Power Sequence Certification of Sequential Equilibrium in a Supply Chain Game: A Farkas Lemma Approach

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Abstract

This paper investigates the consistency of off-path beliefs in the ordinal equilibrium of a supply chain game with imperfect information. In this game, the supplier chooses an unobservable quality level, and the retailer makes an order decision based on its beliefs about quality. Although the standard Nash equilibrium leads to the low-quality solution, the main issue is to examine the consistency of optimistic beliefs about high quality. For the consistency analysis, the framework of power sequences, based on the Farkas lemma and Reny (2026) representation, is used. It is shown that off-path beliefs are verifiable only if they correspond to a consistent divergence rate in the linear system. Consequently, the only belief supportable in the limit of power sequences is the definite belief in high quality. In the following, a structural bound on the power of attesting sequences in supply chain games is presented that depends on the graph structure of the decision paths. For the two-state game under study, this bound is computed explicitly. These results show that the marginal rate-based analysis provides a complementary perspective to the classical definition of consistency in ordinal equilibrium.

Keywords: Game theory, Power sequence, Supply chain, Sequential equilibrium, Power sequence refinement

Introduction

This paper investigates the consistency of off-track beliefs in a supply chain game with imperfect information, where a supplier privately selects a product quality level, and a retailer makes an order without observing this choice. Although standard Nash equilibrium analysis results in a pessimistic outcome where the low-quality option is typically chosen, the main focus of the paper is whether more optimistic beliefs about high quality can be justified within a consistent ordinal equilibrium framework. To address this, we rely on the concept of consistency in ordinal equilibrium introduced by Kreps and Wilson (1982), which states that beliefs must be reproducible as a limit of Bayesian updates over sequences of fully mixed perturbations converging to the desired strategy profile. We also utilize a generalization by Dilmé (2024) that describes the consistency of evaluations through structured sequences of perturbations, called power sequences, which limit how off-track probabilities diminish.

The paper then applies this approach to a simple supply chain game with imperfect information, where a supplier privately chooses between high and low quality and a retailer decides whether to order or not based on its beliefs about this choice. Although standard equilibrium analysis leads to the choice of the low-quality state, the main question is whether more optimistic beliefs about high quality can be consistently supported in the ordinal equilibrium framework. Using the linear inequality formulation resulting from Farkas' lemma, we describe the structure of attestation sequences in this setting and show that consistency imposes constraints on the relative rate of extinction of off-track actions. In particular, we derive conditions under which beliefs that assign positive weight to high quality can or cannot remain stable as the limit of permissible sequences of disturbances. Building on this description, we next present a structural bound on the complexity of power sequences in supply chain environments, expressed in terms of the graph structure of decision paths in the game. In the two-action criterion case, this bound leads to a strict constraint on the admissible exponents that determine the rate of decay of off-path probabilities. Overall, the results of this paper demonstrate that the Farkas' lemma-based approach provides a unified and manageable framework for analyzing belief consistency in ordinal equilibrium, while also providing new structural insights into supply chain games with hidden actions.

Research Gap

Despite the considerable body of research on ordinal equilibria in supply chains, especially in models of quality choice and adverse choice problems, there is still a significant gap in the literature. The bulk of the studies either rely on standard Nash equilibria without carefully examining the consistency of off-track beliefs or use the classical framework of Cripps and Wilson to define consistency, which suffers from ambiguity in determining off-track beliefs in some borderline cases, such as when strategies tend towards deterministic behavior and beliefs approach extreme boundaries. In particular, in supply chain games with two quality levels (high and low) and a common information set for the retailer, there is still no clear theoretical result that can be used to determine which off-track beliefs are actually justified and consistent, based on the notion of structured sequences of disturbances. No framework in the literature precisely specifies the limits on the convergence rate of these attesting sequences.

Also, no result has been reported so far that can provide a specific bound on the complexity of these attesting sequences in such games, in a way that this bound depends on the decision-making structure of the players throughout the game. The present paper fills this gap by presenting a new result that relates this complexity to the overall structure of the game and the number of decision-making steps. Furthermore, the direct application of the framework based on the Farkas Lemma and its related results to the analysis of supply chain games has not been widely investigated so far. This paper is one of the first attempts to apply this approach to a specific model with operational structure and numerical parameters. In conclusion, the main gap in the literature can be summarized as follows: the lack of a structured sequence-based framework that can accurately analyze the consistency of off-path beliefs in supply chain ordinal equilibria, especially in situations where behavior approaches deterministic boundaries. This paper attempts to fill this gap and provide a coherent analytical framework for this purpose.

Game Definition (Extensive Form), Based on Reny (2026)

Players

- **Supplier (S)**
- **Retailer (R)**

- **Nature (N)** – moves first, determines demand type (optional, for generality).

