Collatz-Conjecture Proved and Halemane-Conjecture Proposed

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Abstract

The entire dynamics of the Collatz (3n+1) System can be exactly represented by a dynamically-evolving graded-algebraic-structure of an ideal-based-filtration-scheme; with divisibility-classes for modulo-3 quotient-semiring generated by the primitive root 2; along with a filtration-shifting-global-affine-transformation, f(x)=(3x+1) when x is a positive odd number; achieving a Euclidean-expansion-shift to the topmost coprime-layer of that filtration-scheme; while also avoiding the modulo-multiple-layer (zero-layer) and all the nilpotent-layers with nilpotent-elements (dead-end zero-divisors) like (6m-3) and (6m-0); the trivial-cycle {(1⇐2⇐4)} being bypassed by an initialization-phase for this filtration scheme, starting directly with the modul-9 coprime-layer. The transitive-closure of the Collatz (3n+1) System is exactly represented by the transitive-closure of this dynamically evolving algebraic system which is indeed the complete set of natural numbers. This design for the system-dynamics-framework by itself provides a definitive-proof of the Collatz (3n+1) Conjecture, establishing that the entire Collatz Map is exactly represented by this dynamically-evolving graded-algebraic-structure defined on the set of all natural

numbers; neither missing any natural-number nor including any extraneous-elements; providing both the necessary-&-sufficient condition simultaneously in one single sweep.

Halemane-Conjecture states that the maximum number of odd (3n+1) operations required to reach the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ starting from any given positive integer and moving along the Collatz sequence, is limited by that given number itself; with the triad $\{(31 \Leftarrow 41 \Leftarrow 27)\}$ as an exceptional limiting case.

1. ALGEBRAIC STRUCTURE for COLLATZ FUNCTION

The Collatz function S(n) for any natural number n; is given by:

if (n is even) then $S(n) \coloneqq D(n) \coloneqq (n/2)$; else $S(n) \coloneqq U(n) \coloneqq (3*n+1)$; [Eqn.1]

where S refers to the Collatz successor operator, D refers to the pull-down operator and U refers to the push-up operator.

Note that the Collatz successor operator is a one-to-one mapping.

The algebraic structure for the Collatz function has the following characteristics.

1.1 BASE SEMIRING AND IDEAL

Start with the semiring of non-negative integers, $R = (Z_{\geq 0}, +, .)$;

The modulo-3 quotient semiring is constructed by considering the ideal $I = \langle 3 \rangle$ defined by the modulo-base 3.

1.2 PRIMITIVE-ROOT

The element 2 serves as a primitive-root modulo-3, because $2^2 \equiv 1$ (mod3); allowing us to achieve the division by 2 operations within the semiring structure.
Every division by 2 (whenever the number is even, as per the Collatz function) is achieved through multiplication by the modular inverse of the primitive-root in the specific quotient-semiring; for example, multiplying by 5 in modulo-9 semiring.

1.3 DIVISIBILITY-CLASSES

The divisibility-classes based on the modulo-3 base (used for the creation of the quotient semiring) defines the subring layers in the filtration scheme.

1.4 IDEAL-BASED FILTRATION SCHEME

To define a hierarchical filtration scheme, we create a tower of subrings or ideals, corresponding to the divisibility-order with respect to the modulo-base 3; the bottommost being the zero-layer or the all-absorbing-layer. Each higher layer is created corresponding to every application of the push-up operator, as defined in [Eqn.1] above; the topmost layer at any stage being the units or the coprime-layer forming the principal-orbit. The quotient semirings defined with respect to modulus base 3^k characterize layer k with divisibility-order k in this filtration scheme. The primitive-root 2 generates the multiplicative structure within any given subring; and the elements transition down the filtration hierarchy by exact division by 3, although the Collatz function does not have any operation of division by 3. The elements of the subring within any given layer

Note that the process of repeated division by the primitive-root 2, within the subring of any given filtration layer, produces two partitions of the subring consisting of alternating elements of the orbit in the subring; one belonging to the modulo-3 residue class $[1]_3$ and the other belonging to the modulo-3 residue class $[2]_3$; of almost equal-size; the relative size breakup alternates between these two residue classes as we move up or down the filtration tower.

1.5 EUCLIDEAN EXPANSION SHIFT TO THE TOPMOST COPRIME LAYER

The Collatz odd operator f(x)=(3x+1) when x is a positive odd number; acts as a filtration shifting global affine transformation; achieving a Euclidean expansion shift to the topmost coprime-layer of that filtration-scheme; while also avoiding the modulo multiple layer (zero-layer) and all the intermediate nilpotent-layers with nilpotent elements (dead-end zero-divisors) like (6m-3) and (6m-0). When the Collatz odd operator acts on a positive odd number at the current topmost coprime-layer, we create a new topmost coprime-layer with a higher modulus base of 3^{k+1} instead of the earlier 3^k; leading to a dynamically evolving expansion of the filtration scheme.

1.6 INITIALIZATION PHASE TO BYPASS THE TRIVIAL CYCLE

The trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ can be bypassed by an initialization-phase for this filtration scheme, starting directly with the modul-9 coprime-layer.

