

Halemane's Theory on the Collatz $(3x+1)$ System

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Abstract

This paper presents certain details about the development of Halemane's Theory on the Collatz $(3x+1)$ System. The concept of binary-exponential-ladder BEL has been introduced, to enable effective merging/absorption of repeated application of the Collatz even-operator $(x/2)$ whenever valid, within the corresponding BEL. The set of all binary exponential ladders has a direct bijective mapping with the set of all positive integers. Every positive odd integer corresponds to the unique defining-base-rung of a BEL; whereas every positive even integer corresponds to a non-base higher rung of the corresponding BEL. The immediate-successor relation is defined as a unique one-to-one binary-relation on the set of binary exponential ladders, using the condensed compact Collatz odd-to-odd function. Therefore, the network of binary exponential ladders BELnet provides an exact mathematical representation of the Collatz system. Note that the condensed compact inverse

Collatz (CTUHSK) odd-to-odd function defines the immediate-predecessor relation, which is shown to be a one-to-many mapping.

A structured system framework H has been designed merely as a well-organized condensation of BELnet by exploiting the partial order relation in BELnet; to achieve a linear strict order in the structured system framework H .

An order-isomorphism is established between the relevant component H^s (with an invariant-base-element H_0) of the structured system framework H and the set of positive integers. This provides the necessary condition that starting with any integer in H^s the Collatz sequence converges to the trivial-cycle contained in $BEL(1)$ which is the invariant-base-element or the sink-node H_0 in H^s .

The sufficient condition is provided by a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) that is used to demonstrate domain exhaustion; having already captured all the modular-residue-classes in H^s ; logically excluding the existence of any extraneous elements or objects or sub-systems such as disjoint loops/cycles H^∞ and/or divergent chains $H^\&$ or even any/all non-standard objects, in H .

Some directions for possible future research work on algorithmic / computational characteristics of the Collatz system have also been presented. A situation has been identified wherein the *emergence of global system properties through persistent local subsystem characteristics* can be clearly demonstrated.

Halemane-Conjecture states that the upper limit on the number of odd $(3x+1)$ operations required to reach the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ starting from any given

positive integer and moving along the Collatz sequence, is the given number itself; the triad $\{(31 \Leftarrow 41 \Leftarrow 27)\}$ being an exceptional limiting case.

This paper also presents certain exploratory studies on related systems –

- (1) Halemane's $(1.x+1)/2$ System; (2) Halemane's $(2.x \pm 1)/3$ System;
- (3) Halemane's $(3.x \pm 1)/4$ System; and (4) Halemane's $(3.x+1)/4$ System which is a clone of the Collatz $(3x+1)$ System.

Keywords: Binary-Exponential-Ladder; Modular-Periodicity; Arborescence;

CTUHSK-Theorem; CTUHSK-Generative-Parameters;

Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Sequence;

Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture;

Order-Isomorphism; Dedekind-Peano Axioms; Convergence;

Halemane's $(1.x+1)/2$ System; Halemane's $(2.x \pm 1)/3$ System;

Halemane's $(3.x \pm 1)/4$ System; Halemane's $(3.x+1)/4$ System;

Halemane's Conjecture on the Collatz $(3x+1)$ System.

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1. Introduction

The Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture, simply referred to as the *Collatz Conjecture* [§1] states that the Collatz $(3x+1)$

Sequence, also referred to as the Collatz Sequence, converges to the trivial-cycle, that is, $\{1 \Leftarrow 2 \Leftarrow 4\}$, starting from any positive integer.

Lagarias [§3]&[§4] gives an exhaustive annotated bibliography, whereas Lagarias [§5]&[§6] gives an elaborate overview of the Collatz Problem, also referred to as the “ $3x+1$ problem”, referring to it as “The Ultimate Challenge”. Guy [§2] has been somewhat pessimistic in advising researchers “Don’t Try to Solve These Problems”. Lagarias [§5]&[§6] refers to the statement by the most revered Mathematician and Number Theory Expert Paul Erdos, who said that - “Mathematics is not yet ready for such problems” - about the Collatz Conjecture. The Collatz ($3x+1$) Sequence is indeed a highly complex chaotic system defined or generated with one of the simplest deterministic well-defined equations, that too, just two in number. We will not get into any more of the reported literature except the problem definition; but will simply focus on a rather shockingly simple approach toward a possible resolution of this amazingly notorious long-standing unresolved problem.

