D_{(j)}(G_1) \begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix} > D_{(3-j)}(G_1) \begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix}\}$ for $j \in \{1, 2\}$

<u>&</u>

$\{p_1 \in [0, 1] \mid (p_1 \quad | \quad (1 - p_1)) G_2^{(j)} > (p_1 \quad | \quad (1 - p_1)) G_2^{(3-j)}\} = \{p_1 \in [0, 1] \mid$
$(p_1 \quad | \quad (1 - p_1)) D^{(j)}(G_2) > (p_1 \quad | \quad (1 - p_1)) D^{(3-j)}(G_2)\}$
$(p_1 \quad | \quad (1 - p_1)) D^{(j)}(G_2)$.

Further, for $j \in \{1, 2\}$, $\{q_1 \in [0, 1] \mid G_{1(j)} \begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix} > G_{1(3-j)} \begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix}\}$ and $\{p_1 \in [0, 1] \mid (p_1 \quad | \quad (1 - p_1)) G_2^{(j)} > (p_1 \quad | \quad (1 - p_1)) G_2^{(3-j)}\}$ are sub-intervals of [0, 1].

This follows directly from the convexity of each of the four sets.

Action j$\in$$\{1, 2\}$ is said to be *risk dominant for the row player* if the length of the interval $\{q_1 \in [0, 1] | \ G_{1(j)}\begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix} > G_{1(3-j)}\begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix}\} > \frac{1}{2}$.

Action j$\in$$\{1, 2\}$ is said to be *risk dominant for the column player* if the length of the interval $\{p_1 \in [0, 1] | \ (p_1 \quad | \quad (1-p_1))G_2^{(j)} > (p_1 \quad | \quad (1-p_1))G_2^{(3-j)}\} > \frac{1}{2}$.

An action profile (i, j) is said to be a *risk-dominant action profile* if i is risk dominant for the row player and j is risk dominant for the column player.

Note 3: For h$\in$$\{1, 2\}$, action j is risk dominant for player 'h' in the game (G_1, G_2) <u>if and only if</u> it is risk dominant for player 'h' in the game ($D(G_1)$, $D(G_2)$). Clearly action profile (i, j) is a risk-dominant action profile in the game (G_1, G_2) <u>if and only if</u> (i, j) is a risk-dominant action profile in the game ($D(G_1)$, $D(G_2)$).

The concepts of risk-dominance introduced above are due to Harsanyi and Selten (1988). ∎

An equilibrium action profile (i, j) is said to be a *pay-off dominant equilibrium action profile* if <u>there does not exist</u> an equilibrium action profile (i^+, j^+) different from it (i.e., $(i^+, j^+) \neq (i, j)$) such that $x_h(i^+, j^+) \geq x_h(i, j)$ for h$\in$$\{1, 2\}$, with strict inequality for at least one 'h'.

3. Implications of Risk-Dominance: Olcina and Urbano (1993), appeal to the fact that the set of equilibrium strategy profiles of (G_1, G_2) are equivalent to the set of equilibrium strategy profiles of ($D(G_1)$, $D(G_2)$) and provide a "learning dynamics" that converges to the equilibrium action profile that is risk dominant. However, such a "diagonalization" blurs the conflict between "risk-dominance" and "pay-off dominance".

Example 1: Consider a specific example of "Stag Hunt":

$$(G_1, G_2) = \begin{bmatrix} (5, 5) & (0, 4) \\ (4, 0) & (2, 2) \end{bmatrix}.$$

(G_1, G_2) has two equilibrium action profiles, namely (1, 1) and (2, 2), and an equilibrium strategy profile in $(0, 1)\times(0, 1)$, namely, $(\frac{2}{3}, \frac{2}{3})$.

It is easily verified that action profile (1, 1) is a pay-off dominant equilibrium action profile, and action profile (2, 2) is a risk dominant action profile.