1.7 FIRST ELEGANT PROOF OF THE COLLATZ CONJECTURE

The transitive-closure of the Collatz (3n+1) System is exactly represented by the transitive-closure of the above designed dynamically evolving algebraic system which is indeed the complete set of natural numbers. This design for the system-dynamics-framework by itself provides a definitive-proof of the Collatz (3n+1) Conjecture, establishing that the entire Collatz Map is exactly represented by this dynamically-evolving graded-algebraic-structure defined on the set of all natural numbers; neither missing any natural number nor including any extraneous-elements; providing the necessary & sufficient condition.

2. ALGEBRAIC STRUCTURE for INVERSE COLLATZ FUNCTION

The inverse of the Collatz function is given by:

$$\begin{aligned}&\text{if } (n\equiv 1(\text{mod}6)) \text{ then } && P(n) \coloneqq g(n) \coloneqq (n.2^w -1)/3;\\ &\text{else if } (n\equiv 5(\text{mod}6)) \text{ then } && P(n) \coloneqq h(n) \coloneqq (n.2^v -1)/3; \qquad \text{[Eqn.2]}\end{aligned}$$

with w being a positive even exponent and v being a positive odd exponent; where P refers to the Collatz predecessor operator.

Note that the Collatz predecessor is a one-to-many mapping.

The algebraic structure for the inverse Collatz function has the following characteristics.

2.1 BASE SEMIRING AND IDEAL

Start with the semiring of non-negative integers, R = ($Z_{\geq 0}$, +, .);
The modulo-2 quotient semiring is constructed by considering the ideal I = ⟨2⟩ defined by the modulo-base 2.

2.2 PRIMITIVE-ROOT

The element 3 serves as a primitive-root modulo-3, because $3 \equiv 1$ (mod2);
allowing us to achieve the division by 3 operation within the semiring structure.
Every division by 3 (whenever applicable, as per the inverse Collatz function) is achieved through multiplication by the modular inverse of the primitive-root in the specific quotient-semiring; for example, multiplying by 11 in modulo-16 semiring.

2.3 DIVISIBILITY-CLASSES

The divisibility-classes based on the modulo-2 base (used for the creation of the quotient semiring) defines the subring layers in the filtration scheme.

2.4 IDEAL-BASED FILTRATION SCHEME

To define a hierarchical filtration scheme, we create a tower of subrings or ideals, corresponding to the divisibility-order with respect to the modulo-base 3; the bottommost being the zero-layer or the all-absorbing-layer. Each higher layer is created to accommodate for the possible valid values of the binary exponents w and v in [Eqn.2] above as required in the definition of the inverse Collatz function; the topmost layer being the units or the coprime-layer forming the principal-orbit. The quotient semirings defined with respect to modulus base 2^k characterize layer k with divisibility-order k in this filtration scheme. The primitive-root 3 generates the multiplicative structure within any given subring; and the elements transition down the filtration hierarchy by exact division by 2; although the inverse Collatz function does not have any operation of division by 2.

Note that the process of repeated division by the primitive-root 3, within the subring of any given filtration layer, produces two partitions of the subring consisting of alternating elements of the orbit in the subring; one belonging to the modulo-4 residue class $[1]_4$ and the other belonging to the modulo-4 residue class $[3]_4$; both being of exactly equal size.

2.5 EUCLIDEAN EXPANSION SHIFT TO THE TOPMOST COPRIME LAYER

The inverse Collatz operators g(n) & h(n) as in [Eqn.2] above; act as the filtration shifting global affine transformations, achieving a Euclidean expansion shift to a higher layer; while also avoiding the modulo multiple layer (zero-layer) and all the intermediate nilpotent-layers with nilpotent elements (dead-end zero-divisors) like (4m-2) and (4m-0). This upward jump may be either (1) an even number w of binary exponential hops (based on the chosen value of w) in case of g(n); or (2) an even number w of binary exponential hops (based on the chosen value of w) in case of g(n). The valid values for w and v are determined by the modulo-6 residue class of the operand element, as specified in [Eqn.2] above. When either of these inverse Collatz odd operators of [Eqn.2] act on a valid operand element in the current topmost coprime-layer, or even when g(n) of [Eqn.2] acts on a valid operand element in a penultimate (just beneath the topmost) layer; we create a new topmost coprime-layer with a higher modulus base of 2^{k+1} instead of the earlier 2^k; leading to a dynamically evolving expansion of the filtration scheme.

2.6 INITIALIZATION PHASE TO BYPASS THE TRIVIAL CYCLE

The trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ can be bypassed by an initialization-phase for this filtration scheme, starting directly with the modul-8 coprime-layer.

2.7 SECOND ELEGANT PROOF OF THE COLLATZ CONJECTURE

The transitive-closure of the inverse Collatz System is exactly represented by the transitive-closure of this dynamically evolving algebraic system which is indeed the complete set of natural numbers. This design for the system-dynamics-framework by itself provides a definitive-proof of the Collatz (3n+1) Conjecture, establishing that the entire inverse Collatz Map is exactly represented by this dynamically-evolving graded-algebraic-structure defined on the set of all natural numbers; neither missing any natural number nor including any extraneous-elements; providing the necessary & sufficient condition.

3. Recommended Reading

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4. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an

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5. Author Rights Retention Notice

6. Dedication

To my **ಅಜ್ಜ**(ajja) Karinja Halemane Keshava Bhat & **ಅಜ್ಜಿ**(ajji) Thirumaleshwari, **ಅಪ್ಪ**(appa) Shama Bhat & **ಅಮ್ಮ**(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection*.