This paper presents the *Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani* (CTUHSK) Theorem, asserting the convergence of the Collatz ($3x+1$) Sequence to the trivial-cycle. The proof is two-fold. First is the application of the most fundamental *Dedekind-Peano Axioms* and *Modular Arithmetic* to a meticulously designed *Structured System Framework* of *Binary-Exponential-Ladders* defined on the set of positive odd numbers, and establishing an *order-isomorphism* between the set of natural numbers and the *structured system framework* of *binary-exponential-ladders*. Secondly a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) ensures the non-existence of any extraneous elements.

2. Problem Description

We define the *Collatz Function* $C(n)$ with a positive integer n as its input argument, in terms of a ‘pull-Down’ operator $D(n)$ and a ‘push-Up’ operator $U(n)$ as follows:

$$\text{if } (n \text{ is even}) \ C(n) := D(n) = (n/2); \text{ else } C(n) := U(n) = (3*n+1); \quad [\text{Eqn.1}]$$

where the ‘*pull-Down*’ operator $D(n)$ takes only an even number as its input argument whereas the ‘*push-Up*’ operator $U(n)$ takes only an odd number as its input argument and gives an output that is an even number.

For convenience in our study of the Collatz Sequence, we define the *Compact Collatz Function* $T(m)$ by the repeated application of the ‘pull-down’ operator $D(m)$ wherever applicable, say, $(p \geq 1)$ times, that is, $D^*(m) := D^p(m)$ so as to get an output $D^\#(m)$ that is an odd number:

$$\begin{aligned} \text{if } (m \text{ is even}) \ T(m) &:= D^*(m) := D^p(m) = (m / 2^p) := D^\#(m); \\ \text{else } (m \text{ is odd}) \ T(m) &:= U(m) = (3*m + 1) := U^\#(m); \end{aligned} \quad [\text{Eqn.2}]$$

where $D^\#(m)$ is called the “D-floor number” associated with the input argument m ; and $U^\#(m)$ is called the “U-ceiling number” associated with the input argument m .

The *Compact Collatz Function* $T(m)$ may as well be considered to have been redefined with these newly introduced two operators, the “D-floor operator” $D^\#(m)$ and the “U-ceiling operator” $U^\#(m)$ as given in [Eqn.2] above.

This new definition for the *Compact Collatz Function* $T(m)$ facilitates our study of the corresponding *Compact Collatz Sequence*; which is no different from its equivalent standard Collatz Sequence, once we understand that the repeated application, say, $(p \geq 1)$ times, of the ‘pull-down’ operator $D(m)$ has now been collapsed into an equivalent single “D-floor operator” $D^\#(m)$ giving the D-floor number $D^\#(m)$ as its output. The push-Up operator U has been simply redefined as the “U-ceiling operator” $U^\#$ for uniformity and elegant completeness.

The *Compact Collatz Sequence* is obtained by the repeated sequential application of the *Compact Collatz Function* $T(m)$ starting with the given initial input number m - represented by an alternating series of $D^\#$ number and $U^\#$ number - except possibly the starting initial ‘seed’ number m and the final terminating number, which as per the Collatz Conjecture, is anyway a $D^\#$ number that is unity.

3. Observations on the Pull-Down Operator

The pull-down operator D *always* takes only an even number n as its input argument. Every application of this pull-down operator results in an alternating (toggling) effect on the $n \bmod 3$ property of the input argument number; that is, a $1 \bmod 3$ input gives a $2 \bmod 3$ output and a $2 \bmod 3$ input gives a $1 \bmod 3$ output; whereas a $0 \bmod 3$ input gives a $0 \bmod 3$ output. Repeated application of D , in case applicable, results in a final output that is an odd number and therefore becomes an input for the push-Up operator. In such a case, we call it a “D-floor operator” $D^\#$ as defined in [Eqn.2] above, and its output a “D-floor number” $D^\#(n)$

characterized by being a odd number; $D^\#(n)$ may be in any one of the three possible types:

- (1) a 1MOD6 odd number, being a 1MOD3 odd number of the type $(6m-5)$;
- (2) a 5MOD6 odd number, being a 2MOD3 odd number of the type $(6m-1)$;
- (3) a 3MOD6 odd number, being a 0MOD3 odd number of the type $(6m-3)$.

4. Observations on the Push-Up Operator

The push-Up operator U *always* takes only an odd number m as its input argument, and *always* gives an output that is a 4MOD6 number, being a 1MOD3 even number that is of the type $(6m-2)$; irrespective of whether the input is a 1MOD6 odd number or a 3MOD6 odd number or a 5MOD6 odd number. Note that one single application of the ‘push-Up’ operator U transforms any input odd number m into a 4MOD6 even number that becomes an input to the “D-floor operator $D^\#$ ”. That is why we may as well consider the push-Up operator U as the “U-ceiling operator” $U^\#$ as defined in [Eqn.2] above.

