Lexicographic and Strictly Weighted Equilibrium Strategy Profiles with Applications to Interval-Valued Bi-matrix Games

By

Somdeb Lahiri

Email: somdeb.lahiri@gmail.com

ORCID: https://orcid.org/0000-0002-5247-3497

(Formerly with) PD Energy University, Gandhinagar (EU-G), India.

June 6, 2026.

Thus version: June 11, 2026.

Abstract

We prove the existence of lexicographic equilibrium strategy profiles for pair-valued bi-matrix games under some conditions, which leads to an existence result for lexicographic mean-standard deviation equilibrium strategy profile in the case of interval-valued bi-matrix games. We also provide necessary conditions and sufficient conditions for a strategy profile to be a strictly weighted equilibrium strategy profile with analogous results being applicable for strictly weighted mean-standard deviation equilibrium strategy profiles for interval-valued bi-matrix games.

Keywords: pair-valued bi-matrix games, lexicographic equilibrium strategy profile, strictly weighted equilibrium strategy profile, interval-valued bi-matrix games, lexicographic mean-standard deviation equilibrium strategy profile, strictly weighted mean-standard deviation equilibrium strategy profile

AMS Subject Classifications: 91A05, 91A10, 91B06

JEL Subject Classification: C72.

1. Introduction: In this paper we consider bi-matrix games with outcomes being pairs of real numbers, instead of just one real number. We refer to such games as "*pair-valued bi-matrix games*". We order $\mathbb{R}^2$ using lexicographic preferences (as defined in Varian (1978)) and provide two existence results for "lexicographic equilibrium strategy profile". We first show that a pure strategy strict equilibrium strategy profile for the bi-matrix game determined by the first coordinates of the outcome pairs, is a strict lexicographic equilibrium strategy profile for the pair-valued bi-matrix game. We then prove that under the assumption that for the row player either the matrix of first coordinates <u>or</u> the matrix of second coordinates has identical rows and for the column player either the matrix of first coordinates <u>or</u> the matrix of

second coordinates have identical columns, then a lexicographic equilibrium strategy profile exists for the pair-valued bi-matrix game. We refer to the condition for the row player as the "identical rows condition" and to the condition for the column player as the "identical columns condition". We then show that our existence results are applicable for interval-valued bi-matrix games, which are bi-matrix games with each outcome being a closed and bounded interval of the real line. Preceding the two existence results is an example to show that unconditional existence of lexicographic equilibrium strategy profile for pair-valued bi-matrix games, may not be possible. Interval-valued bi-matrix games along with a very restrictive solution concept called "strong equilibrium" are introduced in Hladik (2010). Analysis of interval-valued bi-matrix games based on a somewhat weaker solution concept called "support set invariancy" is avalable in Hladik (2019). In an appendix of this paper, we provide an example of a interval-valued bi-matrix game that *does not* satisfy support set invariancy and hence has no "strong equilibrium" strategy profile. Discussion on computational methods for finding "strong equilibrium" is available in Savaskan, Or and Haci (2016) and characterization of such and related equilibrium strategy profiles is available in Afreen and Bhurjee (2025). Pioneering techniques (not related to any of the papers cited here) for the study of interval-valued bi-matrix games is available in Nayak and Pal (2009). In the context of two-player zero-sum (matrix) interval games, Radhakrishnan and Saikeerthana (2020), introduce the concept, that we refer to here as a "lexicographic equilibrium strategy profile".
Since, every interval is uniquely defined by its mid-point and length, we use the linear ordering on intervals inherited from lexicographic preferences on $\mathbb{R}^2$, with the first coordinate denoting the mid-point of the interval and the second coordinate denoting *the negative of* half the length of the interval. Using this linear ordering, our existence results in the more general context imply the existence of related equilibrium strategy profiles for interval-valued bi-matrix games.
The classical approach to solving inter-valued bi-matrix game would view it as a Bayesian game and rely on payoffs that are the means of the interval-dependent priors. The approach adopted here departs from the classical approach, in that we consider both the mean and the standard deviation, assuming that the random payoff from each interval is distributed uniformly over the interval.We refer to our solution concept for interval-valued bi-matrix games as "lexicographic mean-standard deviation

equilibrium strategy profile". Since comparing risky prospects using mean and variance- particularly in the context of portfolio choice theory- is one of several accepted and mainstream procedures in decision theory, our approach is not a significant departure from traditional methods used in game theory. Resnik (1987) contains a sufficiently comprehensive discussion on decision theory and game theory for our present purpose. Chapter 6 of Biswas (1997) continues to remain a reliable source of information about the standard theory of portfolio analysis.

It ought to be mentioned, that there has been considerable work on interval-valued matrix (i.e., zero-sum) games that has been cited in Hladik (2010). To the best of our knowledge, the solution concept we discuss here has not been defined in any work other than Radhakrishnan and Saikeerthana (2020). We extend the solution concept to pair-valued bi-matrix games and provide two existence results in this considerably more general context.

We will now provide some clarifications for the lexicographic mean-standard deviation equilibrium strategy profile in the context of interval valued games, assuming that each player views each entry in its outcome matrix as a uniformly distributed random variable over the specified interval. A strategy profile would then lead to what is conventionally known as a "compound lottery". While the expected value of the means as we have calculated would be equal to the expected value of the compound lottery, the expected value of the standard deviations as we have calculated *may not* be equal to the standard deviation of the compound lottery. This observation ought to be factored in when considering the lexicographic mean-standard deviation equilibrium strategy profile as a possible solution concept.

The next- and more crucial-point that ought to be noted is that in behavioral and/or decision sciences, we feel it is "far-fetched" to claim a "one size fits all" theory of human (interactive) behavior/decision making. Suppose an individual is given a choice between a uniformly distributed random monetary return with "mean" INR 1, 00, 000 and standard deviation 1000, and a second uniformly distributed random monetary return with "mean" INR 99, 999 and standard deviation 0 (i.e., getting INR 99, 999 with probability 1), a decision making criteria that gives priority to the mean over uncertainty aversion (as in the case of lexicographic mean-standard deviation equilibrium strategy profile) would lead to the first option being chosen, contrary what appears to be reasonable in reality, namely, choosing the second option would.