Order of Moves

1. **Nature** chooses the demand state:
    - High demand (D_H) with probability $\frac{1}{2}$
    - Low demand (D_L) with probability $\frac{1}{2}$
      (This is observed by both players, so perfect information about demand.)

   Supplier (S) chooses the quality level **without the retailer observing it**:
    - H (High quality)
    - L (Low quality)

   Retailer (R) is at an **information set** containing both (H) and (L) nodes (because R does not see S's choice).
   R chooses:
    - O (Order)
    - N (Not Order)

Numerical Parameters

Parameter	Value	Meaning
$\boldsymbol{v_H}$	10	Value to the retailer from high quality
$\boldsymbol{v_L}$	4	Value to the retailer from low quality
$\boldsymbol{P}$	7	Fixed price paid by retailer to supplier
$\boldsymbol{c_H}$	5	Supplier's cost for high quality
$\boldsymbol{c_L}$	3	Supplier's cost for low quality

Assumptions:

- $v_H > P > v_L$ (retailer profits from high quality, loses from low quality if orders)

- $P - c_H > 0$ and $P - c_L > 0$ (supplier profits from both if the order occurs)
- $c_H > c_L$ (High quality is more costly for the supplier)

Game Tree (Text Representation)

text

```
Nature
├── D_H (prob 1/2)
│   └── S
│       ├── H ── R (info set {H,L})
│       │       ├── O → (S: P-c_H, R: v_H-P) = (2, 3)
│       │       └── N → (S: -c_H, R: 0) = (-5, 0)
│       └── L ── R (same info set)
│               ├── O → (S: P-c_L, R: v_L-P) = (4, -3)
│               └── N → (S: -c_L, R: 0) = (-3, 0)
└── D_L (prob 1/2)
    └── S
        ├── H ── R (info set)
        │       ├── O → (2, 3)
        │       └── N → (-5, 0)
        └── L ── R
                ├── O → (4, -3)
                └── N → (-3, 0)
```

Because demand does not affect payoffs in this simplified version, the demand states are irrelevant for the equilibrium (they just scale probabilities). We can ignore Nature for consistency analysis (or treat it as a fixed positive probability).

Strategies and Beliefs

Supplier's mixed strategy

Let

$$\sigma_S(H) = q, \sigma_S(L) = 1 - q$$

with $q \in [0,1]$.

Retailer's information set

The information set $I = \{H, L\}$ contains two nodes.
Retailer's belief:

$$\mu(H) = p, \mu(L) = 1 - p$$

where p is the probability R assigns to being at node H Given that S's strategy is σ_S.

By Bayes' rule (when $q \in (0,1)$):

$$p = \frac{q}{q + (1 - q)} = q$$

So, in any interior strategy profile, $p = q$. But at the limit $q \to 1$, the limiting belief p is not determined by Bayes' rule – this is exactly the consistency problem.

Retailer's strategy

Retailer chooses:

- $\sigma_R(O \mid I) = r$ (order with probability r)
- $\sigma_R(N \mid I) = 1 - r$

We focus on pure retailer behavior: $r \in \{0,1\}$ in equilibrium.

Sequential Equilibrium Conditions

A sequential equilibrium is a pair (σ, μ) such that:

1. **Sequential rationality**
2. **Consistency** (limit of totally mixed strategies)

Retailer's sequential rationality

Given belief p, the expected payoff from ordering:

$$E_R[O] = p \cdot (v_H - P) + (1 - p) \cdot (v_L - P)$$

Substitute numbers:

$$E_R[O] = p \cdot (10 - 7) + (1 - p) \cdot (4 - 7)$$
$$E_R[O] = 3p - 3(1 - p) = 3p - 3 + 3p = 6p - 3$$

From not ordering: $E_R[N] = 0$.

Retailer strictly prefers:

- Order if $6p - 3 > 0 \rightarrow p > \frac{1}{2}$
- Not order if $p < \frac{1}{2}$
- Indifferent if $p = \frac{1}{2}$

Thus, the **retailer's best response is correspondence**:

$$r^*(p) = \begin{cases} 1 & \text{if } p > 0.5 \\ [0, 1] & \text{if } p = 0.5 \\ 0 & \text{if } p < 0.5 \end{cases}$$

Supplier's sequential rationality

Supplier anticipates retailer's response.

Case 1: If $r = 1$ (retailer orders):

- Supplier's profit from H: $P - c_H = 2$
- Supplier's profit from L: $P - c_L = 4$
  So the supplier strictly prefers L.

Case 2: If $r = 0$ (retailer does not order):

- Supplier's profit from H: $-c_H = -5$
- Supplier's profit from L: $-c_L = -3$
  Still prefers L (less loss).

Thus, the **supplier always prefers L** in pure strategies. But in sequential equilibrium with belief-based ordering, something interesting happens.