5. Observations on the Compact Collatz Function

Start with any positive integer. (1) If the starting initial number n is even, then we apply the D-floor operator $D^\#$ giving an output that is the D-floor number $D^\#(n)$ which is given as input to the U-ceiling operator. Of course, if the starting number is a power of 2 we terminate at unity. Else, we have a D-floor number

$D^\#(n)$ that is an odd number greater than unity, in any non-trivial case; as the initial $D^\#$ node in the compact Collatz sequence. (2) If on the other hand the starting initial seed number n is an odd number, we treat that itself as the initial $D^\#$ node in the compact Collatz sequence.

Having thus obtained the initial $D^\#$ node in the Compact Collatz Sequence, we apply the U-ceiling operator $U^\#$ to get the U-ceiling number $U^\#$ that is a $4\text{MOD}6$ even number. That in turn is given as input to the D-floor operator $D^\#$. Now the process continues.

Note that the compact Collatz sequence can therefore be defined as the *trajectory* generated by an *alternating sequence* of a “D-floor number” $D^\#$ and a “U-ceiling number” $U^\#$, *with its starting initial node being a $D^\#$ number*. The Compact Collatz Function as presented in [Eqn.2] defines the unique link (directed arc) from any given D-floor number $D^\#$ as the predecessor node to its corresponding unique U-ceiling number $U^\#$ as the successor node and also the unique link (directed arc) from any given U-ceiling number $U^\#$ as the predecessor node to its corresponding unique D-floor number $D^\#$ as the successor node. The unique link (directed arc) from a starting initial even “seed” number leading to the first node (D-floor number $D^\#$) in the *trajectory* is similarly defined.

As mentioned earlier, the application of the D-floor operator $D^\#$ on a U-ceiling number $U^\#$ that is a $4\text{MOD}6$ even number of the form $(6m-2)$ can lead to a D-floor number $D^\#$ that is an odd number that can be either: (1) a $1\text{MOD}6$ odd number, being a $1\text{MOD}3$ odd number that is of the type $(6m-5)$; xor (2) a $5\text{MOD}6$ odd number, being a $2\text{MOD}3$ odd number that is of the type $(6m-1)$; but can never be (3) a $3\text{MOD}6$ odd number, that is a $0\text{MOD}3$ odd number of the type $(6m-3)$. The

only exception, when the D-floor operator $D^\#$ gives an output D-floor number $D^\#$ that is a 3MOD6 odd number of the type $(6m-3)$ is the situation when its input is a 0MOD6 even number, which is impossible for any U-ceiling number $U^\#$; although such an input may come in those special cases wherein the starting initial 'seed' number itself is a 0MOD6 even number that is of the form $(6m-3).2^p$ with $p>0$; leading to an output $D^\#$ that is a 3MOD6 odd number of the form $(6m-3)$.

6. Analysis of the Compact Collatz Sequence

From the above observations, it is clear that corresponding to every positive integer n as the starting initial 'seed' number, there is a starting initial node in the trajectory representing the compact Collatz sequence, that is a $D^\#$ number in exactly one of the three possible forms as mentioned above - that can be an input argument to the U-ceiling operator $U^\#$ giving exactly one unique output $U^\#$ which itself can be an input to the D-floor operator $D^\#$ so that the process continues. Successive application of each of these two operators ($U^\#$ and $D^\#$) wherever applicable, traces a unique trajectory, wherein each node is represented by the unique output number of the appropriate operation applied to the input number represented by the preceding node in the trajectory.

The anticipated terminating trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ can be obtained only through a final application of the D-floor operator $D^\#$ on a 4MOD6 even number that is of the form $(6m-2).2^p$ with $m=1$ and $0 \leq p$.

7. Binary-Exponential-Ladder with its Defining-Base-Rung $D^\#$

Here, we present a meticulously designed *Structured System Framework* that *partitions* the *set of positive integers* to facilitate a *general systems analysis* of the *Compact Collatz Sequence*.

Let every positive odd number be associated with a *binary-exponential-ladder*, denoted by $BEL(2m-1)$ and defined as a sequence $\{(2m-1).2^u \mid (u \geq 0)\}$; with its *defining-base-rung* ($u=0$) given by the odd number $(2m-1)$.

$$BEL(2m-1) := \{(2m-1).2^u \mid (u \geq 0); (m \geq 0)\}; \quad [Eqn.3]$$

Thus, we establish an exact one-to-one correspondence (bijection) between the *set of positive odd numbers* that form the $D^\#$ value and the *defining-base-rung* of the corresponding *binary-exponential-ladder* $BEL(D^\#)$.