Reversing the roles of the means and standard deviation could also lead to an unreasonable conclusion where getting INR 1 with probability 1 is preferred to getting a mean return of INR 1, 00, 000 with standard deviation $10^{-10,000}$. This unreasonableness in the second scenario arises due to excessive emphasis on "uncertainty aversion". The unreasonableness persists if we use the "co-efficient of variation" (i.e., mean divided by standard deviation) instead of the pair comprising of the mean and standard deviation.

In order to rectify the paradoxical observations noted in the previous paragraph, we consider the equilibrium strategy profile for the bi-matrix game that results from player-dependent weighted sums of the means and standard deviations with both the mean and the standard deviation having positive weights. We refer to such an equilibrium profile as a strictly weighted mean-standard deviation equilibrium strategy profile for the interval-valued bi-matrix game. This solution concept is an application to interval-valued bi-matrix game of a solution concept for pair-valued bi-matrix game, the latter solution concept being what we refer to as a strictly weighted equilibrium strategy profile. A strictly weighted equilibrium strategy profile for a pair-valued bi-matrix game, is an equilibrium strategy profile for the bi-matrix game that results by considering a player-dependent weighted sum of the two coordinates of pair-valued bi-matrix game with both weights being strictly positive. It is well known -as a basic result for bi-matrix games- that any such game will always have an equilibrium strategy profile. It is easily proved, that for a strategy profile to be a strictly weighted equilibrium strategy profile for a pair-valued bi-matrix game, it is necessary that if for a player the expected value of a coordinate corresponding to a particular action is less than the expected value of the coordinate, then the expected value of the other coordinate corresponding to the same action must be greater than the expected value of this latter coordinate for the same player. Subsequently, we show that if a strategy profile is an equilibrium strategy profile for the bi-matrix game obtained by subtracting coordinate 3-r from coordinate r for some $r \in \{1, 2\}$, and further if whenever the excess of the expected value of coordinate 3-r corresponding to an action over the expected value of coordinate 3-r is positive player, it is also the case that the excess of the expected value of coordinate r over the expected value of coordinate r at the same action is greater than the former excess for the same player, then the strategy profile must be a strictly weighted equilibrium strategy profile for the pair-valued bi-matrix game. This, solution concept when used appropriately in the context of interval-valued bi-matrix game may lead to more "balanced" decisions, as can easily be verified by applying 50% weight to each criterion by the two players in the numerical examples in the previous paragraph.

Thus, the structure of the pair in the entries of the outcome matrices, has an important role to play in determining or expecting which solution concept is or likely to be more realistic. Fortunately, the theory of pair-valued bi-matrix games that we have have developed in sections 2, 4 and 5 of this paper, provide a wide variety of solution concepts to choose from in the context of specific applications.

2. The Framework of Pair-valued Bi-matrix Games: For positive integers m, n consider an interactive decision making context (game) with two players/participants/decision-makers who we denote by 'a' and 'b' respectively, and where player 'a' has m actions to choose from and 'b' has 'n' actions to choose from. A **strategy for player 'a'** is a point in $\Delta^{m-1} = \{x\epsilon\mathbb{R}_+^m | \sum_{i=1}^{m} x_i = 1\}$ and a **strategy for player 'b'** is a point in $\Delta^{n-1} = \{x\epsilon\mathbb{R}_+^n | \sum_{j=1}^{n} y_j = 1\}$.

A strategy for the player 'a' is a probability distribution on {1, ..., m}, with each coordinate of the strategy denoting the probability with which the player 'a' chooses the corresponding action. A strategy for player 'b' is a probability distribution on {1, ..., n}, with each coordinate of the strategy denoting the probability with which player 'b' chooses the corresponding action.

Notation 1: For $i\epsilon\{1, \ldots, m\}$, let $E^{(m,i)}$ be the **m-dimensional i^{th} unit coordinate column vector** (i.e., 1 in its i^{th} coordinate and 0 in all other coordinates) and let $E^{(m)} = \sum_{h=1}^{m} E^{(m,h)}$ be the **m-dimensional sum vector**. For $j\epsilon\{1, \ldots, n\}$, let $E^{(n,j)}$ be the **n-dimensional j^{th} unit coordinate column vector** (i.e., 1 in its j^{th} coordinate and 0 in all other coordinates) and let $E^{(n)} = \sum_{k=1}^{n} E^{(n,k)}$ be the n**-dimensional sum vector**.■

If a strategy x for player 'a' is such that $x = E^{(m,i)}$ for some $i\epsilon\{1, \ldots, m\}$, then x is said to be the **i^{th} pure strategy for the player 'a'**.

If a strategy y for player 'b' is such that $y = E^{(n,j)}$ for some $j\epsilon\{1, \ldots, n\}$, then y is said to be the **j^{th} pure strategy for the player 'b'**.

A pair (x, y) where x is a strategy for player 'a' and y is a strategy for player 'b' is said to be a **strategy profile**.

If both x and y are pure strategies then the strategy profile (x, y) is said to be a **pure strategy profile**.

Note 1: If (x, y) is a pure strategy profile, then we may represent it as (i, j) if x is the i^{th} pure strategy for player 'a' and j is the j^{th} pure strategy for player 'b'.∎

Let $\mathbb{R}^{m\times n}$ be the set of all m×n matrices.

Given $C\epsilon\mathbb{R}^{m\times n}$ and (i, j)ϵ{1, …, m}×{1, ..., n}, let c_{ij} denote the $(i, j)^{th}$ entry of C, C_i denote the i^{th} row of C and C^j denote the j^{th} column of C.

For our purposes here, a member of $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ will be referred to as a **bi-matrix game** and $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ is the **set of all bi-matrix games**.

Given $(A, B)\epsilon\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ and (i, j)ϵ {1, …, m}×{1, ..., n}, if player 'a' (who may also be referred to as the **row player**) chooses row i and player 'a' (who may also be referred to as the **column player**) chooses column j, the payoff received by the row player is a_{ij} and payoff received by the column player is b_{ij}. A is the **payoff matrix of the row player** and B **is the payoff matrix of the column player**.

Given a bi-matrix game (A, B) a strategy profile (x, y) is said to be an **equilibrium strategy profile for (A, B)**, if for all (i, j)ϵ{1, …, m}×{1, ..., n}: $A_i y \le x^T Ay$ and $x^T B^j \le x^T By$.

An equilibrium strategy profile (x, y) for (A, B) is said to be a **pure strategy equilibrium profile for (A, B)** if (x, y) is a pure strategy profile.

