LAY-UP OPTIMALITY CONDITIONS FOR BUCKLING LEVEL MAXIMIZATION OF VAT (STEERED FIBER) COMPOSITE PLATES

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Abstract

In the present paper the flat composite plates in buckling are studied. The plates have a symmetric lay-up and loaded along their contour by the in-plane forces. The consideration employs the von Karman approach. The lay-up optimality conditions for a single-mode (lowest) buckling eigenvalue are derived using the proper variational principles and the variation calculus. Both the bending terms and the terms following from the redistribution of the 2D stresses over the plate are taken into account. The optimality conditions contain two items (corresponding to the bending and to the 2D infinitesimal plane strains). The comparative analysis of the items is performed. The physical meaning of the optimality conditions is demonstrated and explained. An illustration to the derived conditions is presented and analyzed. It is shown that for the optimal lay-up not all lamination parameters are independent on each other (one of them is a linearly dependent one). The analysis leads to a conclusion that the optimization w.r.t. the lamination parameters only is not sufficient for obtaining an optimal plate lay-up.

Keywords: composite plate, buckling, lay-up, steered fibers, optimality

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1. INTRODUCTION

Buckling analysis of composite plates attracts attention of the numerous researchers since 80-ies years of the last century. The account of the effect allows obtaining lighter design solutions and using the capacity of the material more efficiently.

The book of Turvey and Marshall (1995) seems to be the first one summarizing the state-of-the-art for buckling and postbuckling of the composite plates. The more modern reviews may be found in the papers of Ghiasi et al. (2009, 2010) and in the book of Falzon and Aliabadi (2008).

We also mention some monographs important to the considered matter.

Ohsaki and Ikeda (2007) consider mainly the numerical aspects of the design optimization under the eigenvalue constraints.

In the book of Kassapoglou (2010) the modern approaches to the buckling analysis of the composite plates are described. The approaches are used at the Universities to educate the engineers in structural mechanics.

The buckling analysis of the composite plates is performed, as a rule, numerically, using the Rayleigh-Ritz or similar kinematic approaches based on the kinematic variational principles (indicated, e.g., in Washizu 1982).

In several papers (e.g., Ijsselmuiden et al. 2010) it was noticed in the numerical analysis that there is an influence of the 2D stress distribution on the buckling eigenvalue.

The present paper considers the influence theoretically. The theoretical results for the VAT plate buckling optimization are absent now.

The present paper deals with the derivation and analysis of the lay-up optimality conditions for the buckling level maximization of the steered fiber composite plates.

Section 1 presents the short Introduction.

Section 2 describes the main assumptions and the theoretical background.

In Section 3 we derive the first variation of the buckling eigenvalue.

Section 4 is devoted to the result analysis and their discussion.

Section 5 presents the conclusions.

2. THEORETICAL BACKGROUND

In the present paper we consider the laminated composite plate with the symmetric lay-up. The lay-up is composed of the orthotropic fiber-reinforced plies of the same thickness h_{ply} and various point-wise orientation angles. The total number of plies is even and equal to *2K*, the generalization to the odd ply number is straightforward and is not considered in the present paper. The flat plate is located in *XY* plane, it has a piecewise smooth contour *C* with external normal ν and tangent direction *s*; the directions ν, *l* and Z create a right-hand triplet. The coordinate system *XYZ* is a Cartesian one.

The plate is loaded by the in-plane forces p_x, p_y, acting at the part C_1 of the contour *C*. The remaining part C_2 of the contour the plate is not moving in *X* and *Y*. The plate is simply supported, i.e., the *z*-displacement and the normal bending moment M_ν acting perpendicular to the contour *C* are equal to zero. The clamping boundary conditions are not considered in the present paper, the change to the case is straightforward.

The *X, Y, Z* displacements are denoted as *u, v, w*, respectively.