Every *positive even number* in the form $\{(2m-1).2^u \mid (u > 0)\}$; for which there exists its corresponding $D^\#$ value, $D^\#((2m-1).2^u) = (2m-1)$; there exists exactly one corresponding *binary-exponential-ladder* $BEL(2m-1)$ that contains the given even number $(2m-1).2^u$ as one of the higher rungs in $BEL(2m-1)$;

$$B((2m-1), u) := \{(2m-1).2^u \mid (u > 0)\}; \quad [Eqn.4]$$

Therefore, *the set of all binary-exponential-ladders form a partition of the set of all positive integers*; with an *exact one-to-one correspondence (bijection)* between each positive odd number $D^\#$ and the corresponding *binary-exponential-ladder* for which

it is the *defining-base-rung* $D^\#$; whereas each of the positive even numbers correspond to exactly one of the higher rungs of a specific binary-exponential-ladder identified by the D-floor number $D^\#$ associated with that given positive even number.

This partitioned framework of positive integers goes another step deeper because of the fact that the *defining-base-rung* $D^\#$ of a binary-exponential-ladder $BEL(D^\#)$ can itself be in one of the three possible forms $1 \text{MOD} 6$ or $5 \text{MOD} 6$ or $3 \text{MOD} 6$; whereas each of the higher rungs of the binary-exponential-ladder are either (1) alternatingly $2 \text{MOD} 6$ and $4 \text{MOD} 6$; xor (2) all being $0 \text{MOD} 6$ numbers.

The Collatz Conjecture states that every Collatz Sequence, starting from any positive integer, converges to the trivial-cycle $\{1 \Leftarrow 2 \Leftarrow 4\}$ which is in $BEL(1)$ that is uniquely identified by its defining-base-rung $D^\#$ value that is unity. Therefore, our focus will be the binary-exponential-ladders $BEL(1)$ and its relation with every other binary-exponential-ladder $BEL(D^\#)$.

Note that $D^\#$ can be a positive odd number in any one of the three possible forms: either (1) a $1 \text{MOD} 6$ number of the form $(6m-5)$; xor (2) a $5 \text{MOD} 6$ number of the form $(6m-1)$; xor (3) a $3 \text{MOD} 6$ number of the form $(6m-3)$. $BEL(6m-5)$ contains the output of $U^\#$ at a higher rung $(6m-5)2^W$ with w being a positive even exponent; the input of $U^\#$ being given by $\lfloor \{(6m-5).2^W - 1\} / 3 \rfloor$. $BEL(6m-1)$ contains the output of $U^\#$ at a higher rung $(6m-1)2^V$ with v being a positive odd exponent; the input of $U^\#$ being given by $\lfloor \{(6m-1).2^V - 1\} / 3 \rfloor$. However, $BEL(6m-3)$ cannot contain any such output of the U-ceiling operator $U^\#$ irrespective of any input argument.

8. Immediate Neighborhood of a Binary-Exponential-Ladder

The relation between a pair of binary-exponential-ladders $BEL(m)$ and $BEL(n)$ can be considered to be defined & characterized by the relation between the corresponding pair of the *defining-base-rung* $D^\#$ values m and n along with the corresponding pair $U^\#(m)$ and $U^\#(n)$.

The immediate-neighborhood of a given binary-exponential-ladder $BEL(D^\#)$ is defined by the *immediate-predecessors* and *immediate-successors*, considering the U -ceiling operator $U^\#$; since the D -floor operator $D^\#$ is applicable only within a given binary-exponential-ladder and not between a pair of them.

8.1 Single Unique Immediate-Successor – CTUHSK function

It turns out that the only *one single unique immediate-successor* of $BEL(m)$ is $BEL(D^\#(U^\#(m)))$ that contains $U^\#(m)$ as one of its higher rungs; its identifying characteristic D -floor number associated with its defining-base-rung being given by $n := D^\#(U^\#(m))$. That is,

$$S(BEL(m)) = BEL(D^\#(U^\#(m))) := BEL(n); \text{ with } n := D^\#(U^\#(m)); \quad [\text{Eqn.5}]$$

Note that [Eqn.5] describes the forward tracing movement (transfer-function) from one BEL to its immediate-successor BEL defined by the very same Compact Collatz Function [Eqn.2] above; wherein the downward movement ($D^\#$ operator)

from a higher rung to the defining-base-rung within the same BEL is compressed and left as implicit; whereas the upward movement from the defining-base-rung of one BEL to a higher rung of its immediate-successor BEL is kept explicit; thus redefining the compact Collatz sequence by its equivalent, a *condensed compact Collatz* (CTUHSK) *sequence*; which is simply a hop from one BEL to its immediate successor BEL. Here, [Eqn.5] can therefore be considered as the condensed compact Collatz (CTUHSK) function; representing one hop from a positive odd number to its immediate-successor odd number; thus defining the condensed compact Collatz (CTUHSK) sequence, that is equivalent to skipping the intermediate even numbers in the standard Collatz sequence. Our study will focus on the characteristics of this condensed compact Collatz (CTUHSK) sequence defined by the condensed compact Collatz (CTUHSK) function; or equivalently the characteristics of BELnet, that is, the network of binary-exponential-ladders.