Let $\mathbb{R}^{(2,\, m\times n)}$ denote the set of all matrices m×n matrices whose entries are pairs of real number.

If $V\epsilon\mathbb{R}^{(2,\, m\times n)}$, then for all (i, j)ϵ{1, …, m}×{1, ..., n}, the typical (though not invariable) notations are as follows: $(i, j)^{th}$ entry of V will be denoted by $(v_{ij}^{(1)}, v_{ij}^{(2)})$, the i^{th} row of V will be denoted by V_i and the j^{th} column of V will be denoted by V^j.

For our purposes here, a member of $\mathbb{R}^{(2,\, m\times n)} \times \mathbb{R}^{(2,\, m\times n)}$ will be referred to as a **pair-valued bi-matrix game** and $\mathbb{R}^{(2,\, m\times n)} \times \mathbb{R}^{(2,\, m\times n)}$ as the **set of all pair-valued bi-matrix games**.

Let ≽ be a binary relation on $\mathbb{R}^2$ such that for all $(\gamma_1, \eta_1), (\gamma_2, \eta_2)\epsilon\mathbb{R}^2$: $(\gamma_1, \eta_1) \succcurlyeq (\gamma_2, \eta_2)$ <u>if and only if</u> either $\gamma_1 > \gamma_2$ or $\gamma_1 = \gamma_2$ and $\eta_1 \ge \eta_2$.

Clearly ≽ is a linear order on $\mathbb{R}^2$. It is generally known as the **lexicographic preference ordering on** $\mathbb{R}^2$.

Notation 2: Given a pair-valued bi-matrix game (V(a), V(b)), for rϵ{1, 2} and αϵ{a, b}, let $V^{(r)}(\alpha)$ be the m×n real-valued matrix whose $(i, j)^{th}$ term for (i, j)ϵ{1, …, m}×{1, ..., n} is $v_{ij}^{(r)}(\alpha)$. The i^{th} row of $V^{(r)}(\alpha)$ for iϵ{1, …, m} is denoted by $V_i^{(r)}(\alpha)$ and the j^{th} column of $V^{(r)}(\alpha)$ for jϵ{1, …, n} is denoted by$V^{(r)j}(\alpha)$. For $(x, y)\epsilon\Delta^{m-1}\times\Delta^{n-1}$

and $\alpha\epsilon\{a, b\}$, $x^TV(\alpha)y = \sum_{i=1}^{m}\sum_{j=1}^{n} x_iy_j(v_{ij}^{(1)}(\alpha), v_{ij}^{(2)}(\alpha)) =$

$\sum_{i=1}^{m}\sum_{j=1}^{n} (x_iy_jv_{ij}^{(1)}(\alpha), x_iy_jv_{ij}^{(2)}(\alpha))$. ▌

Given a pair-valued bi-matrix game (V(a), V(b)), a strategy profile (x, y) is said to be a **lexicographic equilibrium strategy profile** for (V(a), V(b)) if $x^TV(a)y \succcurlyeq V_i(a)y$ for all $i\epsilon\{1, \ldots, m\}$and $x^TV(b)y \succcurlyeq x^TV^j(b)$ for all $j\epsilon\{1, \ldots, n\}$.

Note 2: It ought to be emphasized that the entries of V(a) and V(b) are *not payoffs*. They are "*outcomes*" that can be linearly ordered. ▌

3. Possibility of Non-existence of Lexicographic Equilibrium Strategy Profile: In this section we will provide an example to show that unconditional existence of lexicographic equilibrium profile may not be possible.

Example 1: Let V(a) = $\begin{bmatrix} (1, -2) & (-1, -2) \\ (-2, -1) & (2, -1) \end{bmatrix}$ and V(b) = $\begin{bmatrix} (-1, -1) & (1, -1) \\ (2, -1) & (-2, -1) \end{bmatrix}$.

Let (x, y) be a strategy profile. For $y_1 > \frac{1}{2}$, $y_1 - (1 - y_1) = 2y_1 - 1 > 0$ and $-2y_1 + 2(1-y_1) = 2 - 4y_1 < 0$.

Thus, the best reply of 'a' to y with $y_1 > \frac{1}{2}$ is (1, 0).

For $y_1 < \frac{1}{2}$, $y_1 - (1 - y_1) = 2y_1 - 1 < 0$ and $-2y_1 + 2(1-y_1) = 2 - 4y_1 > 0$.

Thus, the best reply of 'a' to y with $y_1 < \frac{1}{2}$ is (0, 1).

For $y_1 = \frac{1}{2}$, $y_1 - (1 - y_1) = 2y_1 - 1 = 0$ and $-2y_1 + 2(1-y_1) = 2 - 4y_1 = 0$.

Since, the second coordinates in the second row of V(a) are equal to -1 which is greater than the second coordinate in the first row of V(a), the latter being equal to -2, the best reply of 'a' to y with $y_1 = \frac{1}{2}$ is (0, 1).

However, if 'a' chooses (1, 0), the best reply of 'b' is z = (0, 1), i.e., $z_1 = 0 < \frac{1}{2} < y_1$ and if 'a' chooses (0, 1) the best reply of 'b' is is z = (1, 0), i.e., $z_1 = 1 > \frac{1}{2} \geq y_1$.

Thus, (V(a), V(b)) does not have any lexicographic equilibrium strategy profile.

4. Existence of Lexicographic Equilibrium: We will now prove two existence results for lexicographic equilibrium profile for pair-valued bi-matrix games. The first proposition reduces the existence problem to the existence of an equilibrium strategy profile for the bi-matrix game ($V^{(1)}(a)$, $V^{(1)}(b)$).

Proposition 1: Suppose (V(a), V(b)) is a pair-valued bi-matrix game and suppose (i, j) is a pure strategy equilibrium profile for $(V^{(1)}(a), V^{(1)}(b))$ satisfying $v_{ij}^{(1)}(a) > v_{hj}^{(1)}(a)$ for all h∈{1, …, m}\{i} and $v_{ij}^{(1)}(b) > v_{ik}^{(1)}(b)$ for all k∈{1, …, n}\{j}. Then (i, j) is a pure strategy *strict* lexicographic equilibrium strategy profile for (V(a), V(b)), i.e., unilateral deviations from $(E^{(m,i)}, E^{(n,j)})$ makes the deviator "strictly worse-off". .

