Halemane's Theorem on the Collatz (3n+1) System

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Abstract

Halemane's Theorem on the Collatz (3n+1) System states that the entire dynamics of the Collatz (3n+1) System and therefore the entire Collatz-Map defined by the transitive closure of the Collatz Function on the set of all natural numbers; can be exactly represented by a bounded finite small-sized ideal based graded filtration structure defined on a modulo-3 quotient semiring generated by the primitive-root 2; typically, with three layers; which guarantees the convergence to the trivial-cycle. To facilitate further deeper study, this filtration structure can be subjected to a suitably designed dynamically evolving refinement mechanism. This refinement is achieved with two unique features. The first feature is the application of the global-affine-transformation, f(x)=(3x+1) when x is a positive odd number (as defined in the Collatz System); to achieve a Euclidean expansion shift to the topmost coprime-layer of that filtration-structure; while avoiding the modulo-multiple-layer (zero-layer) and all the intermediate nilpotent-layers with nilpotent-elements (dead-end zero-divisors) like (6m-3) and (6m-0). The second feature

is to bypass the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ of the Collatz-Map by an initialization-phase for this filtration structure, starting directly with the modul-9 coprime-layer. The entire Collatz-Map (transitive-closure of the Collatz function) is exactly represented by each and every one of these dynamically evolving graded algebraic filtration structures; which is itself covers the complete set of all natural numbers; neither missing any natural-number nor including any extraneous-elements; providing both the necessary-&-sufficient condition simultaneously.

Halemane-Conjecture states that the maximum number of odd (3n+1) operations required to reach the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ starting from any given positive integer and moving along the Collatz sequence, is limited by that given number itself; with the triad $\{(31 \Leftarrow 41 \Leftarrow 27)\}$ as an exceptional limiting case.

1. ALGEBRAIC STRUCTURE for COLLATZ FUNCTION

The Collatz function S(n) for any natural number n; is given by:

$$\text{if (n is even) then } S(n) \coloneqq D(n) \coloneqq (n/2);\text{ else } S(n) \coloneqq U(n) \coloneqq (3*n+1); \qquad \text{[Eqn.1]}$$

where S refers to the Collatz successor operator, D refers to the pull-down operator and U refers to the push-up operator.
Note that the Collatz successor operator is a one-to-one mapping.

The algebraic filtration structure for the Collatz system is designed as follows.

1.1 BASE SEMIRING AND IDEAL

Start with the semiring of non-negative integers, $R = (Z_{\geq 0}, +, .)$.
The modulo-3 ideal-based quotient-semiring is constructed by considering the ideal $I=\langle 3\rangle$ defined by the modulo-base 3.

1.2 PRIMITIVE-ROOT

The element 2 serves as a primitive-root modulo-3, because $2^2 \equiv 1 \pmod{3}$. Every division by this primitive-root 2 (whenever the number is even, as per the Collatz function) is achieved through multiplication by the modular (modulo-3) inverse of this primitive-root in the specific quotient-semiring; for example, through a multiplication by 2 in modulo-3 semiring, whereas it is through a multiplication by 5 in modulo-9 semiring, etc.

1.3 DIVISIBILITY-CLASSES and IDEAL-BASED FILTRATION-STRUCTURE

The divisibility-classes based on the modulo-3 ideal-base (used for the creation of the quotient-semiring) defines the subring layers in the filtration-structure. To start with, in the most trivial-case, we have a modulo-multiplier layer (modulo-3^0 layer or zero-layer) at the bottom and also a modulo-3^1 layer (layer-1) as the topmost coprime-layer. This is considered as a trivial one-layer algebraic-filtration-structure F(3, 2; 1). This trivial filtration-structure F(3, 2; 1) by itself is indeed an exact representation of the entire Collatz System Dynamics or the entire Collatz-Map.

To facilitate more detailed analysis of the system dynamics, we define a hierarchical filtration-structure, we create a filtration-tower of subrings or ideals, corresponding to the divisibility-order with respect to the modulo-base 3; the bottommost being the zero-layer or the all-absorbing-layer. Each higher layer is created corresponding to every application of the push-up operator, as defined in [Eqn.1] above. The topmost layer at any stage being the units, that is, say, the modulo-3^m coprime-layer forming the principal-orbit. The quotient semirings defined with respect to modulus base 3^k ($0 \leq k \leq m$) form layer-k in the filtration structure with layer-m being the topmost coprime-layer; denoted by S(F(3, 2; m); k) or simply S(3, 2; m; k) where (m-k) denotes the divisibility-order in this filtration-tower; and the elements in the subring S(3, 2; m; k) consist of the modulo-3^k residue-classes.

