Shared equilibrium strategy profile for a finite number of bi-matrix games

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Abstract

We show that a strategy profile is an equilibrium strategy profile for an arbitrary finite set of bi-matrix games if and only if it solves a bi-linear programming problem and the optimal value of the bi-linear programming problem is zero. An immediate consequence of this result, is that if the payoff matrix for the row player is a convex combination of its payoff matrices in the finite collection and the payoff matrix for the column player is a (possibly different) convex combination of its payoff matrices in the finite collection, then a strategy profile is an equilibrium strategy profile for the pair of matrices if and only if it satisfies the same two conditions.

Keywords: equilibrium strategy profile, finite set of bi-matrix games, bi-linear programming problem, optimal value, convex hull

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1. Introduction: In the seminal work of Mangasarian and Stone (1964), the main result states that a strategy profile of a bi-matrix game (i.e., a randomization for each of the two players- a row player and a column player- over the non-empty finite set of actions available to each) is an equilibrium strategy profile (i.e., no one benefits from unilateral deviations) if and only if it solves a certain bi-linear programming problem. In this note we show that a strategy profile is an equilibrium strategy profile for an arbitrary finite set of bi-matrix games if and only if it solves a generalized form of the bi-linear programming problem in Mangasarian and Stone (1964) and the optimal value of the bi-linear programming problem is zero. We need, this additional condition about the optimal value being zero, since unlike in the case of a single bi-matrix game, there is no guarantee that a non-empty finite set of bi-matrix games with cardinality greater than one, has an equilibrium strategy profile that is shared by all games in the set. The existence of an equilibrium for a bi-matrix game implies that the optimal value of the objective function in the bi-linear programming problem in

Mangasarian and Stone (1964) is zero. An immediate consequence of this result, is that if the payoff matrix for the row player is a convex combination of its payoff matrices in the finite collection and the payoff matrix for the column player is a (possibly different) convex combination of its payoff matrices in the finite collection, then a strategy profile is an equilibrium strategy profile for the pair of matrices if and only if it satisfies the same two conditions.

Bhurjee (2016), attempts to extend the scope of the main result in Mangasarian and Stone (1964) to the case of two bi-matrix games, where each entry in the payoff matrix of each player in the second bi-matrix game is "greater than or equal to" the corresponding entry in the payoff matrix the same player in the first bi-matrix game. In our framework of analysis, there is no such restriction on the bi-matrix games. The possibility of non-existence of a shared equilibrium profile for the kind of bi-matrix games discussed in Bhurjee (2016) is shown in example 2 in (the appendix of) Lahiri (2026).

2. The Model: For positive integers m, n, let $\mathbb{R}^{m\times n}$ denote the set of all real-valued m×n matrices.

For $C\in\mathbb{R}^{m\times n}$ and (i, j)∈{1, …, m}×{1, ..., n}, let C_i denote the i^{th} row (vector) of C and C^j denote the j^{th} column (vector) of C.

Unless otherwise mentioned, all vectors will be assumed to be column vectors and the transpose of a vector z, denoted by z^T is a row vector.

For $C\in\mathbb{R}^{m\times n}$, the transpose of C is denoted by C^T. Clearly, $C^T\in\mathbb{R}^{n\times m}$.

Let $\Delta^{m-1} = \{x\in\mathbb{R}^m_+ | \sum_{i=1}^m x_i = 1\}$ and $\Delta^{n-1} = \{y\in\mathbb{R}^n_+ | \sum_{i=1}^n y_j = 1\}$.

A point in $\Delta^{m-1} \times \Delta^{n-1}$ is said to be a **strategy profile**.

Given $(A, B)\in\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ a strategy profile (x, y) is said to be an **equilibrium strategy profile** for (A, B) if for all (i, j)∈{1, …, m}×{1, ..., n}: $x^TAy \geq A_iy$ and $x^TBy \geq x^TB^j$.

We will for the purpose of this note, refer to each $(A, B)\in\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ as a **bi-matrix game**.

3. Shared Equilibrium Strategy Profile For a Finite Number of Bi-matrix Games:

For a positive integer K ≥ 2, let $\{(A^{(k)}, B^{(k)}) | k = 1, …, K\}$ be a non-empty finite set of bi-matrix games.

