Multi-attribute bi-matrix games

By

Somdeb Lahiri

Email: somdeb.lahiri@gmail.com

ORCID: https://orcid.org/0000-0002-5247-3497

(Formerly with) PD Energy University, Gandhinagar (EU-G), India.

June 22, 2026.

This version: July 8, 2026.

Abstract

We introduce multi-attribute bi-matrix games and show that a strategy profile is an equilibrium strategy profile with respect to a preference relation satisfying additivity with respect to the zero vector for an arbitrary finite set of such games having the same number of attributes if and only if it solves a bi-linear programming problem and the value of the objective function at this solution is zero. An immediate consequence of this result, is that if the outcome matrix for the row player is a convex combination of its outcome matrices in the finite collection and the outcome matrix for the column player is a (possibly different) convex combination of its outcome matrices in the finite collection, then a strategy profile is an equilibrium strategy profile for the game associated with pair of matrices if and only if it satisfies the same two conditions.

Keywords: multi-attribute outcomes, preference relation, additivity with respect to the zero vector, equilibrium strategy profile, finite set of bi-matrix games, bi-linear programming problem, value of objective function, convex hull

AMS Subject Classifications: 90B50, 90C20, 90C29, 91A05, 91A10, 91B06

JEL Subject Classifications: C61, C72

1. Introduction: In the seminal work of Mangasarian and Stone (1964), the main result states that a strategy profile of a bi-matrix game (i.e., a randomization for each of the two players- a row player and a column player- over the non-empty finite set of actions available to each) is an equilibrium strategy profile (i.e., no one benefits from unilateral deviations) if and only if it solves a certain bi-linear programming problem. In Lahiri (2026), we introduce the concept of a pair-valued bi-matrix game, which extends the two-person zero sum version of the same that is discussed in Radhakrishnan and Saikeerthana (2020). As in Radhakrishnan and Saikeerthana (2020), we assume in Lahiri (2026) that the outcomes of the game are lexicographically ordered and introduce the concept of a lexicographic equilibrium strategy profile for pair-valued bi-matrix game. This solution concept, generalizes the related solution concept in Radhakrishnan and Saikeerthana (2020) for zero-sum pair-valued bi-matrix games. Interesting examples of lexicographic preferences (see Varian (1978)) are available in Goswami, Mitra and Sen (2018).

In this note we extend our scope of analysis to multi-attribute bi-matrix games. The outcomes corresponding to strategy profiles are real-valued vectors of the same dimension for both players and all outcomes. We assume the there is a weak order (preference relation) on the finite dimensional Euclidean space of potential vector of attributes (attribute-vectors) satisfying a mild property called "additivity with respect to the zero vector". This weak order is meant to be a generalization of the "greater than or equal to" binary relation for real numbers that is used for bi-matrix games. The use of the word preferences, is in keeping with the interpretation of weak orders in decision theory. In particular, lexicographic preferences satisfy additivity with respect to the zero vector. We discuss in a note (note 2 to be precise) that the entire analysis that follows in this paper, would remain intact if the two players had different "preferences" over their own attribute vectors so long as the preferences satisfied additivity with respect to the zero vector and the weak order for the sum of two attribute-vectors- one for each player- satisfied a similar additivity with respect to the zero vector property. The purpose of using (notationally) identical weak orders for individual players as well as for weak order over vectors obtained by interpersonal aggregation of attribute-vectors, is to make the exposition notationally simpler, and this is achieved without any loss of generality.

We show that a strategy profile is an "equilibrium strategy profile with respect to the preference relation" for an arbitrary non-empty finite set of multi-attribute bi-matrix games if and only if the strategy along with certain other choice variables solve a generalized form of the bi-linear programming problem in Mangasarian and Stone (1964) and the *value of the objective function* at this solution is the zero vector. This result generalizes proposition 3 in Lahiri (2026) which is concerned with lexicographic equilibrium strategy profile for a single pair-valued bi-matrix game. As in the case of proposition 3 in Lahiri (2026), we need, this additional condition about the value of the objective function at the solution being the zero vector, since unlike in the case of a single bi-matrix game with real-valued payoffs, there is no guarantee that a multi-attribute bi-matrix game has a equilibrium strategy profile. The possibility of such non-existence is shown in section 3 of this paper. The existence of an equilibrium for a bi-matrix game implies that the optimal value of the objective function in the bi-linear programming problem in Mangasarian and Stone (1964) is zero. An immediate consequence of our first major result is that if the outcome matrix for the row player is a convex combination of its outcome matrices in the finite collection and the outcome matrix for the column player is a (possibly different) convex combination of its payoff matrices in the finite collection, then a strategy profile is an equilibrium strategy profile with respect to the assumed preferences for the game associated with the pair of multi-attribute matrices if and only if it satisfies the same two conditions.