8.2 Multiple Immediate Predecessors – inverse CTUHSK function

There exists a *set of immediate-predecessors* for each $BEL(D^\#)$ of the form $BEL(6m-5)$ and $BEL(6m-1)$ although none for $BEL(6m-3)$. Note that if $S(BEL(m))$ is $BEL(n)$ than $BEL(m)$ is one of the predecessors of $BEL(n)$.

The *set of all immediate-predecessors* for a given $BEL(n)$ is defined by considering the *inverse of the immediate-successor relation*; as the set of all BELs, each of which having its single unique immediate-successor as $BEL(n)$.

$$\{P(BEL(n))\} := \{BEL(m) \mid BEL(n) = S(BEL(m))\}; \quad [Eqn.6]$$

$BEL(1MOD6)$ or equivalently $BEL(6m-5)$ has, as its set of immediate-predecessors, $\{BEL(|(1MOD6).2^W - 1|/3)\}$ or equivalently $\{BEL(|(6m-5).2^W - 1|/3)\}$ with w being a positive even exponent. Here, the input of $U^\#$ is given by $\{|(1MOD6).2^W - 1|/3\}$ or equivalently $\{|(6m-5).2^W - 1|/3\}$ and the output of $U^\#$ is $\{(1MOD6).2^W\}$ or equivalently $\{(6m-5).2^W\}$ that is contained in $BEL(1MOD6)$ or equivalently $BEL(6m-5)$. Each of the three possible classes of BEL , namely, $BEL(1MOD6)$ and $BEL(5MOD6)$ and $BEL(3MOD6)$ can be the immediate-predecessor of $BEL(1MOD6)$.

$$\{P(BEL(6m-5))\} = \{BEL(|(6m-5).2^W - 1|/3)\}; \quad [Eqn.7]$$

$BEL(5MOD6)$ or equivalently $BEL(6m-1)$ has, as its set of immediate-predecessors, $\{BEL(|(5MOD6).2^V - 1|/3)\}$ or equivalently $\{BEL(|(6m-1).2^V - 1|/3)\}$ with v being a positive odd exponent. Here, the input of $U^\#$ is given by $\{|(5MOD6).2^V - 1|/3\}$ or equivalently $\{|(6m-1).2^V - 1|/3\}$ and the output of $U^\#$ is $\{(5MOD6).2^V\}$ or equivalently $\{(6m-1).2^V\}$ that is contained in $BEL(5MOD6)$ or equivalently $BEL(6m-1)$. Each of the three possible classes of BEL , namely, $BEL(1MOD6)$ and $BEL(5MOD6)$ and $BEL(3MOD6)$ can be the immediate-predecessor of $BEL(5MOD6)$.

$$\{P(BEL(6m-1))\} = \{BEL(|(6m-1).2^V - 1|/3)\}; \quad [Eqn.8]$$

$BEL(3MOD6)$ or equivalently $BEL(6m-3)$ has no immediate-predecessors.

$$\{P(BEL(6m-3))\} = \phi; \quad [Eqn.9]$$

Note that [Eqn.6] or equivalently, [Eqn.7]&[Eqn.8]&[Eqn.9] together define the algebra of the condensed compact inverse CTUHSK odd-to-odd function.

The CTUHSK function in [Eqn.5] along with its inverse in [Eqn.6] or equivalently in [Eqn.7]&[Eqn.8]&[Eqn.9] together provide a complete mathematical model for the network of binary-exponential-ladders.

8.3 Quaternary-Exponential-Ladder

The above-mentioned property, that *only* the alternating rungs, defined by $\{(1 \bmod 6).4^u \mid u > 0\}$ of $BEL(1 \bmod 6)$ or $\{(5 \bmod 6).2.4^u \mid u \geq 0\}$ of $BEL(5 \bmod 6)$ are the ‘active’ higher rungs of BEL forming the nodes in the CTUHSK-Sequence; makes it convenient to define a system of *Quaternary-Exponential-Ladders* QEL wherein every rung of QEL becomes an active node in the CTUHSK-Sequence. This concept has not been directly used for proving the convergence of the Collatz Sequence, and therefore will not be taken up for further study in this paper.