Proof: $v_{ij}^{(1)}(a) > v_{hj}^{(1)}(a)$ for all h∈{1, …, m}\{i}implies $E^{m,i)T}V^{(1)}(a)E^{(n,j)} > V_h^{(i)}(a)E^{(n,j)}$ for all h∈{1, …, m}\{i} and thus $E^{m,i)T}V(a)E^{(n,j)} \succcurlyeq V_h(a)E^{(n,j)}$ for all h∈{1, …, m}\{i}. Further, $E^{m,i)T}V^{(1)}(a)E^{(n,j)} > V_h^{(i)}(a)E^{(n,j)}$ for all h∈{1, …, m}\{i} implies that *there does not exist* $x \in \Delta^{m-1}\backslash\{E^{(m,i)}\}$ such that $x^T V(a)E^{(n,j)} \succcurlyeq E^{m,i)T}V(a)E^{(n,j)}$.

$v_{ij}^{(1)}(b) > v_{ik}^{(1)}(b)$ for all k∈{1, …, n}\{j} implies $E^{m,i)T}V^{(1)}(b)E^{(n,j)} > E^{m,i)T}V^{(1)k}(b)$ for all k∈{1, …, n}\{j} and thus $E^{m,i)T}V(b)E^{(n,j)} \succcurlyeq E^{m,i)T}V^k(b)$ for all k∈{1, …, n}\{j}. Further, $E^{m,i)T}V^{(1)}(b)E^{(n,j)} > E^{m,i)T}V^{(1)k}(b)$ for all k∈{1, …, n}\{j} implies that there does not exist $y \in \Delta^{n-1}\backslash\{E^{(n,i)}\}$ such that $E^{(m,i)T}V(b) \succcurlyeq E^{m,i)T}V(b)E^{(n,j)}$

Thus, (i, j) is a strict lexicographic equilibrium strategy profile for (V(a), V(b)).
Q.E.D.

V(a) is said to satisfy **the identical rows condition** if for some r∈{1, 2}, $V_i^{(r)}(a) = V_h^{(r)}(a)$ for all i, h∈{1, …, m}.

V(b) is said to satisfy **the identical columns condition** if for some s∈{1, 2}, $V^{(s)j}(b) = V^{(s)k}(b)$ for all j, k∈{1, …, n}

Proposition 2: Suppose (V(a), V(b)) is a pair-valued bi-matrix game such that V(a) satisfies the identical rows condition and V(b) satisfies the identical columns condition. Then, there exists a lexicographic equilibrium strategy profile for (V(a), V(b)).

Proof: Consider the bi-matrix game $(V^{(1)}(a) + V^{(2)}(a), V^{(1)}(b) + V^{(2)}(b))$ and let (x^*, y^*) be an equilibrium strategy profile for this bi-matrix game. That such an equilibrium profile exists is well known (see for instance Chandrasekaran (nd)).

[For r, s∈{1, 2} all i, h∈{1, …, m}, $V_i^{(r)}(a) = V_h^{(r)}(a)$ and for all j, k∈{1, …, n}, $V^{(s)j}(b) = V^{(s)k}(b)$] implies [For r, s∈{1, 2} and all $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$: $x^T V^{(r)}(a)y = V_i^{(r)}(a)y = V_1^{(r)}(a)y$ for all i∈{1, …, m} and $x^T V^{(s)}(b)y = x^T V^{(s)j}(b) = x^T V^{(s)1}(b)$ for all j∈{1, …, m}].

Thus, $x^{*T}V^{(3-r)}(a)y^* + V_1^{(r)}(a)y^* = x^{*T}(V^{(3-r)}(a) + V^{(r)}(a))y^* \geq V_i^{(3-r)}(a)y^* + V_i^{(r)}(a)y^* = V_i^{(3-r)}(a)y^* + V_1^{(r)}(a)y^*$ for all $i\in\{1, \ldots, m\}$ and $x^{*T}V^{(3-s)}(b)y^* + x^{*T}V^{(s)1}(b) = x^{*T}(V^{(3-s)}(b) + V^{(s)}(b))y^* \geq x^{*T}V^{(3-s)j}(b) + x^{*T}V^{(s)j}(b) = x^{*T}V^{(3-s)j}(b) + x^{*T}V^{(s)1}(b)$ for all $j\in\{1, \ldots, n\}$.

Thus, $x^{*T}V^{(3-r)}(a)y^* \geq V_i^{(3-r)}(a)y^*$ and $x^{*T}V^{(r)}(a)y^* = V_1^{(r)}(a)y^* = V_i^{(r)}(a)y^*$ for all $i\in\{1, \ldots, m\}$ and $x^{*T}V^{(3-s)}(b)y^* \geq x^{*T}V^{(3-s)j}(b)$ and $x^{*T}V^{(s)}(b)y^* = x^{*T}V^{(s)1}(b) = x^{*T}V^{(s)j}(b)$ for all $j\in\{1, \ldots, n\}$.

Thus, $x^{*T}V(a)y^* \succcurlyeq V_i(a)y^*$ for all $i\in\{1, \ldots, m\}$ and $x^{*T}V(b)y^* \succcurlyeq x^{*T}V^j(b)$ for all $j\in\{1, \ldots, n\}$.

Thus, (x^*, y^*) is a lexicographic equilibrium strategy profile for (V(a), V(b)). Q.E.D.

5. Strictly Weighted Equilibrium Strategy Profile For Pair-valued Bi-matrix Games: In this section we discuss an alternative solution concept for pair-valued bi-matrix games.

Given a pair-valued bi-matrix game (V(a), V(b)), a strategy profile (x, y) is said to be a **strictly weighted equilibrium strategy profile** for (V(a), V(b)) if there exists a pair of real numbers $(\theta(a), \theta(b))\in(0, 1)\times(0, 1)$ such that (x, y) is an equilibrium strategy profile for the bi-matrix game $(\theta(a)V^{(1)}(a) + (1- \theta(a))V^{(2)}(a), \theta(b)V^{(1)}(b) + (1- \theta(b))V^{(2)}(b))$.

The following proposition is of some interest in the context of pair-valued bi-matrix games.