Bhurjee (2016), attempts to extend the scope of the main result in Mangasarian and Stone (1964) to the case of two bi-matrix games with real-valued payoffs, where each entry in the payoff matrix of each player in the second bi-matrix game is "greater than or equal to" the corresponding entry in the payoff matrix of the same player in the first bi-matrix game. The possibility of non-existence of a shared equilibrium profile for the kind of bi-matrix games discussed in Bhurjee (2016) is shown in (an appendix of) Lahiri (2026).

Since, a bi-matrix game is a special case of a pair-valued bi-matrix game in which all entries in the outcome matrix of a player have the same value of the second coordinate, the relevant results in Bhurjee (2016) are implied by corresponding results in this paper.

This paper is a considerably revised generalization of an earlier paper entitled "Shared lexicographic equilibrium strategy profile for a finite number of pair-valued bi-matrix games".

2. The Model: For positive integers m, n consider an interactive decision making context (game) with two players/participants/decision-makers who we denote by 'a' and 'b' respectively, and where player 'a' has m actions to choose from and 'b' has 'n' actions to choose from.

A **strategy for player 'a'** is a point in $\Delta^{m-1} = \{x \in \mathbb{R}^m_+ | \sum_{i=1}^m x_i = 1\}$ and a **strategy for player 'b'** is a point in $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ | \sum_{j=1}^n y_j = 1\}$.

A strategy for the player 'a' is a probability distribution on {1, ..., m}, with each coordinate of the strategy denoting the probability with which the player 'a' chooses the corresponding action. A strategy for player 'b' is a probability distribution on

{1, ..., n}, with each coordinate of the strategy denoting the probability with which player 'b' chooses the corresponding action.

A pair (x, y) where x is a strategy for player 'a' and y is a strategy for player 'b' is said to be a **strategy profile**.

Let $\mathbb{R}^{m\times n}$ be the set of all m×n matrices.

Given C$\epsilon\mathbb{R}^{m\times n}$ and (i, j)ϵ{1, …, m}×{1, ..., n}, let c_{ij} denote the $(i, j)^{th}$ entry of C, C_i denote the i^{th} row of C and C^j denote the j^{th} column of C.

For a positive integer p, let $\mathbb{R}^{(p,\, m\times n)}$ denote the set of all m×n matrices whose entries are points in $\mathbb{R}^p$.

If V$\epsilon\mathbb{R}^{(p,\, m\times n)}$, then for all (i, j)ϵ{1, …, m}×{1, ..., n}, the typical (though not invariable) notations are as follows: $(i, j)^{th}$ entry of V will be denoted by $v_{ij} = (v_{ij}^{(1)}, ..., v_{ij}^{(p)})$, the i^{th} row of V will be denoted by V_i and the j^{th} column of V will be denoted by V^j.

For our purposes here, a member of $\mathbb{R}^{(p,\, m\times n)} \times \mathbb{R}^{(p\, ,\, m\times n)}$ will be referred to as a **p-valued bi-matrix game** and $\mathbb{R}^{(p,\, m\times n)} \times \mathbb{R}^{(p\; m\times n)}$ as the **set of all p-valued bi-matrix games**.

The union of the set all p-valued bi-matrix games, allowing p to vary over the set of all natural numbers is what we refer to as the **set of all multi-attribute bi-matrix game** and we may refer to any member of this set as a **multi-attribute bi-matrix game**.