8.4 $BEL(1)$ as the Central Focus

Considering $BEL(1)$ as our central focus of interest, which itself belongs to the type $BEL(1 \bmod 6)$ or equivalently $BEL(6m-5)$; it is interesting to note that it has its *single unique immediate-successor*; as $S(BEL(1)) = BEL(D^\#(U^\#(1))) = BEL(1)$; that is, $BEL(1)$ itself is its single unique immediate-successor, and that it has no

other immediate-successor distinct from itself; because of the very fact that the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ contained within $BEL(1)$.

8.5. BELnet : Network of Binary-Exponential-Ladders

The above discussion about the successor predecessor relations among the binary-exponential-ladders and its neighborhood leads to the observation that the *network of binary-exponential-ladders*, BELnet, has countably infinite number of each of the three classes/types of nodes: (1) $BEL(1 \text{MOD} 6)$ or equivalently $BEL(6m-5)$; (2) $BEL(5 \text{MOD} 6)$ or equivalently $BEL(6m-1)$; and (3) $BEL(3 \text{MOD} 6)$ or equivalently $BEL(6m-3)$. Each BEL being a node of the BELnet has a single unique outward directed arc that points towards its single unique immediate-successor, specifically linking onto some higher rung. Multiple (countably infinite number of) inward directed arcs, each linked onto some specific higher rung of a given BEL, emanate from each of its immediate-predecessors. $BEL(1)$ is an invariant-base-element or equivalently a sink-node in BELnet, the network of binary-exponential-ladders.

Thus, the set of binary-exponential-ladders is an exact representation of the Collatz-domain and the network of binary-exponential-ladders BELnet is an exact representation of the Collatz-map.

The *connectedness* of the network of binary-exponential-ladders BELnet will be analyzed from the *design of a structured system framework* consisting of the entire set of binary-exponential-ladders, merely as a re-organized *condensation* of the very same BELnet, as presented below.

9. Modular-Periodicity in BELnet

A study of the algebra (arithmetic) of the condensed compact inverse CTUHSK odd-to-odd function given by [Eqn.7]&[Eqn.8]&[Eqn.9] reveals that BELnet, the network of binary-exponential-ladders, exhibits modular-periodicity at three levels; that is, (1) the input odd number that is the defining-base-rung of a BEL in question and (2) the binary-exponent value that is associated with (3) the output odd number that is the defining-base-rung of the corresponding predecessor BEL.

To illustrate the modular-periodicity property, let us rewrite the condensed compact inverse CTUHSK odd-to-odd function as follows:

input odd number $x := 6a + x_6$;

with $0 \leq a$; $a_3 := a \pmod{3} \in \{1, 2, 0\}$; $x_3 := x \pmod{3} \in \{1, 2\}$ and $x_6 := x \pmod{6} \in \{1, 5\}$;

binary-exponent-value $k := 2j - (x_6 - x_3)/3$;

for the j^{th} jump; $1 \leq j$; $j_3 := j \pmod{3} \in \{1, 2, 0\}$; k is the binary-exponent-value;

$k := w := 2j$; for $x_6 := x \pmod{6} = 1$; OR $k := w := 2j - 1$; for $x_6 := x \pmod{6} = 5$;

output odd number $y := 6b + y_6$;

with $0 \leq b$; $b_3 := b \pmod{3} \in \{1, 2, 0\}$; $y_3 := y \pmod{3} \in \{1, 2, 0\}$ and $y_6 := y \pmod{6} \in \{1, 5, 3\}$;

Table-1 lists all the 54 distinctly different possible combinations of these three parameters. Table-2 lists the same parameters while restricting the observations only to the first non-base rung of the respective binary-exponential-ladder, which

demonstrates that the modular-periodicity property is spread across the three parameters.

j=9; w=18	(1,3)	(0,5)	(1,5)	(2,3)	(2,1)	(1,1)	j=9; v=17
j=8; w=16	(1,5)	(0,1)	(1,1)	(2,5)	(0,3)	(2,3)	j=8; v=15
j=7; w=14	(1,1)	(1,3)	(2,3)	(2,1)	(0,5)	(2,5)	j=7; v=13
j=6; w=12	(2,3)	(1,5)	(2,5)	(0,3)	(0,1)	(2,1)	j=6; v=11
j=5; w=10	(2,5)	(1,1)	(2,1)	(0,5)	(1,3)	(0,3)	j=5; v=9
j=4; w=8	(2,1)	(2,3)	(0,3)	(0,1)	(1,5)	(0,5)	j=4; v=7
j=3; w=6	(0,3)	(2,5)	(0,5)	(1,3)	(1,1)	(0,1)	j=3; v=5
j=2; w=4	(0,5)	(2,1)	(0,1)	(1,5)	(2,3)	(1,3)	j=2; v=3
j=1; w=2	(0,1)	(0,3)	(1,3)	(1,1)	(2,5)	(1,5)	j=1; v=1
$[b]_3, [x]_6 =$	(0,1)	(0,5)	(1,1)	(1,5)	(2,1)	(2,5)	$= [b]_3, [x]_6$
Table-1 : Modular-Periodicity							