Proposition 3: For a pair-valued bi-matrix game (V(a), V(b)) and any $(\theta(a), \theta(b))\in(0, 1)\times(0, 1)$, the bi-matrix game $(\theta(a)V^{(1)}(a) + (1- \theta(a))V^{(2)}(a), \theta(b)V^{(1)}(b) + (1- \theta(b))V^{(2)}(b))$ has an equilibrium strategy profile. Further, if (x, y) is a strictly weighted equilibrium strategy profile for (V(a), V(b)) then (i) for $i\in\{1, \ldots, m\}$: $\min\{x^TV^{(1)}(a)y - V_i^{(1)}(a)y, x^TV^{(2)}(a)y - V_i^{(2)}(a)y\} < 0$ <u>implies</u> $\max\{x^TV^{(1)}(a)y - V_i^{(1)}(a)y, x^TV^{(2)}(a)y - V_i^{(2)}(a)y\} > 0$, and (ii) for $j\in\{1, \ldots, n\}$: $[\min\{x^TV^{(1)}(b)y - x^TV^{(1)j}(b), x^TV^{(2)}(b)y - x^TV^{(2)j}(b)\} < 0$ <u>implies</u> $\max\{x^TV^{(1)}(b)y - x^TV^{(1)j}(b), x^TV^{(2)}(b)y - x^TV^{(2)j}(b)\} \geq 0]$.

Proof: The first part of the proposition follows from the known result about existence of equilibrium strategy profile for bi-matrix games (see Chandrasekaran (nd)). Hence let us proceed to the proof of the second part.

Note that for $(\theta(a), \theta(b))\in(0, 1)\times(0, 1)$:

For all $i\in\{1, \ldots, m\}$, $x^T(\theta(a)V^{(1)}(a) + (1- \theta(a))V^{(2)}(a)y - (\theta(a)V_i^{(1)}(a) + (1- \theta(a))V_i^{(2)}(a))y = \theta(a)(x^TV^{(1)}(a)y - V_i^{(1)}(a)y) + (1- \theta(a))(x^TV^{(2)}(a)y - V_i^{(2)}(a)y)$;

&

For all j∈{1, …, n}: $x^T(\theta(b)V^{(1)}(b) + (1- \theta(b))V^{(2)}(b))y - x^T(\theta(b)V^{(1)j}(b) + (1- \theta(b))V^{(2)j}(b)) = \theta(b)(x^TV^{(1)}(b)y - x^TV^{(1)j}) + (1- \theta(b))(x^TV^{(2)}(b)y - x^TV^{(2)j})$.

If for some i∈{1, …, m}, $x^TV^{(1)}(a)y - V_i^{(1)}(a)y \le 0$ and $x^TV^{(2)}(a)y - V_i^{(2)}(a)y \le 0$ with at least one strict inequality, then $x^T(\theta(a)V^{(1)}(a) + (1- \theta(a))V^{(2)}(a)y - (\theta(a)V_i^{(1)}(a) + (1- \theta(a))V_i^{(2)}(a))y = \theta(a)(x^TV^{(1)}(a)y - V_i^{(1)}(a)y) + (1- \theta(a))(x^TV^{(2)}(a)y - V_i^{(2)}(a)y) < 0$, contradicting (x, y) is an equilibrium strategy profile for $(\theta(a)V^{(1)}(a) + (1- \theta(a))V^{(2)}(a), \theta(b)V^{(1)}(b) + (1- \theta(b))V^{(2)}(b))$.

This proves (i).

An almost similar argument proves (ii). Q. E. D.

The following proposition provides sufficient conditions for a strategy profile (x, y) to be a strictly weighted equilibrium strategy profile for a pair-valued bi-matrix game (V(a), V(b)).

Proposition 4: Let (V(a), V(b)) be a pair-valued bi-matrix game.

Suppose that for some r∈{1, 2}, (x, y) is an equilibrium strategy profile for the bi-matrix game $(V^{(r)}(a) - V^{(3-r)}(a), V^{(r)}(b) - V^{(3-r)}(b))$ such that for all i∈{1, …, m}, satisfying $V_i^{(3-r)}(a)y > x^TV^{(3-r)}(a)y$, it is the case that $x^TV^{(r)}(a)y > V_i^{(r)}(a)y$ and for all j∈{1, ..., n}, satisfying $x^TV^{(3-r)j}(b) > x^TV^{(3-r)}(b)y$, it is the case that $x^TV^{(r)}(b) > x^TV^{(r)j}$.

Then (x, y) is a strictly weighted equilibrium strategy profile for (V(a), V(b)).

Proof: Let r∈{1, 2} and suppose (x, y) is a strategy profile that satisfies the following properties for (i, j) ∈{1, …, m}×{1, ..., n}:

(I) $V_i^{(3-r)}(a)y > x^TV^{(3-r)}(a)y$ implies $x^TV^{(r)}(a)y > V_i^{(r)}(a)y$.

(II) $x^TV^{(3-r)j}(b) > x^TV^{(3-r)}(b)y$ implies $x^TV^{(r)}(b)y > x^TV^{(r)j}$.

$x^TV^{(r)}(a)y > V_i^{(r)}(a)y$ if and only if $x^TV^{(r)}(a)y - V_i^{(r)}(a)y > 0$.

Thus, $x^TV^{(r)}(a)y > V_i^{(r)}(a)y$ if and only if $x^T(V^{(r)}(a) - V^{(3-r)}(a))y - (V_i^{(r)}(a) - V_i^{(3-r)}(a))y = (x^TV^{(r)}(a)y - V_i^{(r)}(a)y) - (x^TV^{(3-r)}(a)y - V_i^{(3-r)}(a)y) > - (x^TV^{(3-r)}(a)y - V_i^{(3-r)}(a)y) = V_i^{(3-r)}(a)y - x^TV^{(3-r)}(a)y$.

Thus (I) is equivalent to the following statement:

(I*) $V_i^{(3-r)}(a)y > x^TV^{(3-r)}(a)y$ implies $x^T(V^{(r)}(a) - V^{(3-r)}(a))y - (V_i^{(r)}(a) - V_i^{(3-r)}(a))y > V_i^{(3-r)}(a)y - x^TV^{(3-r)}(a)y$.

$x^TV^{(r)}(b)y > x^TV^{(r)j}$ if and only if $x^TV^{(r)}(b)y - x^TV^{(r)j} > 0$.