The primitive-root 2 generates the multiplicative structure within any given subring; and the elements transition down the filtration hierarchy by exact division by the ideal-base 3, although the Collatz function does not have any explicit operation of division by 3 and therefore we do not explicitly use this feature. However, note that when we refine the trivial structure F(3,2;1) by expansion; introducing a modulo-9 coprime layer as layer-2 to construct F(3,2;2); the elements $[1]_9$ $[4]_9$ and $[7]_9$ of S(3,2;2;2) together form a refinement by expansion of the earlier element $[1]_3$ in S(3,2;1;1) of the trivial structure F(3,2;1); whereas S(3,2;2;1) that is, the new layer-1 of this newly constructed F(3,2;2) has its elements updated by 3.S(3,2;1;1) that is, $3.\{[1]_3; [2]_3\}$. That shows the implicit result of the exact division by the ideal-base 3 at every step as we move down the tower.

Note that the process of repeated division by the primitive-root 2, within the subring of any given filtration layer, produces two partitions of the subring consisting of alternating elements of the orbit in the subring; one belonging to the modulo-3 residue-class $[1]_3$ and the other belonging to the modulo-3 residue class $[2]_3$; of almost equal-size; the relative size breakup alternates between these two residue classes as we move up or down the filtration tower.

1.4 INITIALIZATION PHASE TO BYPASS THE TRIVIAL CYCLE

The trivial-cycle {(1⇐2⇐4)} can be bypassed by an initialization-phase for this filtration structure, starting directly with the modul-9 coprime-layer; that is, F(3,2;2).

1.5 EUCLIDEAN EXPANSION SHIFT TO THE TOPMOST COPRIME LAYER

The Collatz odd operator f(x)=(3x+1) when x is a positive odd number; acts as a filtration shifting global affine transformation; achieving a Euclidean expansion shift to the topmost coprime-layer of that filtration structure; while also avoiding the modulo multiple layer (zero-layer) and all the intermediate nilpotent-layers with nilpotent elements (dead-end zero-divisors) like (6m-3) and (6m-0). When the Collatz odd operator U acts on a positive odd number at the current topmost coprime-layer, we create a new topmost coprime-layer with a higher modulus-base of 3^{k+1} instead of the earlier 3^k; leading to a dynamically evolving refinement-by-expansion of the filtration-structure.

The output of this affine transformation results in a number that is always a positive even number of the form (6m-2) that belongs to the modulo-3 residue class $[1]_3$ in the newly created coprime-layer defined by modulo-3^{k+1}; with three-fold increase in orbit-size compared to the earlier one with modulo-base 3^k; irrespective of whether the input odd number belongs to either the modulo-3 residue-class $[1]_3$ or the modulo-3 residue-class $[2]_3$.

Note that there is absolute self-similar symmetry governed by modular-periodicity (modulo-3) in this graded algebraic filtration-structure, with respect to every parameter.

As a result of the above design and construction, we formulate the following theorem:

2. Halemane's Theorem on the Collatz (3n+1) System

the entire dynamics of the Collatz (3n+1) System and therefore the entire Collatz-Map defined by the transitive closure of the Collatz Function on the set of all natural numbers; can be exactly represented by a bounded finite small-sized ideal based graded filtration structure defined on a modulo-3 quotient semiring generated by the primitive-root 2; typically, by F(3,2;2) with three layers as mentioned above; which guarantees the convergence to the trivial-cycle.

2.1 Proof

First consider the trivial case of a two-layer filtration structure F(3,2;1) with the topmost coprime-layer being a modulo-3 quotient semiring with the principal orbit consisting of the two modulo-3 residue-classes $[1]_3$ and $[2]_3$ and the zero-layer consisting of the modulo-3 residue-class $[0]_3$.

For any positive even number in the subring S(3,2;1;1) of F(3,2;1); it being in either of the modulo-3 residue classes, $[1]_3$ or $[2]_3$; a division by 2 (as per the Collatz function) resulting in an alternating cycle between these two modulo-3 residue classes until finally we arrive at an odd number that may belong to any one of these classes. Any positive even number that is an even multiple of 3 after repeated divisions by 2 will result in an odd multiple of 3. Therefore, we may have an odd number that may be either an odd multiple of 3 in $[0]_3$ or it may be in either of the two modulo-3 residue classes $[1]_3$ or $[2]_3$.