The signs of the loads are the following. The flows N_x, N_y (along *X* and along *Y*) are positive in tension, the shear flow N_{xy} is positive when it decreases the 90° angle between the orthogonal lines at the plate surface. The flows are in equilibrium. The Composite Lamination

Plate Theory (CLPT) is used for description of plate deflections (see the book of Gibson 1994).

The Classical Lamination Plate Theory (CLPT), based on linear elastic relations for every layer and Bernoulli's hypotheses, is used for obtaining the equation for the deflections *w* (see the books of Gibson 1994, Lekhnitzky 1956, Reddy 2004). The equation is written as follows:

$$
\begin{aligned}
&D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}-\\
&-N_x\frac{\partial^2 w}{\partial x^2}-N_y\frac{\partial^2 w}{\partial y^2}-2N_{xy}\frac{\partial^2 w}{\partial x\partial y}=0
\end{aligned}
\qquad (1)
$$

where $D_{ij}, i=1,2,6; j=1,2,6,$ are the elements of the bending stiffness matrix *D*, coupling the bending/twisting moments and various second derivatives of the deflection *w* with respect to *x* and *y*. The coupling is written as follows:

$$
\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x\partial y} \end{bmatrix} \qquad (2)
$$

We denote the left-hand side of (2) as a vector-column $\vec{M}$, and as $\vec{k}$ the vector-column at the right-hand side (which is multiplied to *D* matrix). Also we use the notations

$$
k_x=-\frac{\partial^2 w}{\partial x^2};k_y=-\frac{\partial^2 w}{\partial y^2};k_{xy}=-\frac{\partial^2 w}{\partial x\partial y} \qquad (3)
$$

Then (2) is rewritten in the matrix-vector form as:

$$
\vec{M}=D\vec{k} \qquad (4)
$$

With the boundary conditions indicated above, the boundary-value problem is completed.

Now we discuss the energy balance for the plate element *dxdy*. For a feasible deflection δw we have the variation of the structural work of the internal forces (see the book of Volmir 1967)

$$\delta A = -\iint dS\left(M_x \delta k_x + M_y \delta k_y + 2M_{xy}\delta k_{xy}\right) \tag{5}$$

where δ is the variation symbol. Then the increase of the total strain potential energy $\tilde{\Pi}$ is

$$\delta\tilde{\Pi} = -\delta A \tag{6}$$

The value of the total strain potential energy and its variation are written as follows:

$$\begin{aligned}\tilde{\Pi} = \iint dS\Bigg(&\frac{1}{2}D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \\ &+ 2D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial y} + 2D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial y} + 2D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\Bigg)\end{aligned} \tag{7}$$

$$\begin{aligned}\delta\tilde{\Pi} = \delta\iint dS\Bigg(&\frac{1}{2}D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \\ &+ 2D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial y} + 2D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x \partial y} + 2D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\Bigg)\end{aligned} \tag{8}$$

The work, performed by the external forces, is

$$W = -\frac{1}{2}\iint dS\left[N_x\left(\frac{\partial w}{\partial x}\right)^2 + N_y\left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \tag{9}$$

The total energy of the system

$$U = \tilde{\Pi} - W \tag{10}$$

Because of energy balance at the buckling state *w* the energy variation δU is equal to zero:

$$\delta U = \delta\tilde{\Pi} - \delta W = 0 \tag{11}$$

Making formal variations in (8)-(10), integrating by parts and taking into account the boundary conditions of the considered types and in-plane equilibrium, we obtain the equilibrium equation (1). Hence, the stationary conditions of (10) with respect to *w*, under the proper boundary conditions, lead to the buckling equation (1). The statement is a content of the kinematic variational principle for the plate buckling.