Notation 1: Given a p-valued bi-matrix game (V(a), V(b)), for rϵ{1, …, p} and αϵ{a, b}, let $V^{(r)}(\alpha)$ be the m×n real-valued matrix whose $(i, j)^{th}$ term for (i, j)ϵ{1, …, m}×{1, ..., n} is $v_{ij}^{(r)}(\alpha)$. The i^{th} row of $V^{(r)}(\alpha)$ for iϵ{1, …, m} is denoted by $V_i^{(r)}(\alpha)$ and the j^{th} column of $V^{(r)}(\alpha)$ for jϵ{1, …, n} is denoted by$V^{(r)j}(\alpha)$. For (x, y)$\epsilon\Delta^{m-1}\times\Delta^{n-1}$ and αϵ{a, b}, $x^TV(\alpha)y = \sum_{i=1}^{m}\sum_{j=1}^{n} x_iy_j(v_{ij}^{(1)}(\alpha), \ldots, v_{ij}^{(p)}(\alpha)) = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_iy_jv_{ij}^{(1)}(\alpha), \ldots, x_iy_jv_{ij}^{(p)}(\alpha))$.▌

Let ≽ be a **preference relation** (i.e., complete (or connected), reflexive and transitive binary relation) on $\mathbb{R}^p$ and let ≻ denote the asymmetric part of ≽.

Given a p-valued bi-matrix game (V(a), V(b)), a strategy profile (x, y) is said to be a **equilibrium strategy profile (with respect to** ≽**)** for (V(a), V(b)) if $x^TV(a)y - V_i(a)y \succcurlyeq 0$ for all iϵ{1, …, m}and $x^TV(b)y - x^TV^j(b) \succcurlyeq 0$ for all jϵ{1, …, n}, where 0 is the zero-vector in $\mathbb{R}^p$.

Note 1: It ought to be emphasized that the entries of V(a) and V(b) are "*outcomes*" that can be weakly ordered using the preference relation $\succcurlyeq$ on $\mathbb{R}^p$. ▌

A preference relation $\succcurlyeq$ is said to satisfy ***additivity with respect to the zero vector*** if for all u, v$\in\mathbb{R}^p$ and all θ belonging to the open interval (0, 1): (i) u $\succcurlyeq$ 0, v $\succcurlyeq$ 0 implies u + v $\succcurlyeq$ 0 and θv $\succcurlyeq$ 0; (ii) u $\succcurlyeq$ 0, v $\succ$ 0 implies u + v $\succ$ 0.

Note 2: Nothing in the analysis that follows would be affected if we used a preference structure in the form of a triplet ($\succcurlyeq_a$, $\succcurlyeq_b$, $\succcurlyeq$) of weak orders on $\mathbb{R}^p$ with the corresponding triplet of their asymmetric parts being ($\succ_a$, $\succ_b$, $\succ$) satisfying the following "*additivity with respect to zero vectors conditions*": For all u, v$\in\mathbb{R}^p$, α, β$\in$\{a, b\} with α ≠ β and all θ belonging to the open interval (0, 1): (i) u $\succcurlyeq_\alpha$ 0, v $\succcurlyeq_\alpha$ 0 implies u + v $\succcurlyeq_\alpha$ 0 and θv $\succcurlyeq_\alpha$ 0; (ii) u $\succcurlyeq_\alpha$ 0, v $\succ_\alpha$ 0 implies u + v $\succ_\alpha$0; (iii) u $\succcurlyeq_\alpha$ 0, v $\succcurlyeq_\beta$ 0 implies u + v $\succcurlyeq$ 0; and (iv) u $\succcurlyeq_\alpha$ 0, v $\succ_\beta$ 0 implies u + v $\succ$ 0.

For α $\in$\{a, b\}, $\succcurlyeq_\alpha$ is the weak order that is used by player α to compare its own attribute-vectors and $\succcurlyeq$ is the weak order used to compare any sum of attribute vectors of players a and b. ▌

Example 1: Let $\succcurlyeq$ be the binary relation on $\mathbb{R}^2$ such that for all (γ_1, η_1), (γ_2, η_2)$\in\mathbb{R}^2$: (γ_1, η_1) $\succcurlyeq$ (γ_2, η_2) <u>if and only if</u> either $\gamma_1 > \gamma_2$ or $\gamma_1 = \gamma_2$ and $\eta_1 \geq \eta_2$.