The Collatz odd function when applied to $[0]_3$ or $[1]_3$ or $[2]_3$ always gives a result that is in $[1]_3$. However, as pointed out earlier, when we refine the trivial structure F(3,2;1) by expansion; introducing a modulo-9 coprime layer as layer-2 to construct F(3,2;2); the elements $[1]_9$ $[4]_9$ and $[7]_9$ of S(3,2;2;2) together form a refinement-by-expansion of the earlier element $[1]_3$ in S(3,2;1;1) of the trivial structure F(3,2;1); whereas elements $[8]_9$ $[2]_9$ and $[5]_9$ of S(3,2;2;2) together form a refinement-by-expansion of the earlier element $[2]_3$ in S(3,2;1;1) of the trivial structure F(3,2;1). Therefore, with this new algebraic filtration structure F(3,2;2) we are able to distinguish between odd numbers of the form (6m-5) and (6m-1) in S(3,2;2) and also those of the form (6m-3) and (6m-0) in S(3,2;1).

The transitive-closure of the Collatz (3n+1) System, that is, the Collatz-Map, is exactly represented by the transitive-closure of the above designed dynamically evolving algebraic system which is indeed the complete set of natural numbers. This design for the system-dynamics-framework by itself provides a definitive-proof of the Collatz (3n+1) Conjecture, establishing that the entire Collatz-Map is exactly represented by this dynamically-evolving graded-algebraic-structure defined on the set of all natural numbers; neither missing any natural number nor including any extraneous-elements; providing the necessary & sufficient condition.

Here is further justification based on some specific numerical behavior. The self-similar symmetry structure governed by the modular-periodicity creates equal possibilities of the occurrence of all the possible modulo residue classes corresponding to each of the parameters at every step, requiring the application of the corresponding operators as in

[Eqn.1] above. The worst-case scenario that needs to be studied corresponds to the case of a subsequence in the condensed compact Collatz sequence (of positive odd numbers) associated with a triad of the form {(18m-5)⇐(24m-7)⇐(16m-5)}; which gets generated by a subsequence of operations of the form (UD) & (UDD). Equivalently, we may as well consider the subsequence {(6m-1)⇐(4m-1)} for (UD) operation and the subsequence {(6m-5)⇐(8m-7)} for (UDD) operation. A study of such worst-case scenarios leads to the limiting critical values for the number of (UDD) operations, that is, nUDD required to compensate for the increased numerical value resulting from the (UD) operations, that is, nUD; to be given by -

$$(\log(3/2)/\log(4/3)) \leq nUDD/nUD \leq (\log(17/11)/\log(17/13));$$

that is,

$$1.409421 \leq nUDD/nUD \leq 1.622722; \qquad \text{[Eqn.2]}$$

It is clear that the self-similar symmetry of the Collatz-Map governed by the modular-periodicity with respect to each of the defining parameters therein provides for equal distribution of the operand numbers as well as the resulting numbers in every operation (remember that every U-operation must necessarily be immediately followed by at least one instance of D-operation). Also, we have not accounted for the situation with repeated multiple application of the D-operator as for example $((U^1D^1)D^p)$ for p>1; which will certainly result in immediate further decrease in the numerical value of the operand. Equal distribution among the three modulo-3 residue classes would result in an average limiting critical value of 3/2; which is certainly within the range determined above.

The real reason for convergence is of course the self-similar symmetry governed by the modular-periodicity together with the closure property covering all the modulo-3 modular residue classes for all the defining parameters of the Collatz system.

3. Recommended Reading

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4. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an Independent Researcher, and that I have not received any funding from any source other than my own savings, and I do not have any obligations or encumbrances of any kind, neither financial nor legal nor of any other kind, regarding the contents of the manuscript - of which I am the original author and creator. Also, I hereby declare there is no conflict of interests.

5. Author Rights Retention Notice

6. Dedication

To my **ಅಜ್ಜ**(ajja) Karinja Halemane Keshava Bhat & **ಅಜ್ಜಿ**(ajji) Thirumaleshwari, **ಅಪ್ಪ**(appa) Shama Bhat & **ಅಮ್ಮ**(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection.*