Following the book of Washizu (1982), the kinematic variational principle may be rewritten as the stationary form of the following ratio (under the same boundary conditions as before):

$$\delta\left[\frac{\tilde{\Pi}}{W^{(0)}}\right]=0 \tag{12}$$

where

$$W^{(0)}=-\iint dS\left[\frac{1}{2}N_x^{(0)}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}N_y^{(0)}\left(\frac{\partial w}{\partial y}\right)^2+N_{xy}^{(0)}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \tag{13}$$

and the upper superscript (0) corresponds to some constant initial (pre-buckling) force flows. The flows should be multiplied to the eigenvalue λ for obtaining the buckling force flows.

For found buckling mode the eigenvalue satisfies the Railey relation:

$$\lambda=\frac{\tilde{\Pi}}{W^{(0)}} \tag{14}$$

The distribution of the in-plane stress flows are given by the relations:

$$\vec{N}=A\begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x} \end{bmatrix} \tag{15}$$

where A is the in-plane stiffness matrix, and the column-vector

$$\vec{N}=\left(N_x, N_y, N_{xy}\right)^T \tag{16}$$

T means transposing, and index after comma means the differentiation w.r.t. the corresponding coordinate.

The components of the $\vec{N}$ must satisfy the equilibrium equations, and the in-plane displacements must satisfy the corresponding boundary conditions.

We will also use below some relations for the laminated plate presented in Selyugin (2019a, 2019b).

For the orthotropic lamina, we follow the approach and the notations of (Gibson 1994, Section 2.6).

Fig. 1 shows the orthotropic lamina. The stress tensor components within the lamina in the local coordinates (*x, y*) are:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{pmatrix} \tag{17}$$

where $\overline{Q}$ is the lamina stiffness matrix, and $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ are the lamina infinitesimal strain tensor components.

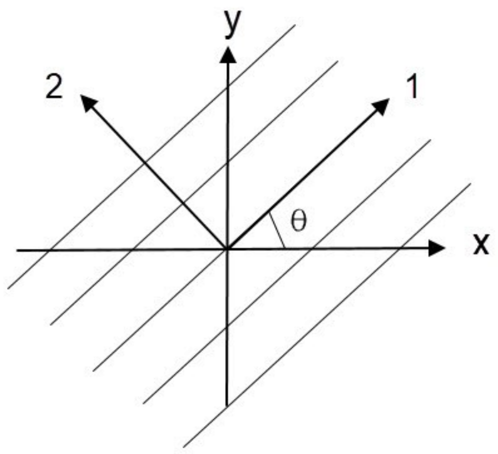

Fig. 1. Orthotropic lamina with material orientation (1, 2).

The components of the lamina stiffness matrix are (see Gibson 1994, Section 2.6):

$$\begin{aligned}
\overline{Q}_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \\
\overline{Q}_{12} &= U_4 - U_3 \cos 4\theta \\
\overline{Q}_{22} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta \\
\overline{Q}_{16} &= \frac{1}{2} U_2 \sin 2\theta + U_3 \sin 4\theta \\
\overline{Q}_{26} &= \frac{1}{2} U_2 \sin 2\theta - U_3 \sin 4\theta \\
\overline{Q}_{66} &= \frac{1}{2} (U_1 - U_4) - U_3 \cos 4\theta
\end{aligned} \tag{18}$$

where U_1, U_2, U_3, U_4 are the elastic constants of the orthotropic layer. The stress tensor components within the lamina in the coordinates (*x, y*) are:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{pmatrix} \tag{19}$$

where $\overline{Q}$ is the lamina stiffness matrix, and $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ are the lamina infinitesimal strain tensor components. The components of the lamina stiffness matrix are the known algebraic functions of the four layer elastic constants U_1, U_2, U_3, U_4 and of the trigonometric functions of the angle *Θ.* The angle is the angle between the fiber direction and the *x*-axis (measured in the direction of the counter-clockwise rotation from *x* to *y*).