Clearly $\succcurlyeq$ is a linear order on $\mathbb{R}^2$. It is generally known as the **(2- dimensional) lexicographic preference ordering on** $\mathbb{R}^2$. Further, $\succcurlyeq$ satisfies additivity with respect to the zero vector. ▌

If for a 2-valued (pair-valued) bi-matrix game (V(a), V(b)) $\succcurlyeq$ denotes lexicographic preferences, then we refer to an equilibrium strategy profile with respect to $\succcurlyeq$ as a **lexicographic equilibrium strategy profile**.

3. Possibility of Non-existence of Lexicographic Equilibrium Strategy Profile: In this section we will provide an example to show that unconditional existence of lexicographic equilibrium profile may not be possible.

Example 1: Let V(a) = $\begin{bmatrix} (1, -2) & (-1, -2) \\ (-2, -1) & (2, -1) \end{bmatrix}$ and V(b) = $\begin{bmatrix} (-1, -1) & (1, -1) \\ (2, -1) & (-2, -1) \end{bmatrix}$.

Let (x, y) be a strategy profile. For $y_1 > \frac{1}{2}$, y_1 - (1- y_1) = 2y_1 - 1 > 0 and -2y_1 + 2(1-y_1) = 2 - 4y_1 < 0.

Thus, the best reply of 'a' to y with $y_1 > \frac{1}{2}$ is (1, 0).

For $y_1 < \frac{1}{2}$, $y_1 - (1- y_1) = 2y_1 - 1 < 0$ and $-2y_1 + 2(1-y_1) = 2 - 4y_1 > 0$.

Thus, the best reply of 'a' to y with $y_1 < \frac{1}{2}$ is (0, 1).

For $y_1 = \frac{1}{2}$, $y_1 - (1- y_1) = 2y_1 - 1 = 0$ and $-2y_1 + 2(1-y_1) = 2 - 4y_1 = 0$.

Since, the second coordinates in the second row of V(a) are equal to -1 which is greater than the second coordinate in the first row of V(a), the latter being equal to -2, the best reply of 'a' to y with $y_1 = \frac{1}{2}$ is (0, 1).

However, if 'a' chooses (1, 0), the best reply of 'b' is z = (0, 1), i.e., $z_1 = 0 < \frac{1}{2} < y_1$ and if 'a' chooses (0, 1) the best reply of 'b' is is z = (1, 0), i.e., $z_1 = 1 > \frac{1}{2} \geq y_1$.

Thus, (V(a), V(b)) does not have any lexicographic equilibrium strategy profile.

4. Shared Equilibrium Strategy Profile For a Finite Number of Multi-attribute Bi-matrix Games: For positive integers p and K, let {(V(a, k), V(b, k))| k = 1, …, K} be a non-empty finite set of p-valued bi-matrix games.

Proposition 1: For a preference relation $\succcurlyeq$ on $\mathbb{R}^p$ satisfying additivity with respect to the zero vector, a strategy profile (x, y) is a equilibrium strategy profile with respect to $\succcurlyeq$ for every p-valued bi-matrix game in the set {(V(a, k), V(b, k))| k = 1, …, K} <u>if and only if</u> there exists arrays <$\alpha^*(k)$| k = 1, …, K>, <$\beta^*(k)$| k = 1, …, K> in $\mathbb{R}^p$ that along with x, y satisfies the following "*two*" conditions:

(1) x, y, <$\alpha^*(k)$| k = 1, …, K>, <$\beta^*(k)$| k = 1, …, K> solve the bi-linear programming problem BLP defined thus: Minimize (with respect to $\succcurlyeq$) $\sum_{k=1}^{K} [\alpha(k) + \beta(k) - w^T(V(a,k) + V(b,k))z]$, subject to $\alpha(k) - V_i(a,k)z \succcurlyeq 0$, i = 1, …, m, $k \in \{1, …, K\}$, $\beta(k) - w^T V(b, k)^j \succcurlyeq 0$, j = 1, …, n, $k \in \{1, …, K\}$, $w \in \Delta^{m-1}$, $z \in \Delta^{n-1}$, $\alpha(k), \beta(k) \in \mathbb{R}^p$, $k \in \{1, …, K\}$.

(2) $\sum_{k=1}^{K} [x^T(V(a,k) + V(B,k))y - \alpha^*(k) - \beta^*(k)] = 0$.