Proposition 1: A strategy profile (x, y) is an equilibrium strategy profile for every bi-matrix game in the set $\{(A^{(k)}, B^{(k)}) | k = 1, …, K\}$ <u>if and only if</u> there exists arrays of real numbers $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ that along with x, y satisfies the following "*two*" conditions:

(1) x, y, $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ solve the bi-linear programming problem BLP defined thus: Maximize $\sum_{k=1}^{K} [w^T(A^{(k)} + B^{(k)})z - \alpha^{(k)} - \beta^{(k)}]$, subject to $A_i^{(k)}z - \alpha^{(k)} \leq 0$, i = 1, …, m, $k \epsilon \{1, \ldots, K\}$, $(B^{(k)j})^T w - \beta^{(k)} \leq 0$, j = 1, …, n, $k \epsilon \{1, \ldots, K\}$, $w \epsilon \Delta^{m-1}$, $z \epsilon \Delta^{n-1}$.

(2) $\sum_{k=1}^{K} [x^T(A^{(k)} + B^{(k)})y - \alpha^{(k)*} - \beta^{(k)*}] = 0$.

Proof: First note that if z, w, $<\alpha^{(k)}| k = 1, \ldots, K>$, $<\beta^{(k)}| k = 1, \ldots, K>$ satisfies the constraints of BLP, then it must be the case that $\sum_{k=1}^{K} [w^T(A^{(k)} + B^{(k)})z - \alpha^{(k)} - \beta^{(k)}] \leq 0$.

Suppose (x, y) is an equilibrium strategy profile for every bi-matrix game in the set $\{(A^{(k)}, B^{(k)})| k = 1, \ldots, K\}$.

For $k \epsilon \{1, \ldots, K\}$, let $\alpha^{(k)*} = x^T A^{(k)} y$ and $\beta^{(k)*} = x^T B^{(k)} y$.

Then, x, y, $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ satisfy all the constraints of BLP and $\sum_{k=1}^{K} [x^T(A^{(k)} + B^{(k)})y - \alpha^{(k)*} - \beta^{(k)*}] = 0$.

Thus, x, y, $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ solve BLP and satisfy (2).

Now suppose x, y, $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ solve BLP and satisfy (2).

Since, x, y, $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ satisfy the constraints of BLP, it must be the case that for all $k \epsilon \{1, \ldots, K\}$, $A_i^{(k)}y - \alpha^{(k)*} \leq 0$, i = 1, …, m and $(B^{(k)j})^T x - \beta^{(k)*} \leq 0$, j = 1, …, n.

Since $(x, y) \epsilon \Delta^{m-1} \times \Delta^{n-1}$, it must be the case that for all $k \epsilon \{1, \ldots, K\}$, $x^T A^{(k)} y \leq \alpha^{(k)*}$ and $x^T B^{(k)} y \leq \beta^{(k)*}$.

From (2) we know that, $\sum_{k=1}^{K} [x^T(A^{(k)} + B^{(k)})y - \alpha^{(k)*} - \beta^{(k)*}] = 0$.

Thus, it must be the case that $k \epsilon \{1, \ldots, K\}$, $x^T A^{(k)} y = \alpha^{(k)*}$ and $x^T B^{(k)} y = \beta^{(k)*}$.

Since, for all $k \epsilon \{1, \ldots, K\}$, $A_i^{(k)}y - \alpha^{(k)*} \leq 0$, i = 1, …, m and $(B^{(k)j})^T x - \beta^{(k)*} \leq 0$, j = 1, …, n, it follows that for all $k \epsilon \{1, \ldots, K\}$, $A_i^{(k)}y \leq x^T A^{(k)} y$, i = 1, …, m and $x^T B^{(k)j} \leq x^T B^{(k)} y$, j = 1, …, n.

Thus, (x, y) must be an equilibrium strategy profile for every bi-matrix game in the set $\{(A^{(k)}, B^{(k)})| k = 1, \ldots, K\}$. Q.E.D.