It is this conflict between "risk-dominance" and "pay-off dominance" that led to the work reported in Harsanyi and Selten (1988). In Harsanyi and Selten (1988) it is conjectured that the pay-off dominant equilibrium action profile would be chosen, but subsequently, in Harsanyi (1995), it is argued that the risk dominant action profile would be chosen.

However, if we consider (D(G_1), D(G_2)) = $\begin{bmatrix} (1, 1) & (0, 0) \\ (0, 0) & (2, 2) \end{bmatrix}$, then action profile (2, 2) is a pay-off dominant equilibrium action profile as well as a risk dominant action profile.

In fact, the observation can be generalized as follows.

Proposition 1: Let (G_1, G_2) = $\begin{bmatrix} (a_1, b_1) & (0, 0) \\ (0, 0) & (a_2, b_2) \end{bmatrix}$ be a 2×2 bi-matrix game satisfying $a_1, b_1, a_2, b_2 > 0$. Then,

(i) (G_1, G_2) has two equilibrium action profiles, namely (1, 1) and (2, 2) and one equilibrium strategy profile in (0, 1)×(0, 1), namely, $(\frac{a_2}{a_1 + a_2}, \frac{b_2}{b_1 + b_2})$. It does not have any equilibrium strategy profile in ((0, 1)×{0, 1})∪({0, 1}×(0, 1)).

(ii) action 1 is risk dominant for the row player <u>if and only if</u> $a_1 > a_2$.

(iii) action 1 is risk dominant for the column player <u>if and only if</u> $b_1 > b_2$.

(iv) action 2 is risk dominant for the row player <u>if and only if</u> $a_2 > a_1$.

(v) action 2 is risk dominant for the column player <u>if and only if</u> $b_2 > b_1$.

Proof: (i) is well known.

We will prove (ii), the remaining proofs being similar.

Action 1 is risk dominant for the row player <u>if and only if</u> the length of the interval $\{q_1 \in [0, 1] \mid q_1 a_1 > (1 - q_1) a_2\} > \frac{1}{2}$.

This condition is equivalent to the condition that the length of the interval $\{q_1 \in [0, 1] \mid q_1 > \frac{a_2}{a_1 + a_2}\} > \frac{1}{2}$, i.e., the length of the interval $(\frac{a_2}{a_1 + a_2}, 1] > \frac{1}{2}$.

The length of the interval $(\frac{a_2}{a_1 + a_2}, 1]$ is $1 - \frac{a_2}{a_1 + a_2} = \frac{a_1}{a_1 + a_2}$.

$\frac{a_1}{a_1 + a_2} > \frac{1}{2}$ <u>if and only if</u> $a_1 > a_2$.

Thus, action 1 is risk dominant for the row player <u>if and only if</u> $a_1 > a_2$. Q.E.D.

4. Generalized Risk-Dominance for 2×2 Games: We now generalize the concept of "risk-dominance", so that the conflict between "risk-dominance" and "pay-off dominance" can be recovered using this generalized notion of risk-dominance.

For $h \in \{1, 2\}$, F_h: [0, 1]→[0, 1] is said to be an **almost uniform probability distribution function (AUPDF)** if for some real number $f_h \in (0, 2)$: $F_h(\alpha) = f_h \alpha$ for all $\alpha \in [0, \frac{1}{2}]$ and $F_h(\alpha) = \frac{1}{2} f_h + (2 - f_h)(\alpha - \frac{1}{2})$ for $\alpha \in (\frac{1}{2}, 1]$.

Clearly, if $f_h = 2 - f_h$, then it must be the case that $f_h = 1$, in which case the AUPDF is the uniform probability distribution function.