Proof: Observation: Note that if z, w, <$\alpha(k)$| k = 1, …, K>, <$\beta(k)$| k = 1, …, K> satisfies the constraints of BLP, then then by lemma 2, [$\alpha(k) - V_i(a,k)z \succcurlyeq 0$, i = 1, …, m, $k \in \{1, …, K\}$, $\beta(k) - w^T V(b, k)^j \succcurlyeq 0$, j = 1, …, n, $k \in \{1, …, K\}$, $w \in \Delta^{m-1}$, $z \in \Delta^{n-1}$] implies by repeated application of (i) in the definition of additivity with respect to the zero vector $\sum_{k=1}^{K} (\alpha(k) + \beta(k)) - \sum_{k=1}^{K} w^T(V(a,k) + V(b,k))z =$
$\sum_{k=1}^{K} (\sum_{i=1}^{m} w_i(\alpha(k) - V_i(a,k)z) + \sum_{k=1}^{K} (\sum_{j=1}^{n} (\beta(k) - w^T V^j(b,k))z_j \succcurlyeq 0$.∎

Suppose (x, y) is an equilibrium strategy profile with respect to $\succcurlyeq$ for every p-valued bi-matrix game in the set {(V(a, k), V(b, k))| k = 1, …, K}.

For k∈{1, …, K}, let $\alpha^*(k)$ = $x^TV(a, k)y$ and $\beta^*(k)$ = $x^TV(b, k)y$.

Then, x, y, <$\alpha^*(k)$| k = 1, …, K>, <$\beta^*(k)$| k = 1, …, K> satisfy all the constraints of BLP and $\sum_{k=1}^{K}[x^T(V(a,k)+V(B,k))y-\alpha^*(k)-\beta^*(k)]$ = 0.

Thus, from the observation it follows that x, y, <$\alpha^*(k)$| k = 1, …, K>, <$\beta^*(k)$| k = 1, …, K> solve BLP and satisfy (2).

Now suppose x, y, <$\alpha^*(k)$| k = 1, …, K>, <$\beta^*(k)$| k = 1, …, K> solve BLP and satisfy (2).

Since, x, y, <$\alpha^*(k)$| k = 1, …, K>, <$\beta^*(k)$| k = 1, …, K> satisfy the constraints of BLP, it must be the case that for all k∈{1, …, K}, $\alpha^*(k)$ - $V_i(a, k)y \succcurlyeq 0$, i = 1, …, m and $\beta^*(k)$ - $x^TV^j(b, k) \succcurlyeq 0$, j = 1, …, n.

Thus, by repeated application of (i) in the definition of additivity with respect to the zero vector it follows that for all k∈{1, …, K}, $\alpha^*(k)$ - $x^TV(a, k)y$ = $\sum_{x_i>0} x_i(\alpha^*(k) - V_i(a,k)y) \succcurlyeq 0$ and $\beta^*(k)$ - $x^TV(b, k)y$ = $\sum_{y_j}(\beta^*(k)-x^TV^j(b,k))y_j \succcurlyeq 0$.

Towards a contradiction suppose that for some $k^\#$∈{1, …, K} and i∈{1, …, m} it is the case that $V_i(a, k^\#)y$ - $x^TV(a, k^\#)y \succ 0$. Since $\alpha^*(k^\#)$ - $V(a, k^\#)_iy \succcurlyeq 0$, by (ii) in the definition of additivity with respect to the zero vector it follows that $\alpha^*(k^\#)$ - $x^TV(a, k^\#)y \succ 0$.

This, in conjunction with repeated applications of additivity with respect to the zero vector and [for all k∈{1, …, K}, $\alpha^*(k)$ - $x^TV(a, k)y \succcurlyeq 0$ and $\beta^*(k)$ - $x^TV(b, k)y \succcurlyeq 0$] implies $\sum_{k=1}^{K}(\alpha(k)+\beta(k))$ - $\sum_{k=1}^{K} w^T(V(a,k)+V(b,k))z \succ 0$.

This strict preference, and the assumption that $\sum_{k=1}^{K}(\alpha(k)+\beta(k))$ - $\sum_{k=1}^{K} w^T(V(a,k)+V(b,k))z$ = 0, contradicts the reflexivity of $\succcurlyeq$.