According to (Gibson 1994), the derivatives of the lamina stiffness matrix $\overline{Q}_{ij}$ w.r.t. the orientation angle *Θ* are:

$$
\begin{aligned}
\frac{d\overline{Q}_{11}}{d\theta} &= -2U_2 \sin 2\theta - 4U_3 \sin 4\theta = -4\overline{Q}_{16} \\
\frac{d\overline{Q}_{22}}{d\theta} &= 2U_2 \sin 2\theta - 4U_3 \sin 4\theta = 4\overline{Q}_{26} \\
\frac{d\overline{Q}_{12}}{d\theta} &= 4U_3 \sin 4\theta = 2(\overline{Q}_{16} - \overline{Q}_{26}) \\
\frac{d\overline{Q}_{16}}{d\theta} &= U_2 \cos 2\theta + 4U_3 \cos 4\theta \\
\frac{d\overline{Q}_{26}}{d\theta} &= U_2 \cos 2\theta - 4U_3 \cos 4\theta \\
\frac{d\overline{Q}_{66}}{d\theta} &= 4U_3 \sin 4\theta
\end{aligned}
\tag{20}
$$

The determination of A_{ij} and D_{ij} via the lamina parameters is performed according to the general formulas (Gibson 1994), *ij=11, 12, 22, 16, 26, 66*:

$$
A_{ij} = \sum_{k=1}^{K} \left(\overline{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right) \tag{21}
$$

$$
D_{ij} = \frac{1}{3} \sum_{k=1}^{K} \left(\overline{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right) \tag{22}
$$

where z_k, z_{k-1} are the top and the bottom *z*-coordinates of the layer (*z*=0 corresponds to the mid-plane).

The derivatives of the stiffness matrices *A, D* w.r.t. the lamina angles θ_k are, *ij=11, 12, 22, 16, 26, 66; k=1,…,K*:

$$
\begin{aligned}
\frac{dA_{ij}}{d\theta_k} &= (z_k - z_{k-1}) \frac{d\left(\overline{Q}_{ij}\right)_k}{d\theta_k} \\
\frac{dD_{ij}}{d\theta_k} &= \frac{1}{3}(z_k^3 - z_{k-1}^3) \frac{d\left(\overline{Q}_{ij}\right)_k}{d\theta_k}
\end{aligned}
\tag{23}
$$

where θ_k is the layer orientation angle.

3. FIRST VARIATION OF THE BUCKLING EIGENVALUE

Now we proceed with the determination of the variation of the eigenvalue. It is supposed that the eigenvalue is a single-modal one. Our goal is to design a lay-up maximizing the first (lowest) eigenvalue.

The eigenvalue variation (to be equal to zero for the optimal design), using the Lagrange multiplier technique, may be written in the form:

$$\delta\lambda = \delta_w\left[\frac{\tilde{\Pi}}{W^{(0)}}\right] + \delta_D\left[\frac{\tilde{\Pi}}{W^{(0)}}\right] + \delta_{\vec{N}}\left[\frac{\tilde{\Pi}}{W^{(0)}}\right] + \delta_{\vec{N},u,v}\left[\int_S dS\vec{\alpha}^T\left(A^{-1}\vec{N} - \left(u_{,x}; v_{,y}; u_{,y} + v_{,x}\right)^T\right)\right] + \\ + \sum_{m=1}^{K}\delta_{\theta_m}\int_S dS\vec{\alpha}^T A^{-1}\vec{N} \tag{24}$$

where the first item is equal to zero due to the above-indicated variational principle, the second item is due to the bending stiffness variation and is derived in Selyugin (2013) (the influence of the in-plane force variation was considered as small and negligible there), the third item is due to the in-plane force variations influencing the eigenvalue, the fourth item corresponds to the in-plane equilibrium, the fifth item corresponds to the orientation angle θ_m variation for the m-th ply, $m=1,\dots,K$. $\vec{\alpha}$ is a vector-column of the Lagrange multipliers.