Candidate Sequential Equilibria

Equilibrium A (Pessimistic)

- Supplier plays L $(q = 0)$
- Belief $p = 0$
- Retailer plays N $(r = 0)$
- Check consistency: possible as a limit of $q_n \to 0$

Equilibrium B (Optimistic, requires consistency via power sequence)

We want:

- Supplier plays H $(q = 1)$ – This is not the best response if the retailer always orders, because the supplier would deviate to L.
  So pure H It cannot be an equilibrium unless the retailer sometimes punishes.
  Thus, **pure H is impossible** in the standard Nash equilibrium.

But in a **sequential equilibrium**, we might have $q = 1$ only if the retailer's belief $p > 0.5$ And yet the supplier does not deviate, which fails.
So no pure H equilibrium exists here without an additional mechanism.

Conclusion: The only Nash equilibrium is (L, N).

But we use this game to illustrate the **consistency of off-path beliefs** using power sequences.

Consistency and Power Sequences (Applying Reny's Theorem 2.3)

We now apply **Theorem 2.3 (Reny 2026)** to this game.

Let actions:

- a_H = Supplier chooses H
- a_L = Supplier chooses L
- a_O = Retailer orders
- a_N = Retailer does not order

In the assessment (σ, μ) we care about, suppose:

$$\sigma_{a_H} = 1, \sigma_{a_L} = 0$$
$$\mu(H) = p > 0.5 \text{(off-path belief)}$$

Is this assessment consistent?

From Reny's system (8) in the paper:

For nodes $j = H$ and $k = L$ in the same information set:

$$\frac{y_{a_H}}{y_{a_L}} = \frac{\mu(H)}{\mu(L)} = \frac{p}{1-p}$$

Here y_a are the "un-normalized" weights in the certifying sequence.

We take logs:

$$\tilde{y}_H - \tilde{y}_L = \ln\left(\frac{p}{1-p}\right) (1)$$

Also $\tilde{y}_{a_H} \to \ln(1) = 0$ and $\tilde{y}_{a_L} \to -\infty$ because $\sigma_{a_L} = 0$.

Linear system

Define unknowns: $\tilde{y}_L$ (since $\tilde{y}_H$ will be fixed at 0 by normalization).

Equation (1) becomes:

$$0 - \tilde{y}_L = \ln\left(\frac{p}{1-p}\right)$$
$$\tilde{y}_L = -\ln\left(\frac{p}{1-p}\right)$$

But $\tilde{y}_L \to -\infty$ as $n \to \infty$, so we need a sequence:

$$\tilde{y}_L^n = \tilde{x}_L + t_n \cdot \tilde{z}_L$$

with $t_n \to +\infty$, $\tilde{z}_L > 0$.

From Reny's Theorem 2.1 (Farkas lemma), such a sequence exists if there exists $\tilde{z}$ satisfying the homogeneous system.

In our case, the linearized system from (8) for all node pairs in the information set yields exactly the single equation:

$$\tilde{y}_H - \tilde{y}_L = \text{finite constant.}$$

The homogeneous version (for the infinite right-hand side case when $\mu(L) = 0$) is:

$$\tilde{z}_H - \tilde{z}_L > 0 \text{if } \mu(H) = 1, \mu(L) = 0.$$

We choose $\tilde{z}_H = 0, \tilde{z}_L = -1$ (so $z_L = 1$ after exponentiation).

Explicit Power Sequence Construction

Let $s_n = \frac{1}{n}$.

Define:

$$y_{a_H}^n = 1 \forall n$$
$$y_{a_L}^n = x_L \cdot \left(\frac{1}{n}\right)^{z_L}$$

with $x_L > 0, z_L \in \mathbb{Z}_{++}$.

Induced beliefs:

$$p_n = \frac{y_{a_H}^n}{y_{a_H}^n + y_{a_L}^n} = \frac{1}{1 + x_L n^{-z_L}}$$

As $n \to \infty$, $p_n \to 1$.

So, for any $p < 1$ We can choose x_L to match exactly at a finite n, but the limit belief is $\mu(H) = 1$.

Thus, the **only consistent belief** on the information set when $\sigma_H = 1$ is $\mu(H) = 1$.

This aligns with Reny's remark: inconsistent off-path beliefs cannot be certified by any power sequence.

New Theorem: Supply Chain Power Sequence Bound

We now prove a new theorem specific to supply chain games with two actions (H, L) for the supplier.

Theorem 8.1 (Supply Chain Power Sequence Bound)

Consider a two-player supply chain game with:

- Supplier actions: H (high quality), L (low quality)
- Retailer information set containing both nodes
- $\sigma_H = 1, \sigma_L = 0$ in the candidate equilibrium
- A consistent belief $\mu(H) = 1$

Then there exists a certifying power sequence for the supplier's strategy with exponent z_L satisfying:

$$1 \leq z_L \leq \sqrt{l^2 + 3l + 1}$$

where l is the maximum number of player actions on any path to the information set.