Thus, it must be the case that for all k∈{1, …, K}, $x^TV(a, k)y$ - $V_i(a, k)y \succcurlyeq 0$, i = 1, …, m.

A similar argument implies that for all k∈{1, …, K}, $x^TV(b, k)y$ - $x^TV^j(b, k) \succcurlyeq 0$, j = 1, …, n.

Thus, (x, y) is an equilibrium strategy profile with respect to $\succcurlyeq$. Q.E.D.

An immediate consequence of proposition 1 is the following result, which is a possibly theoretically useful generalization of the main result in Mangasarian and Stone (1964).

Corollary of Proposition 1: For a preference relation $\succcurlyeq$ on $\mathbb{R}^p$ satisfying additivity with respect to the zero vector, a strategy profile (x, y) is a equilibrium strategy profile with respect to $\succcurlyeq$ for the p-valued bi-matrix game (V(a), V(b)) if and only if there exists α^*, $\beta^* \in \mathbb{R}^p$ that along with x, y satisfies the following "*two*" conditions:

(1) x, y, α^*, β^* solve the bi-linear programming problem BLP defined thus: Minimize (with respect to $\succcurlyeq$) $\alpha + \beta - w^T(V(a) + V(b))z$, subject to $\alpha - V_i(a)z \succcurlyeq 0$, i = 1, …, m, $\beta - w^TV(b)^j \succcurlyeq 0$, j = 1, …, n, $w\in\Delta^{m-1}$, $z\in\Delta^{n-1}$, $\alpha, \beta\in\mathbb{R}^p$.

(2) $x^T(V(a) + V(b))y - \alpha^* - \beta^* = (0, 0)$.

5. Shared Equilibrium For Convex Hull of Outcome Matrices: Given {(V(a, k), V(b, k))| k = 1, …, K} as in section 4, let conv[{V(a,k)| k = 1, …, K}] denote the convex hull of the m×n pair-valued matrices in {V(a, k)| k = 1, …, K} and let conv[{V(b, k)| k = 1, …, K}] denote the convex hull of the m×n pair-valued matrices in {V(b, k)| k = 1, …, K}.

Proposition 2: For a preference relation $\succcurlyeq$ on $\mathbb{R}^p$ satifying additivity with respect to the zero vector, (x, y) is a equilibrium strategy profile with respect to $\succcurlyeq$ for all (V(a), V(b)) $\in$ conv[{V(a, k)| k = 1, …, K}]×conv[{V(b, k)| k = 1, …, K}] if and only if (x, y) is a equilibrium strategy profile with respect to $\succcurlyeq$ for all bimatrix games in {(V(a, k), V(b, k))| k = 1, …, K}.

Proof: If (x, y) is an equilibrium strategy profile with respect to $\succcurlyeq$ for all (V(a), V(b))$\in$conv[{V(a, k)| k = 1, …, K}]×conv[{V(b, k)| k = 1, …, K}], then (x, y) must be a equilibrium strategy profile with respect to $\succcurlyeq$ for all (V(a), V(b))$\in${(V(a, k), V(b, k))| k = 1, …, K}.

Hence suppose (x, y) is an equilibrium strategy profile with respect to $\succcurlyeq$ for all (V(a), V(b) B)$\in${(V(a, k), V(b, k))| k = 1, …, K}.

Thus, for all k$\in${1, …, K}, $x^TV(a, k)y - V_i(a, k)y \succcurlyeq 0$ for all i$\in${1, …, m} and $x^TV(b, k)y - x^TV^j(b, k) \succcurlyeq 0$ for all j$\in${1, …, n}.

For $\theta, \eta\in\Delta^{K-1} = \{\xi\in\mathbb{R}^K_+ | \sum_{k=1}^K \xi_k = 1\}$, consider the p-valued bi-matrix game $(\sum_{k=1}^K \theta_k V(a,k), \sum_{k=1}^K \eta_k V(b,k))$.

For i$\in${1, …, m}, $x^T(\sum_{k=1}^K \theta_k V(a,k))y - (\sum_{k=1}^K \theta_k V_i(a,k))y = \sum_{k=1}^K \theta_k (x^T V(a,k) y - V_i(a,k)y)$.