Performing the usual transformations based on the Gauss divergence theorem, we obtain:

$$\delta_{\vec{N}}\left[\frac{\tilde{\Pi}}{W^{(0)}}\right] = -\frac{\lambda}{W^{(0)}}\int_S dS\left(w_{,x}^2\delta N_x + w_{,y}^2\delta N_y + 2w_{,x}w_{,y}\delta N_{xy}\right) \tag{25}$$

$$\delta_{\vec{N}}\left[\int_S dS\vec{\alpha}^T\left(A^{-1}\vec{N} - \left(u_{,x}; v_{,y}; u_{,y} + v_{,x}\right)^T\right)\right] = \int_S dS\vec{\alpha}^T A^{-1}\delta\vec{N} \tag{26}$$

$$\delta_{u,v}\left[\int_S dS\vec{\alpha}^T\left(A^{-1}\vec{N} - \left(u_{,x}; v_{,y}; u_{,y} + v_{,x}\right)^T\right)\right] = \oint dl\left[\left(\alpha_1\delta u + \alpha_3\delta v\right)\cos(x,\nu) + \\ + \left(\alpha_2\delta v + \alpha_3\delta u\right)\cos(y,\nu)\right] - \int_S dS\left[\delta u\left(\alpha_{1,x} + \alpha_{3,y}\right) + \delta v\left(\alpha_{2,y} + \alpha_{3,x}\right)\right] \tag{27}$$

We make (27) equal to zero by imposing the following conditions on $\vec{\alpha}$:

$$\alpha_{1,x} + \alpha_{3,y} = 0 \ ; \ \alpha_{2,y} + \alpha_{3,x} = 0 \qquad \text{in S} \tag{28}$$

$$\alpha_1 \cos(x,\nu) + \alpha_3 \cos(y,\nu) = 0 \quad ; \quad \alpha_3 \cos(x,\nu) + \alpha_2 \cos(y,\nu) = 0 \quad \text{on } C_1 \tag{29}$$

$$\delta u = \delta v = 0 \quad \text{on } C_2 \tag{30}$$

Further:

$$\delta_{\theta_m} \int_S dS \vec{\alpha}^T A^{-1} \vec{N} = -\int_S dS \vec{\alpha}^T A^{-1} \frac{dA}{d\theta} A^{-1} \vec{N} \delta\theta_m \quad , m=1,\ldots,K \tag{31}$$

Analyzing (25) and (26), we obtain

$$\vec{\alpha}^T A^{-1} = \frac{\lambda}{W^{(0)}} \left(w_{,x}^2 + w_{,y}^2 + 2 w_{,x} w_{,y} \right) \tag{32}$$

Finally the Lagrange multipliers are

$$\vec{\alpha} = \frac{\lambda}{W^{(0)}} A \begin{pmatrix} w_{,x}^2 \\ w_{,y}^2 \\ 2 w_{,x} w_{,y} \end{pmatrix} \tag{33}$$

Now we determine $\frac{dA}{d\theta_m}$. We follow the approach of Selyugin (2019a). For doing that, we use the principal infinitesimal 2D strain $\varepsilon_1, \varepsilon_2$ axes as the *X, Y* ones (where $\varepsilon_1 \geq \varepsilon_2$).

After the cumbersome but the simple transformations, we obtain

$$\begin{aligned} \vec{\alpha}^T A^{-1} \frac{dA}{d\theta_m} A^{-1} \vec{N} = \frac{\lambda}{W^{(0)}} \Bigg\{ & w_{,x}^2 8U_3 \sin 2\theta_m \left[\left(-\frac{U_2}{4U_3} - \cos 2\theta_m \right) \varepsilon_1 + 2\left(2U_3 \sin 4\theta_m\right) \varepsilon_2 \right] + \\ & + w_{,y}^2 8U_3 \sin 2\theta_m \left[\cos 2\theta_m \varepsilon_1 + \left(\frac{U_2}{4U_3} - 2\cos 2\theta_m \right) \varepsilon_2 \right] + \\ & + 8 w_{,x} w_{,y} U_3 \left[\left(\frac{U_2}{4U_3} \cos 2\theta_m + \cos 4\theta_m \right) \varepsilon_1 + \left(\frac{U_2}{4U_3} \cos 2\theta_m - \cos 4\theta_m \right) \varepsilon_2 \right] \Bigg\} \end{aligned} \tag{34}$$