j=1; w=2	(0,1) 1	3 (0,3)	(1,3) 9	7 (1,1)	(2,5) 17	11 (1,5)	j=1; v=1
$[b]_3, [x]_6 =$	(0,1)	(0,5)	(1,1)	(1,5)	(2,1)	(2,5)	$= [b]_3, [x]_6$
Table-2a : Modular-Periodicity							

j=1; w=2	(1,1) 25	15 (2,3)	(2,3) 33	19 (0,1)	(0,5) 41	23 (0,5)	j=1; v=1
$[b]_3, [x]_6 =$	(0,1)	(0,5)	(1,1)	(1,5)	(2,1)	(2,5)	$= [b]_3, [x]_6$
Table-2b : Modular-Periodicity							

$[b]_3, [x]_6 =$	(0,1)	(0,5)	(1,1)	(1,5)	(2,1)	(2,5)	$= [b]_3, [x]_6$
j=1; w=2	(2,1) 49	27 (1,3)	(0,3) 57	31 (2,1)	(1,5) 65	35 (2,5)	j=1; v=1
$[b]_3, [x]_6 =$	(0,1)	(0,5)	(1,1)	(1,5)	(2,1)	(2,5)	$= [b]_3, [x]_6$
Table-2c : Modular-Periodicity							

Observe that the condensed compact inverse CTUHSK odd-to-odd function covers all the relevant modular (modulo-3 / modulo-6) residue classes and also that the transitive closure of the subset $\{1, 5, 3\}$ under this condensed compact inverse CTUHSK odd-to-odd function is the entire set of all the positive odd numbers. This is a critical observation in establishing the convergence of the Collatz sequence to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

10. Structured System Framework H

From the above discussion we find that it is convenient for our study to consider a *Structured System Framework* H as an ordered infinite linear sequence of terms each of which being a set of BELs; that is, $H := \{ \dots H_{k-1} < H_k < H_{k+1} \dots \}$ wherein the *ordering relation* between the adjacent terms of the sequence is derived from the successor predecessor relations among the BELs that form the member elements of these adjacent terms in the sequence.

Specifically, H_k is defined as the set formed by the unique immediate-successor of each BEL belonging to H_{k+1} and also the set of immediate-predecessors of each BEL belonging to H_{k-1} ; as defined in [Eqn.5] & [Eqn.6]; that is,

$$H := \{ \dots H_{k-1} < H_k < H_{k+1} \dots \}; \quad [\text{Eqn.10}]$$

and

$$H_k := \{S(\text{BEL}(m)) \mid \text{BEL}(m) \in H_{k+1}\} \cup \{P(\text{BEL}(n)) \mid \text{BEL}(n) \in H_{k-1}\}; \quad [\text{Eqn.11}]$$

Note that the second part of [Eqn.11] here is required to ensure that BELs of the class/type BEL(3MOD6) can be included in each term H_k since each of them have immediate-successor in H_{k+1} although none of them have any predecessors in H_{k+1} .

Now, we may as well define the predecessor relation as the inverse of the above defined successor relation, as –

$$H_{k-1} := S(H_k) \quad \text{and} \quad H_k := S(H_{k+1}); \quad [\text{Eqn.12}]$$

and

$$P(H_{k-1}) := H_k \quad \text{and} \quad P(H_k) := H_{k+1}; \quad [\text{Eqn.13}]$$

The multiplicity of the *immediate-predecessor* relation among the BELs requires that the set of all immediate-predecessors of every element of H_{k-1} form the elements of the set H_k ; so as to guarantee that the structured system framework H is ordered infinite linear sequence; that is, $H_{k-1} < H_k < H_{k+1}$ among these sets; in spite of only a partial ordering relation among the BELs; and also to guarantee that the entire set of all the BELs are present in H thus making it as merely a *re-organized condensation* of BELnet.