Note 4: The probability distribution function corresponding to F_h for the choice probability for action 2 is given by $(2-f_h)\alpha$ for all $\alpha \epsilon [0, \frac{1}{2}]$ and $\frac{1}{2}(2 - f_h) + f_h(\alpha - \frac{1}{2})$ for $\alpha \epsilon (\frac{1}{2}, 1]$. ▌

For $h \epsilon \{1, 2\}$ and $\alpha \epsilon [0, 1]$, $F_h(\alpha)$ is the probability assigned by player 'h' to the event that player 3-h will choose action 1 with probability less than or equal to α.

Note 5: If $f_h < 1$, then $F_h(\alpha) < \alpha$ for all $\alpha \epsilon (0, 1)$ and hence for all $\alpha \epsilon (0, 1)$, the probability assigned by F_h to the event "player 3-h will choose action 1 with probability greater than α" is *strictly greater* than the probability assigned to the same same event by the uniform probability distribution. Thus, in this case the AUPDF F_h *stochastically dominates* the uniform probability distribution function.

On the other hand, if $f_h > 1$, then $F_h(\alpha) > \alpha$ for all $\alpha \epsilon (0, 1)$ and hence for all $\alpha \epsilon (0, 1)$, the probability assigned by F_h to the event "player 3-h will choose action 1 with probability greater than α" is *strictly less* than the probability assigned to the same same event by the uniform probability distribution. Thus, in this case the uniform probability distribution *stochastically dominates* the AUPDF F_h. ▌

Action $i \epsilon \{1, 2\}$ is said to be *risk dominant for the row player with respect to* F_1 if the probability of the event $\{q_1 \epsilon [0, 1] | G_{1(j)}\begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix} > G_{1(3\text{-}j)}\begin{pmatrix} q_1 \\ 1 - q_1 \end{pmatrix}\}$ determined by F_1 is strictly greater than $\frac{1}{2}$.

Action $j \epsilon \{1, 2\}$ is said to be *risk dominant for the column player with respect to* F_2 if the probability of the event $\{p_1 \epsilon [0, 1] | (p_1 \quad | \quad (1 - p_1))G_2^{(j)} >$ $(p_1 \quad | \quad (1 - p_1))G_2^{(3-j)}\}$ determined by F_2 is strictly greater than $\frac{1}{2}$.

An action profile (i, j) is said to be a *risk-dominant action profile with respect to* (F_1, F_2) if i is risk dominant for the row player with respect to F_1 and j is risk dominant for the column player with respect to F_2.

Risk-dominance as defined in Harsanyi-Selten (1988) assumes $F_h(\alpha) = \alpha$ for all $\alpha \epsilon [0, 1]$, i.e., each player's belief about the probability with which the other player chooses action 1, is uniformly distributed. However, that need not always be the case.

Example 2: Let $(G_1, G_2) = \begin{bmatrix} (a_1, b_1) & (0, 0) \\ (0, 0) & (a_2, b_2) \end{bmatrix}$ satisfying $a_1 > 0$, $a_2 > 0$, $b_1 > 0$ and $b_2 > 0$ be a 2×2 bi-matrix game.

Suppose, that for $h \epsilon \{1, 2\}$, there exists $f_h \epsilon (0, 2)$ such that $F_h(\alpha) = f_h\alpha$ for all $\alpha \epsilon [0, \frac{1}{2}]$ and $F_h(\alpha) = \frac{1}{2}f_h + (\alpha - \frac{1}{2})(2 - f_h)$ for all $\alpha \epsilon (\frac{1}{2}, 1]$.

The event $\{q_1 \epsilon [0, 1] | q_1a_1 > (1- q_1)a_2\} = (\frac{a_2}{a_1+ a_2}, 1]$ and $\{q_1 \epsilon [0, 1] | q_1a_1 < (1- q_1)a_2\} = [0, \frac{a_2}{a_1+ a_2})$.

If $F_1(\frac{a_2}{a_1+ a_2}) < \frac{1}{2}$, then action 1 is risk-dominant for the row player with respect to F_1.

If $F_1(\frac{a_2}{a_1+ a_2}) > \frac{1}{2}$, then action 2 is risk-dominant for the row player with respect to F_1.