Now we write the λ derivative due to the orientation angle variation of the *m*-th layer at the point *x, y* (removing the mutual multipliers). The derivative must be equal to zero in the optimum.

$$\frac{d\lambda}{d\theta_m}\frac{\left(-W^{(0)}\right)}{8U_3h_{ply}} = \sin 2(\theta_m-\psi)\left[\frac{U_2}{4U_3}\left(k_1^2-k_2^2\right)+\left(k_1-k_2\right)^2\cos 2(\theta_m-\psi)\right]\frac{1}{3}\left(z_m^2+z_mz_{m-1}+z_{m-1}^2\right)+$$

$$+\lambda\left\{w_{,x}^2\sin 2(\theta_m-\varphi)\left[\left(-\frac{U_2}{4U_3}-\cos 2(\theta_m-\varphi)\right)\varepsilon_1+\cos 2(\theta_m-\varphi)\varepsilon_2\right]+\right.$$

$$+w_{,y}^2\sin 2(\theta_m-\varphi)\left[\cos 2(\theta_m-\varphi)\varepsilon_1+\left(\frac{U_2}{4U_3}-\cos 2(\theta_m-\varphi)\right)\varepsilon_2\right]+$$

$$\left.+w_{,x}w_{,y}\left[\begin{array}{l}\left(\frac{U_2}{4U_3}\cos 2(\theta_m-\varphi)+\cos 4(\theta_m-\varphi)\right)\varepsilon_1+\\+\left(\frac{U_2}{4U_3}\cos 2(\theta_m-\varphi)-\cos 4(\theta_m-\varphi)\right)\varepsilon_2\end{array}\right]\right\}=0$$

(35)

where *m=1,…,K*, and φ, ψ, k_1, k_2 are the orientation angle of the ε_1 axis, the orientation angle of the k_1 axis, the largest and the lowest principal curvature values, respectively. Remind that the derivatives in the figured brackets are calculated in the principal infinitesimal 2D strain axes. The obtained relations (35) are the first order necessary local optimality conditions for the lay-up determination.

If we sum (35) for *m=1,…,K*, then, after the simple algebraic transformations, we obtain the relation with the clear physical meaning:

$$\Delta^*\widetilde{\Pi}+\lambda\Delta^*W^{(0)}=0 \qquad (36)$$

where Δ^* means the increment of the corresponding quantity due to the (small) rotation of the whole plate with the frozen deflections and the frozen in-plane displacements. The relation (36) looks similar to the variational principle (Washizu 1982), describing the plate buckling.

4. ANALYSIS AND DISCUSSION

The above-obtained results clearly demonstrate that there is an influence of the 2D stress distribution on the buckling eigenvalue.

Comparing the items of $\frac{d\lambda}{d\theta_m}$ in a way similar to the paper of Selyugin (2019a), we see that for the outermost layers the bending item divided to 2D infinitesimal strain items leads to a value about squared ratio of the plate thickness and the maximal deflection or K^2 (supposing the buckling mode deflection to be of the same order of magnitude as the ply thickness). For the innermost layers the bending and the 2D-strain items are of the same order of magnitude.

Analogously to the paper of Selyugin (2019b), we come to a conclusion that there is a relation between the lamination parameters of the optimal solution. Indeed, using the formulas for the sine and the cosine of a difference and summing (35) for all layers (i.e., $m=1,...,K$), we obtain that a linear combination of the lamination parameters R_1, R_2, R_3, R_4 (used for creating the *A* matrix) and $R_9, R_{10}, R_{11}, R_{12}$ (used for creating the *D* matrix) with some factors is equal to zero. Hence, among the eight lamination parameters only seven are linearly independent. It is obvious, that the lay-up optimization w.r.t. the lamination parameters only is not sufficient for obtaining an optimal plate lay-up.