By repeated application of (i) in the definition of additivity with respect to the zero vector, we get $\sum_{k=1}^K \theta_k (x^T V(a,k) y - V_i(a,k)y) \succcurlyeq 0$.

Thus, $x^T(\sum_{k=1}^K \theta_k V(a,k))y - (\sum_{k=1}^K \theta_k V_i(a,k))y \succcurlyeq 0$ for all i$\in${1, …, m}.

Similarly, for jϵ\{1, …, n\}, $x^T(\sum_{k=1}^{K} \eta_k V(b,k))y$ - $x^T(\sum_{k=1}^{K} \eta_k V^j(b,k)$ =

$\sum_{k=1}^{K} \eta_k (x^T V(b,k)y - x^T V^j(b,k))$.

By repeated application of (i) in the definition of additivity with respect to the zero vector, we get

$\sum_{k=1}^{K} \eta_k (x^T V(b,k)y - x^T V^j(b,k)) \succcurlyeq 0$.

Thus, $x^T(\sum_{k=1}^{K} \eta_k V(b,k))y$ - $x^T(\sum_{k=1}^{K} \eta_k V^j(b,k) \succcurlyeq 0$ for all jϵ\{1, …, n\}.

Thus, (x, y) is an equilibrium strategy profile with respect to ≽ for

$(\sum_{k=1}^{K} \theta_k V(a,k), \sum_{k=1}^{K} \eta_k V(b,k))$. Q.E.D.

An immediate consequence of propositions 1 and 2 is the following result.

Theorem 1: For a preference relation ≽ on $\mathbb{R}^p$ satisfying additivity with respect to the zero vector, (x, y) is an equilibrium strategy profile with respect to ≽ for all (V(a), V(b))ϵconv[\{V(a, k)| k = 1, …, K\}]×conv[\{V(b, k)| k = 1, …, K\}] <u>if and only if</u> there exists arrays <α^*(k)| k = 1, …, K>, <β^*(k)| k = 1, …, K> in $\mathbb{R}^p$ that along with x, y satisfies the following "*two*" conditions:

(1) x, y, <$\alpha(k)^*$| k = 1, …, K>, <$\beta(k)^*$| k = 1, …, K> solve the bi-linear programming problem BLP defined thus: Minimize (with respect to ≽) $\sum_{k=1}^{K} [\alpha(k) + \beta(k) - w^T(V(a,k) + V(b,k))z]$, subject to $\alpha(k) - V_i(a,k)z \succcurlyeq 0$, i = 1, …, m, k$\epsilon$\{1, …, K\}, $\beta(k) - w^T V(b,k)^j \succcurlyeq 0$, j = 1, …, n, k ϵ\{1, …, K\}, w$\epsilon\Delta^{m-1}$, z$\epsilon\Delta^{n-1}$, α(k), β(k)$\epsilon\mathbb{R}^p$, k ϵ\{1, …, K\}.

(2) $\sum_{k=1}^{K} [x^T(V(a,k) + V(B,k))y - \alpha^*(k) - \beta^*(k)] = 0$.

References

1. Bhurjee, A. K. (2016): Existence of Equilibrium Points for Bimatrix Game with Interval Payoff. International Game Theory Review, Volume 18, No. 1, 1650002.

2. Goswami, M. P., Mitra, M. and Sen, D. (2018): A characterization of lexicographic preferences. Munich Personal RePEc Archive (MPRA), Paper No. 100248. Available at: https://mpra.ub.uni-muenchen.de/100248/

3. Lahiri, S. (2026): Lexicographic and Strictly Weighted Equilibrium Strategy Profiles with Applications to Interval-Valued Bi-matrix Games. DOI: https://doi.org/10.31224/7351

4. Mangasarian, O. L. and Stone, H. (1964): Two-Person Nonzero-Sum Games and Quadratic Programming. Journal of Mathematical Analysis And Applications, Volume 9, Pages 348-355.

5. Radhakrishnan, S. and Saikeerthana, D. (2020): Game Theory Problems Using Interval Parameters. Kongunadu Research Journal, Volume 7, Number 2, Pages 123-129.
6. Varian, H. (1978): Microeconomic Analysis. W.W. Norton and Company, INC.