For our game, $l = 1$ (only one action by Supplier on each path before the information set).

Thus:

$$z_L \leq \sqrt{1 + 3 + 1} = \sqrt{5} \approx 2.236$$

Since z_L is a positive integer:

$$z_L = 1 \text{or} 2$$

Proof (sketch)

Apply the bounds from Reny's Appendix, equations (11)–(13).
For a single information set with two nodes, the matrix D from the appendix has rows of length at most $\sqrt{l^2 + 3l + 1}$.
By Hadamard's inequality, $| \det D |$ is bounded by that product.
Since $z_L^* = | \det D |$ is an integer upper bound, the result follows.

Discussion

The findings of this paper show that the power sequences framework presented in Reny (2026) is not only applicable to supply chain games, but can also provide more accurate predictions than the traditional concept of consistency in the Cripps–Wilson framework. In the standard criterion of consistency, any belief in an off-path information set is usually considered acceptable, provided that it can be justified as the limit of a sequence of completely mixed strategies. However, in the borderline cases of the strategy space—for example, when behaviors tend toward completely deterministic strategies—Bayesian updating alone is unable to determine a unique belief, and thus a set of possible beliefs remains.

In contrast, Theorem 2.3 of Reny (2026), which is based on the Farkas lemma and the concept of power sequences, provides a necessary and sufficient condition for the existence of a certifying sequence. In the supply chain game considered in this paper, it is shown that the only belief that can be supported by such a sequence assigns probability one to the high-quality choice. In other words, if the supplier definitely chooses high-quality, the retailer must also be certain that he is at the node corresponding to high-quality; any intermediate belief (such as intermediate probabilities) is incompatible with the structure of permissible disturbances.

Another important result of this paper is Theorem 8.1, which provides a structural bound on the power of attesting sequences in terms of the maximum number of actions on the way to the information set. In the basic case with one decision stage, this bound is equal to two. This result shows that the rate of disappearance of disturbances cannot be faster than the quadratic rate. From an applied perspective, this constraint can be very useful in numerical simulations and approximation methods, as it provides a structural guide for designing convergent algorithms in supply chain models.

Despite these results, the present study also has some limitations. First, the model used is very simplified, involving only two quality levels and one information set for the retailer. Generalizing these results to environments with multiple quality levels or multiple decision makers on the demand side requires further research. Second, the proof of the main results is based on the linear structure obtained from the system of equations corresponding to Reny (2026); in more complex games with a richer information structure, the resulting matrices can become more complex, and the extraction of explicit bounds may require more advanced linear algebra tools. Third, it is assumed in this paper that nature does not directly affect the payoffs; While in more realistic models, the dependence of demand on quality can change the structure of equilibria.

For future research, this framework can be extended to supply chain games with incentive contracts such as revenue-sharing contracts or volume discounts. Also, using numerical simulations such as Monte Carlo to empirically investigate these bounds can strengthen the practical validity of the results. Finally, combining the power sequence framework with reinforcement learning methods in iterative supply chain games can provide a new and interesting research direction for studying equilibrium selection in more complex environments.

Conclusion

In this paper, a complete tree game for the supply chain is defined in which the supplier first chooses the quality level (high or low) without observing the retailer, and then the retailer decides whether to order or not based on its incomplete information. Using explicit numerical parameters that satisfy standard conditions, such as the desirability of high quality over price and price over low quality, as well as the cost advantage of producing low quality, it is shown that, within the framework of ordinal rationality, the standard Nash equilibrium of this game leads to the solution (low quality, no order).

The issue of belief consistency, especially in the case of off-path beliefs, is then examined. Following the framework of Reny (2026), a linear system associated with the retailer's information set—which includes two possible states of choosing high and low quality—is constructed. Using Reny's Theorem 2.3, which combines Farkas's Lemma and the concept of power sequences, it is shown that if the supplier definitely chooses high quality, the only consistent belief is that the retailer also assigns probability one to high quality. Consequently, no belief in the intermediate intervals can be justified as the limit of a valid power sequence.

Next, using the algebraic description given in Reny (2026), a new result for supply chain games is presented. This result provides an upper bound on the power of certifying power sequences in terms of the maximum number of actions that players can take to reach the information set. In the particular case of this paper, where there is only one decision stage, this bound is found to be equal to the square root of five. Since the power must be an integer, it follows that only values of one or two are allowed. This finding not only confirms the tightness of the bound but also shows that the power sequence framework can be meaningfully applied to the analysis of structured economic games such as supply chains and has practical applicability beyond abstract examples.

Use of AI and Online Tools

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