To further illustrate the obtained results of the present paper, we consider the simply supported, specially orthotropic plate under the compressive uniaxial in-plane loading (see Fig. 2). The plate has a quadratic shape with the size *a*a*. The plate is compressed along X with the compressive load $N_x = -N$, where $N > 0$.

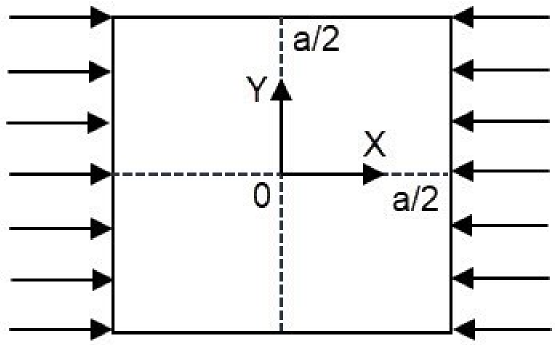

Fig. 2 Square plate loaded by the 1-D uniform in-plane compression

The plate has the simple support boundary conditions along all edges. The origin of the *X*-*Y* coordinate system is located at the plate center. The dashed lines connect the midpoints of the plate opposite sides. The lines coincide the coordinate system. It is assumed that the plate is made of a specially orthotropic material (see Gibson 1994) composed of the symmetrically stacked 0° and 90° tape layers, where the 0° direction is the bending-strongest direction coinciding the *X*-axis.

The buckling solution for the orthotropic plate is known (Gibson 1994). The eigenmode deflection *w* is equal to a product of the corresponding sines.

It is assumed that for the considered mode there is one half-wave of the deflections.

Observing the plate behavior, one may say that the plate (*v, w*) displacements are symmetric relative to the *Y*-axis. The out-of-plane displacements *w* are also symmetric relative to the *X*-axis. The *X*-direction displacements *u* are anti-symmetric relative to the *Y*-

axis. The Y-direction displacements v (parallel to the Y-axis) are anti-symmetric relative to the X-axis. At the dashed line coinciding the X-axis the following equalities are valid

$$v = 0 \quad ; \quad \frac{\partial w}{\partial y} = 0 \tag{37}$$

At the dashed line coinciding the Y-axis we also have

$$u = 0 \quad ; \quad \frac{\partial w}{\partial x} = 0 \tag{38}$$

The indicated observations, after simple analysis, lead to a conclusion that the dashed lines are the principal curvature lines and the principal infinitesimal 2D strain lines. Using the features of the specially orthotropic material, we obtain that at the dashed lines the optimality conditions (35) are valid. Indeed, as the ply orientation directions coincide the X or Y directions, we obtain that at the dashed lines the corresponding sines in (35), as well as one of the deflection slope, are equal to zero. Hence, at the lines the layer first order necessary optimality conditions (35) are valid.

It should be noted that the lay-up solution with the above features may not be unique. Other combinations of 0° and 90° may lead to the same results at the dashed lines. It is explained by the fact that the analysis of this Section is based on the first order necessary conditions of local optimality.

5. CONCLUSIONS

- The first order necessary local lay-up optimality conditions for the plate buckling level maximization are derived;
- The clear physical meaning of the derived conditions is demonstrated and explained;
- A comparative analysis of the bending-related and the 2D-strain-related items is performed. For the outermost plies the bending items are larger than the 2D-strain

items. For the innermost layers the both groups of items are of the same order of magnitude;

- The derived optimality conditions may be used for obtaining the proper lay-up sensitivities;
- The lamination parameters (after possible optimization of the plate w.r.t. them only) may give some preliminary information about the whole plate lay-up. The further optimization of the ply angles is necessary;
- Only seven components of the lamination parameter vector-column are linearly independent for the optimal lay-up solution.

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