$\frac{a_2}{a_1+ a_2} > \frac{1}{2}$ <u>if and only if</u> $a_2 > a_1$.

If $\frac{a_2}{a_1+ a_2} > \frac{1}{2}$, then the probability of the event $\{q_1 \epsilon [0, 1] | q_1a_1 < (1- q_1)a_2\}$(i.e., $(\frac{a_2}{a_1+ a_2}, 1])$ determined by F_1 is $F_1(\frac{a_2}{a_1+ a_2}) = \frac{1}{2}f_1 + (\frac{a_2}{a_1+ a_2} - \frac{1}{2})(2 - f_1)$.

$\frac{1}{2}f_1 + (\frac{a_2}{a_1+ a_2} - \frac{1}{2})(2- f_1) = (1 - \frac{a_2}{a_1+ a_2})f_1 - 1 + \frac{2a_2}{a_1+ a_2} = \frac{a_1}{a_1+ a_2}f_1 + \frac{a_2 - a_1}{a_1+ a_2} = \frac{a_2 - (1-f_1)a_1}{a_1+ a_2}$.

$\frac{a_2}{a_1+ a_2} > \frac{1}{2}$ <u>if and only if</u> $a_2 > a_1$.

Suppose $3a_1 > a_2 > a_1$, so that $0 < \frac{3a_1 - a_2}{2a_1} = \frac{3}{2} - \frac{a_2}{2a_1} < \frac{3}{2} - \frac{1}{2} = 1$.

Let $f_1 \epsilon (0, \frac{3a_1 - a_2}{2a_1})$. Thus, $f_1 < 1$.

The associated probability distribution function *stochastically dominates* the uniform probability distribution function.

Further, $\frac{a_2 - (1-f_1)a_1}{a_1+ a_2} < \frac{1}{2}$.

The probability of the event $\{q_1 \epsilon [0, 1] | q_1a_1 > (1- q_1)a_2\}$ determined by F_1 is $1- F_1(\frac{a_2}{a_1+ a_2}) > \frac{1}{2}$.

Thus, while action 2 is risk dominant for the row player (in the sense of Harsanyi and Selten (1988)), action 1 is risk dominant for the row player with respect to F_1.

If we use the diagonal bi-matrix game of example 1, i.e., $\begin{bmatrix} (1, 1) & (0, 0) \\ (0, 0) & (2, 2) \end{bmatrix}$, then the condition $3a_1 > a_2 > a_1$ is satisfied since $3 > 2 > 1$.

$\frac{a_2}{a_1+ a_2} = \frac{2}{3}$

If $f_1 = \frac{1}{4}$ then the probability of the event $[0, \frac{a_2}{a_1+ a_2}) = \frac{1}{8} + \frac{3}{4}(\frac{2}{3} - \frac{1}{2}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} < \frac{1}{2}$.

Thus, action 1 (and not action 2) is risk dominant for the row player with respect to F_1.

If $F_2 = F_1$, then action 1 (and not action 2) is risk dominant with respect to F_2 for the column player.

Thus, although (2, 2) is a risk dominant action profile, action profile (1, 1) is risk dominant with respect to (F_1, F_2).

Note 6: If $(G_1, G_2) = \begin{bmatrix} (5,5) & (0,4) \\ (4,0) & (2,2) \end{bmatrix}$, then $(D(G_1), D(G_2)) = \begin{bmatrix} (1,1) & (0,0) \\ (0,0) & (2,2) \end{bmatrix}$. Thus, (1, 1) *is a risk dominant profile for* $\begin{bmatrix} (5,5) & (0,4) \\ (4,0) & (2,2) \end{bmatrix}$, *with respect to* (F_1, F_2) *that is formulated above for* $\begin{bmatrix} (1,1) & (0,0) \\ (0,0) & (2,2) \end{bmatrix}$.