4. Shared Equilibrium For Convex Hull of Payoff Matrices: Given $\{(A^{(k)}, B^{(k)})| k = 1, \ldots, K\}$ as above, let $conv[\{A^{(k)}| k = 1, \ldots, K\}]$ denote the convex hull of the m×n matrices in $\{A^{(k)}| k = 1, \ldots, K\}$ and let conv[$\{B^{(k)}| k = 1, \ldots, K\}$] denote the convex hull of the m×n matrices in $\{B^{(k)}| k = 1, \ldots, K\}$.

Proposition 2: (x, y) is an equilibrium strategy for all (A, B)ϵconv[$\{A^{(k)}| k = 1, \ldots, K\}$]×conv[$\{B^{(k)}| k = 1, \ldots, K\}$] <u>if and only if</u> (x, y) is an equilibrium strategy profile for all bimatrix games in $\{(A^{(k)}, B^{(k)})| k = 1, \ldots, K\}$.

Proof: If (x, y) is an equilibrium strategy profile for all (A, B)ϵconv[$\{A^{(k)}| k = 1, \ldots, K\}$]×conv[$\{B^{(k)}| k = 1, \ldots, K\}$], then (x, y) must be an equilibrium strategy profile for all (A, B)ϵ$\{(A^{(k)}, B^{(k)})| k = 1, \ldots, K\}$.

Hence suppose (x, y) is an equilibrium strategy profile for all (A, B)ϵ$\{(A^{(k)}, B^{(k)})| k = 1, \ldots, K\}$.

Thus, for all kϵ{1, …, K}, $x^TA^{(k)}y \geq A_i^{(k)}y$ for all iϵ{1, …, m} and $x^TB^{(k)}y \geq x^TB^{(k)j}$ for all jϵ{1, …, n}.

For θ, ηϵ$\Delta^{K-1} = \{\xi\epsilon\mathbb{R}_+^K| \sum_{k=1}^{K} \xi_k = 1\}$, consider the bi-matrix game $(\sum_{k=1}^{K} \theta_k A^{(k)}, \sum_{k=1}^{K} \eta_k B^{(k)})$.

Thus, $x^T(\sum_{k=1}^{K} \theta_k A^{(k)})y = \sum_{k=1}^{K} \theta_k x^T A^{(k)} y \geq \sum_{k=1}^{K} \theta_k A_i^{(k)} y = (\sum_{k=1}^{K} \theta_k A_i^{(k)})y$ for all for all iϵ{1, …, m} and $x^T(\sum_{k=1}^{K} \eta_k B^{(k)})y = \sum_{k=1}^{K} \eta_k x^T B^{(k)} y \geq \sum_{k=1}^{K} \eta_k x^T B^{(k)j} = x^T(\sum_{k=1}^{K} \eta_k B^{(k)j})$ for all jϵ{1, …, n}.

Thus, (x, y) is an equilibrium strategy profile for $(\sum_{k=1}^{K} \theta_k A^{(k)}, \sum_{k=1}^{K} \eta_k B^{(k)})$.

An immediate consequence of propositions 1 and 2 is the following result.

Theorem 1: (x, y) is an equilibrium strategy for all (A, B)ϵconv[$\{A^{(k)}| k = 1, \ldots, K\}$]×conv[$\{B^{(k)}| k = 1, \ldots, K\}$] <u>if and only if</u> there exists arrays of real numbers $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ that along with x, y satisfies the following "*two*" conditions:

(1) x, y, $<\alpha^{(k)*}| k = 1, \ldots, K>$, $<\beta^{(k)*}| k = 1, \ldots, K>$ solve the bi-linear programming problem BLP defined thus: Maximize $\sum_{k=1}^{K} [w^T(A^{(k)} + B^{(k)})z - \alpha^{(k)} - \beta^{(k)}]$, subject to $A_i^{(k)}z - \alpha^{(k)} \leq 0$, i = 1, …, m, kϵ{1, …, K}, $(B^{(k)j})^Tw - \beta^{(k)} \leq 0$, j = 1, …, n, k ϵ{1, …, K}, wϵΔ^{m-1}, zϵΔ^{n-1}.

(2) $\sum_{k=1}^{K} [x^T(A^{(k)} + B^{(k)})y - \alpha^{(k)*} - \beta^{(k)*}] = 0$.

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