Thus, $x^TV^{(r)}(b)y > x^TV^{(r)j}$ <u>if and only if</u> $x^T(V^{(r)} - V^{(3-r)j})y - x^T(V^{(r)j} - V^{(3-r)j}) = (x^TV^{(r)}y - x^TV^{(r)j}) - (x^TV^{(3-r)}y - x^TV^{(3-r)j}) > - (x^TV^{(3-r)}y - x^TV^{(3-r)j}) = x^TV^{(3-r)j} - x^TV^{(3-r)}y$.

Thus, (II) is equivalent to the following statement:

(II*) $x^TV^{(3-r)j}(b) > x^TV^{(3-r)}(b)y$ <u>implies</u> $x^T(V^{(r)} - V^{(3-r)j})y - x^T(V^{(r)j} - V^{(3-r)j}) > x^TV^{(3-r)j} - x^TV^{(3-r)}y$.

Without loss of generality suppose r = 1. (The proof for r = 2 is completely analogous).

Suppose (x, y) is an equilibrium strategy profile for the bi-matrix game $(V^{(1)}(a) - V^{(2)}(a), V^{(1)}(b) - V^{(2)}(b))$.

Thus, for all $i\epsilon\{1, \ldots, m\}$ and $j\epsilon\{1, \ldots, n\}$ it must be the case that $x^T(V^{(1)}(a) - V^{(2)}(a))y - (V_i^{(1)}(a) - V_i^{(2)}(a))y \geq 0$ and $x^T(V^{(1)}(b) - V^{(2)}(b))y - x^T(V^{(1)j}(b) - V^{(2)j}(b)) \geq 0$.

If for some $i\epsilon\{1, \ldots, m\}$, $V_i^{(2)}(a)y - x^TV^{(2)}(a)y \leq 0$ then for all $\theta\epsilon[0, 1]$, $\theta[x^T(V^{(1)}(a) - V^{(2)}(a))y - (V_i^{(1)}(a) - V_i^{(2)}(a))y] \geq V_i^{(2)}(a)y - x^TV^{(2)}(a)y$ whence for all $\theta\epsilon[0, 1]$, $x^T(\theta V^{(1)}(a) + (1-\theta)V^{(2)}(a))y - (\theta V_i^{(1)}(a) + (1-\theta)V_i^{(2)}(a))y \geq 0$.

If $\{i|V_i^{(2)}(a)y - x^TV^{(2)}(a)y > 0\} = \phi$, then let $\theta(a)\epsilon(0, 1)$. Clearly, $x^T(\theta(a)V^{(1)}(a) + (1-\theta(a))V^{(2)}(a))y - (\theta(a)V_i^{(1)}(a) + (1-\theta(a))V_i^{(2)}(a))y \geq 0$ for all $i\epsilon\{1, \ldots, m\}$.

Hence suppose, $\{i|V_i^{(2)}(a)y - x^TV^{(2)}(a)y > 0\} \neq \phi$.

Let $\theta(a)\epsilon(0, 1)$ be such that $\theta(a) > \max\{\frac{V_i^{(2)}(a)y - x^TV^{(2)}(a)y}{x^T(V^{(1)}(a) - V^{(2)}(a)y - (V_i^{(1)}(a) - V_i^{(2)}(a))y} \mid V_i^{(2)}(a)y - x^TV^{(2)}(a)y > 0\}$.

Such a $\theta(a)$ exists since by hypothesis (I*), if $V_i^{(2)}(a)y - x^TV^{(2)}(a)y > 0$ then $x^T(V^{(1)}(a) - V^{(2)}(a))y - (V_i^{(1)}(a) - V_i^{(2)}(a))y > V_i^{(2)}(a)y - x^TV^{(2)}(a)y > 0$.

Thus, $\theta(a)[x^T(V^{(1)}(a) - V^{(2)}(a))y - (V_i^{(1)}(a) - V_i^{(2)}(a))y] > V_i^{(2)}(a)y - x^TV^{(2)}(a)y$ for all $i\epsilon\{1, \ldots, m\}$ satisfying $V_i^{(2)}(a)y - x^TV^{(2)}(a)y > 0$.

Hence, $x^T(\theta(a)V^{(1)}(a) + (1-\theta(a))V^{(2)}(a))y - (\theta(a)V_i^{(1)}(a) + (1-\theta(a))V_i^{(2)}(a))y > 0$ for all $i\epsilon\{1, \ldots, m\}$ satisfying $V_i^{(2)}(a)y - x^TV^{(2)}(a)y > 0$.

Further, as we have already noted, $x^T(\theta(a)V^{(1)}(a) + (1-\theta(a))V^{(2)}(a))y - (\theta(a)V_i^{(1)}(a) + (1-\theta(a))V_i^{(2)}(a))y \geq 0$ for all $i\epsilon\{1, \ldots, m\}$ satisfying $V_i^{(2)}(a)y - x^TV^{(2)}(a)y \leq 0$.

Thus, $x^T(\theta(a)V^{(1)}(a) + (1-\theta(a))V^{(2)}(a))y - (\theta(a)V_i^{(1)}(a) + (1-\theta(a))V_i^{(2)}(a))y \geq 0$ for all $i\epsilon\{1, \ldots, m\}$.

Similarly, $x^T(V^{(1)}(b) - V^{(2)}(b))y - x^T(V^{(1)j}(b) - V^{(2)j}(b)) \geq 0$ for all $j \in \{1, \ldots, n\}$ implies $\theta[x^T(V^{(1)}(b) - V^{(2)}(b))y - x^T(V^{(1)j}(b) - V^{(2)j}(b))] \geq x^T V^{(2)j}(b) - x^T V^{(2)}(b)y$ for all $\theta \in [0, 1]$ whenever $x^T V^{(2)j}(b) - x^T V^{(2)}(b)y \leq 0$, whence for all $\theta \in [0, 1]$, $x^T(\theta V^{(1)}(b) + (1-\theta)V^{(2)}(b))y - x^T(\theta V^{(1)j}(b) + (1-\theta)V^{(2)j}(b)) \geq 0$ if $x^T V^{(2)j}(b) - x^T V^{(2)}(b)y \leq 0$.

If $\{j | x^T V^{(2)j}(b) - x^T V^{(2)}(b)y > 0\} = \phi$, then let $\theta(b) \in (0, 1)$.