5. Battle of the Sexes: The problem with risk-dominance, as in Harsanyi and Selten (1988), is that for bi-matrix games of the form $\begin{bmatrix} (a_1,b_1) & (0,0) \\ (0,0) & (a_2,b_2) \end{bmatrix}$ satisfying $a_1 > a_2 > 0$ and $b_2 > b_1 > 0$, application of risk-dominance may lead to a violation of stability, i.e., choice of equilibrium action profiles. In this case $\frac{a_1}{a_1+a_2} > \frac{1}{2} > \frac{a_2}{a_1+a_2}$ and $\frac{b_1}{b_1+b_2} < \frac{1}{2} < \frac{b_2}{b_1+b_2}$.

Suppose, both players believe that the probability with which the "other" player will choose action 1 follows a uniform distribution. Then, action 1 would be a risk dominant action for the row player, where as action 2 would be a risk dominant action for the column player, which if implemented would lead to the action profile (1, 2). Clearly, (1, 2) is not an equilibrium action profile.

Suppose the row player's beliefs about the probability with which the column player would choose action 1 is uniformly distributed, but the column player has reasons to believe that the row player is "strongly biased" in favor of action 1. Then in a situation such as "battle of the sexes" it is quite plausible that the distribution function F_2 conjectured by the column player, would tend to assign higher weight to the row player preferring strategies with greater probability for action 1, i.e., an F_2 that stochastically dominates the uniform probability distribution function.

The set $\{p_1 \epsilon [0, 1] | p_1 b_1 > (1 - p_1) b_2\} = (\frac{b_2}{b_1+b_2}, 1]$, where $\frac{b_2}{b_1+b_2} > \frac{1}{2}$

Let $f_2 \epsilon (0, 1)$ be such that $\frac{1}{2} f_2 + (\frac{b_2}{b_1+b_2} - \frac{1}{2})(2 - f_2) < \frac{1}{2}$.

$\frac{1}{2} f_2 + (\frac{b_2}{b_1+b_2} - \frac{1}{2})(2 - f_2) = f_2(1 - \frac{b_2}{b_1+b_2}) - 1 + \frac{2b_2}{b_1+b_2} = f_2 \frac{b_1}{b_1+b_2} + \frac{b_2-b_1}{b_1+b_2} = \frac{b_2-(1-f_2)b_1}{b_1+b_2}$.

Suppose, $3b_1 > b_2 > b_1$, so that $0 < \frac{3b_1-b_2}{2b_1} = \frac{3}{2} - \frac{1}{2}\frac{b_2}{b_1} < \frac{3}{2} - \frac{1}{2} = 1$.

Let $f_2 \epsilon\ (0, \frac{3b_1 - b_2}{2b_1})$. Thus, $\frac{b_2 - (1-f_2)b_1}{b_1+b_2} < \frac{1}{2}$.

Let F_2:[0, 1]→[0, 1] be such that for $\alpha \epsilon [0, \frac{1}{2}]$, $F_2(\alpha) = f_2\alpha$ and for $\alpha \in (\frac{1}{2}, 1]$, $F_2(\alpha) =$

$f_2\frac{b_2}{b_1+b_2} + (2 - f_2)(\alpha - \frac{b_2}{b_1+b_2})$.

Since $f_2 < 1$, F_2 stochastically dominates the uniform distribution, thereby reflecting the column player's belief that the row player is strongly biased in favour of action 1. In this case action 1 will be a risk dominant action for the column player with respect to F_2, since probability of the event $\{p_1 \epsilon [0, 1] | p_1 b_1 > 1 - p_1) b_2\}$ (i.e., $(\frac{b_2}{b_1+b_2}, 1]) = 1 - F_2(\frac{b_2}{b_1+b_2}) > \frac{1}{2}$ so that (1, 1) is a risk dominant action profile with respect to (F_1, F_2), with F_1 being the uniform probability distribution on [0, 1].

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