Clearly, $x^T(\theta(b)V^{(1)}(b) + (1-\theta(b))V^{(2)}(b))y - x^T(\theta(b)V^{(1)j}(b) + (1-\theta(b))V^{(2)j}(b)) \geq 0$ for all $j \in \{1, \ldots, n\}$.

Hence suppose, $\{j | x^T V^{(2)j}(b) - x^T V^{(2)}(b)y > 0\} \neq \phi$.

Let $\theta(b) \in (0, 1)$ be such that $\theta(b) > \max\{ \frac{x^T V^{(2)j}(b) - x^T V^{(2)}(b)y}{x^T(V^{(1)}(b) - V^{(2)}(b)y - x^T(V^{(1)j}(b) - V^{(2)j}(b))}$ $| x^T V^{(2)j}(b) - x^T V^{(2)}(b)y > 0\}$.

Such a $\theta(b)$ exists, since by hypothesis (II*), $x^T V^{(2)j}(b) - x^T V^{(2)}(b)y > 0$ implies $x^T(V^{(1)}(b) - V^{(2)}(b))y - x^T(V^{(1)j}(b) - V^{(2)j}(b)) > x^T V^{(2)j}(b) - x^T V^{(2)}(b)y > 0$.

Thus, $\theta(b)[x^T(V^{(1)}(b) - V^{(2)}(b)y - x^T(V^{(1)j}(b) - V^{(2)j}(b))] > x^T V^{(2)j}(b) - x^T V^{(2)}(b)y$ whenever $x^T V^{(2)j}(b) - x^T V^{(2)}(b)y > 0$.

Hence, $x^T(\theta(b)V^{(1)}(b) + (1-\theta(b))V^{(2)}(b))y - x^T(\theta(b)V^{(1)j}(b) + (1-\theta(b))V^{(2)j}(b)) > 0$ whenever $x^T V^{(2)j}(b) - x^T V^{(2)}(b)y > 0$.

Further, as we have already noted, $x^T(\theta V^{(1)}(b) + (1-\theta)V^{(2)}(b))y - x^T(\theta V^{(1)j}(b) + (1-\theta)V^{(2)j}(b)) \geq 0$ if $x^T V^{(2)j}(b) - x^T V^{(2)}(b)y \leq 0$.

Thus, $x^T(\theta V^{(1)}(b) + (1-\theta)V^{(2)}(b))y - (\theta V^{(1)j}(b) + (1-\theta)V^{(2)j}(b)) \geq 0$ for all $j \in \{1, \ldots, n\}$.

Thus, (x, y) is an equilibrium strategy profile for the bi-matrix game $(\theta(a)V^{(1)}(a) + (1-\theta(a))V^{(2)}(a), \theta(b)V^{(1)}(b) + (1- \theta(b))V^{(2)}(b))$.

Thus, (x, y) is a strictly weighted equilibrium profile for the pair-valued bi-matrix game (V(a), V(b)). Q. E. D.

6. Interval-valued Bi-matrix Games: In this section we consider inter-valued bi-matrix games and associated solution concepts to which propositions 1 to 4 can be applied.

Let $\overline{A}, \underline{A}, \overline{B}, \underline{B} \in \mathbb{R}^{m \times n}$ be such that for all $(i, j) \in \{1, \ldots, m\} \times \{1, ..., n\}$, $\overline{a}_{ij} \geq \underline{a}_{ij}$ and $\overline{b}_{ij} \geq \underline{b}_{ij}$.

The **interval matrix** $[\underline{A}, \overline{A}] = \{A |$ for all $(i, j) \in \{1, \ldots, m\} \times \{1, ..., n\}$, $\overline{a}_{ij} \geq a_{ij} \geq \underline{a}_{ij}\}$.

The **interval matrix** $[\underline{B}, \overline{B}] = \{B |$ for all $(i, j) \in \{1, \ldots, m\} \times \{1, ..., n\}$, $\overline{b}_{ij} \geq b_{ij} \geq \underline{b}_{ij}\}$.

For (i, j)ϵ{1, …, m}×{1, ..., n} the (i, j)th entry of $[\underline{A}, \overline{A}]$ is denoted by $[\underline{a}_{ij}, \overline{a}_{ij}]$ and the (i, j)th entry of $[\underline{B}, \overline{B}]$ is denoted by $[\underline{b}_{ij}, \overline{b}_{ij}]$.

The pair $[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$ = {(A, B)| for all (i, j)ϵ{1, …, m}×{1, ..., n}, $\overline{a}_{ij} \geq a_{ij} \geq \underline{a}_{ij}$ and $\overline{b}_{ij} \geq b_{ij} \geq \underline{b}_{ij}$} is said to be a **interval bi-matrix game**, with $[\underline{A}, \overline{A}]$ denoting the *interval-valued payoff matrix for player 'a' (the row player)* and $[\underline{B}, \overline{B}]$ denoting the *interval-valued payoff matrix for player 'b' (the column player).*

We may denote the (i, j)th entry of $[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$ by the pair $([\underline{a}_{ij}, \overline{a}_{ij}], [\underline{b}_{ij}, \overline{b}_{ij}])$.

Let [α, β] be a closed and bounded interval in the real line. Let $[\alpha, \beta]_c = \frac{\alpha+\beta}{2}$ and $[\alpha, \beta]_w = \frac{\beta-\alpha}{2}$. Thus, $\alpha = [\alpha, \beta]_c - [\alpha, \beta]_w$ and $\beta = [\alpha, \beta]_c + [\alpha, \beta]_w$.

Thus, every closed and bounded interval in the real real line is uniquely defined by a pair of real numbers whose first coordinate is the mid-point (centre) of the interval and the second coordinate is half the length of the interval.

Thus, $([\underline{A}, \overline{A}], [\underline{B}, \overline{B}])$ is uniquely determined by the *pair of pairs* $((\frac{\underline{A} + \overline{A}}{2}, \frac{\overline{A} - \underline{A}}{2}), (\frac{\overline{B} + \underline{B}}{2}, \frac{\overline{B} - \underline{B}}{2})) \epsilon (\mathbb{R} \times \mathbb{R}_+)^{m\times n} \times (\mathbb{R} \times \mathbb{R}_+)^{m\times n}$ where for (i, j)ϵ{1, …, m}×{1, ..., n} the (i, j)th entry of $(\frac{\underline{A} + \overline{A}}{2}, \frac{\overline{A} - \underline{A}}{2})$ is $(\frac{\overline{a}_{ij}+\underline{a}_{ij}}{2}, \frac{\overline{a}_{ij}-\underline{a}_{ij}}{2})$ and the (i, j)th entry of $(\frac{\overline{B} + \underline{B}}{2}, \frac{\overline{B} - \underline{B}}{2})$ is $(\frac{\overline{b}_{ij}+\underline{b}_{ij}}{2}, \frac{\overline{b}_{ij}-\underline{b}_{ij}}{2})$.

For $([\underline{A}, \overline{A}], [\underline{B}, \overline{B}])$ and (i, j)ϵ{1, …, m}×{1, ..., n}, let $c_{ij}(a) = \frac{\overline{a}_{ij}+\underline{a}_{ij}}{2}$, $w_{ij}(a) = \frac{\overline{a}_{ij}-\underline{a}_{ij}}{2}$, $c_{ij}(b) = \frac{\overline{b}_{ij}+\underline{b}_{ij}}{2}$ and $w_{ij}(b) = \frac{\overline{b}_{ij}-\underline{b}_{ij}}{2}$.

Let $C(a)\epsilon\mathbb{R}^{m\times n}$ be such that for (i, j)ϵ{1, …, m}×{1, ..., n}, its (i, j)th entry is $c_{ij}(a)$, $W(a)\epsilon\mathbb{R}^{m\times n}$ be such that for (i, j)ϵ{1, …, m}×{1, ..., n}, its (i, j)th entry is $-w_{ij}(b)$, $C(a)\epsilon\mathbb{R}^{m\times n}$ be such that for (i, j)ϵ{1, …, m}×{1, ..., n}, its (i, j)th entry is $c_{ij}(b)$ and $W(b)\epsilon\mathbb{R}^{m\times n}$ be such that for (i, j)ϵ{1, …, m}×{1, ..., n}, its (i, j)th entry is $-w_{ij}(b)$.

Note 3: Each term in W(a) and W(b) is *non-positive*. This reflects *aversion* towards uncertainty associated with the length of the interval. ▌

For iϵ{1, …, m}, let $C_i(a)$ and $W_i(a)$ denote the ith row of C(a) and W(b) respectively. For jϵ{1, …, n}, let $C^j(b)$ and $W^j(b)$ denote the jth column of C(b) and W(b) respectively. ▌

Let CW(a)$\epsilon(\mathbb{R} \times \mathbb{R}_+)^{m\times n}$ be such that for (i, j)ϵ{1, …, m}×{1, ..., n}, its (i, j)th entry is (c_{ij}(a), -w_{ij}(a)) and let CW(b)$\epsilon(\mathbb{R} \times \mathbb{R}_+)^{m\times n}$ be such that for (i, j)ϵ{1, …, m}×{1, ..., n}, its (i, j)th entry is (c_{ij}(b), -w_{ij}(b)).

Note 4: (i, j)ϵ{1, …, m}×{1, ..., n}, the effect of the outcome $[\underline{a}_{ij}, \overline{a}_{ij}]$ is summarized in a pair whose first component is the expected value of the uniform distribution on $[\underline{a}_{ij}, \overline{a}_{ij}]$ and the second component is $(-\sqrt{3})\times$ the standard deviation. The same interpretation applies to $[\underline{b}_{ij}, \overline{b}_{ij}]$.∎

A lexicographic equilibrium strategy profile for (CW(a), CW(b)) is said to be a **lexicographic mean-standard deviation equilibrium strategy profile** for $([\underline{A}, \overline{A}], [\underline{B}, \overline{B}])$.

A strictly weighted equilibrium strategy profile for (CW(a), CW(b)) is said to be a **strictly weighted mean-standard deviation equilibrium strategy profile** for $([\underline{A}, \overline{A}], [\underline{B}, \overline{B}])$.

Acknowledgment: Thanks to Himansu Barnawal for a correspondence, a response to which led to the fourth and fifth paragraphs in the introductory section of this paper.

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Appendix

A strategy profile (x, y) is said to be a **strong equilibrium strategy profile for $[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$** if for all (A, B)$\epsilon[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$, (x, y) is an equilibrium strategy profile for (A, B).

$[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$ is said to satisfy **support set invariancy** if there exists non-empty subsets S(a) and S(b) of {1, ..., m} and {1, ..., n} respectively such that for all (A, B)$\epsilon[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$, there exists (x, y)ϵ σ(A, B) (i.e., the set of all equilibrium strategy profiles for (A, B)), for which $\{i| x_i > 0\}\times\{j| y_j > 0\}$ = S(a)×S(b), holds.

Clearly the existence of a strong equilibrium strategy profile for $[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$ implies that $[\underline{A}, \overline{A}]\times[\underline{B}, \overline{B}]$ satisfies support set invariancy.

The following example shows that there exists an interval bi-matrix game that does not satisfy support set invariancy.

Example 2: Let $[\underline{A}, \overline{A}] = [\underline{B}, \overline{B}] = \begin{bmatrix} [-1, 1] & 0 \\ 0 & [-1, 0] \end{bmatrix}$.

The bi-matrix game $(\underline{A}, \underline{B}) = \begin{bmatrix} (-1, -1) & (0, 0) \\ (0, 0) & (-1, -1) \end{bmatrix}$ has three equilibrium strategy profiles, ((1, 0), (0, 1)), ((0, 1), (1, 0)) and $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

The bi-matrix game $(\overline{A}, \overline{B}) = \begin{bmatrix} (1, 1) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix}$ has the unique equilibrium strategy profile ((1, 0), (1, 0)) which is a weakly dominant strategy equilibrium.

Since the unique equilibrium strategy profile for $(\overline{A}, \overline{B})$ is ((1, 0), (1, 0)), if $([\underline{A}, \overline{A}] \times [\underline{B}, \overline{B}])$ were to satisfy support set invariancy, then it would be necessary that S(a) = {1} and S(b) = {1}.

However, there is no equilibrium strategy profile (x, y) for $(\underline{A}, \underline{B})$ that satisfies $\{i| x_i > 0\}\times\{j| y_j > 0\} = \{1\}\times\{1\}$.

Thus, the interval bi-matrix game $([\underline{A}, \overline{A}], [\underline{B}, \overline{B}])$ does not satisfy support set invariancy and hence does not have any strong equilibrium strategy profile. $\blacksquare$