10.1 Closed-Chains and Unbounded-Chains and Sink-Nodes in H

The design of the structured system framework H can *in general* allow for the existence of sink nodes (invariant-base-elements) and/or unbounded open chains and/or closed chains (loops). That is, the structured system framework H can in general be *partitioned into mutually disjoint and independent components*;

$$H := H^s \cup H^{\&} \cup H^\infty \quad [\text{Eqn.14}]$$

where (1) H^s corresponds to the set of all possible terms in H connected with sink nodes; (2) $H^{\&}$ corresponds to the set of all possible terms in H connected with unbounded open chains; and (3) H^∞ corresponds to the set of all possible terms in H connected with closed chains (loops). In such a situation, each of these components, H^s and $H^{\&}$ and H^∞ needs to satisfy the ordering conditions expressed above in [Eqn.10], [Eqn.11], [Eqn.12] & [Eqn.13].

10.2 A Sink-Node H_0 in H^s

We have observed earlier that $BEL(1)$ itself is its single unique immediate-successor and does not have any immediate-successor distinct from itself, although it has multiple immediate-predecessors. That is, $BEL(1)$ is an invariant-base-element or equivalently a sink-node in BELnet. Therefore, the component H^s must necessarily have a term H_0 as its *invariant-base-element* or equivalently a *sink-node*, that is, $H_0 := \{BEL(1)\}$; with $BEL(1)$ being its singleton member element. That is,

$$H^s := \{H_0, H_1, H_2, \dots\}; \quad [\text{Eqn.15}]$$

From the above discussion we observe that $H^s := \{H_0, H_1, H_2, \dots\}$ is, by its very design, a well-ordered infinite linear sequence of terms, with a distinct smallest/least element H_0 . Each term $H_k \in H^s$ is a countably infinite set of BELs with an exception that the invariant-base-element or the ‘root’ $H_0 := \{BEL(1)\}$ is

a singleton set. The set of k^{th} immediate predecessors of $BEL(1)$ form the set H_k at tier- k level in the hierarchy, if one wishes to consider it as a hierarchy.

Note that the entire Collatz Domain has been mapped to the structured system framework H through a two-stage bijective mapping.

11. Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani Theorem

STATEMENT OF THE CTUHSK THEOREM

The Collatz Sequence converges to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

PROOF

There are two parts to the proof – a *necessary condition* and a *sufficient condition*. The necessary condition is provided by establishing an order-isomorphism between H^s and the set of natural numbers, by the application of the Dedekind-Peano axioms; thus proving the convergence of the CTUHSK sequence, starting from any given positive integer contained in any BEL that is in H^s to the trivial-cycle at the base of $BEL(1)$ which itself is in $H_0 \in H^s$. The sufficient condition is provided by a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) that establishes *domain exhaustion*; having already captured all the modular residue-classes in H^s ; and guarantees the non-existence of any extraneous elements.

11.1 Proof of CTUHSK Theorem - Necessary Condition

We show that H^s satisfies the Dedekind-Peano's axioms (replacing the 'successor' by the 'predecessor') and therefore H^s is order-isomorphic with the set of natural numbers; and satisfies the necessary condition for the above convergence statement.

DEDEKIND-PEANO AXIOM : Existence of 1 as the invariant-base-element.

$H_0 \in H^s$. H_0 is the invariant-base-element of H^s .

The trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\} \in \text{BEL}(1)$ is contained in $H_0 \in H^s$.

DEDEKIND-PEANO AXIOM : Existence of a *successor function*.

By the very design of $H^s := \{H_0, H_1, H_2, \dots\}$, for every positive integer k ,

$H_k \in H^s$ is the *predecessor* of $H_{k+1} \in H^s$.

Application of the CTUHSK function with the input from numbers contained in some BEL that is a member of H_k yields the single unique output number contained in some immediate-successor BEL that is a member of H_{k+1} ; because of the definition of the successor predecessor relation between H_k and H_{k+1} .

DEDEKIND-PEANO AXIOM : 1 is not a successor; 1 has no predecessor; 1 is a source node in the sequence of natural numbers.

H_0 is not a predecessor to any other H_k . There *does not exist any* $H_k \in H^s$, $k \neq 0$; that is distinct from H_0 ; with $H_k \neq H_0$; such that H_0 is the predecessor of H_k .

H_0 does not have any successor distinct from itself.

H_0 is a sink-node in the sequence $H^s := \{H_0, H_1, H_2, \dots\}$.

Once the CTUHSK Sequence reaches the trivial-cycle/sink there is no exit from it.

DEDEKIND-PEANO AXIOM : Successor function is a unique one-to-one mapping.

If H_u is the predecessor of H_v and also H_u is the predecessor of H_w ;

then it necessarily implies $H_v = H_w$ by the very design of H^s ;

and, also,

If H_v is the predecessor of H_u and also H_w is the predecessor of H_u ;

then it necessarily implies $H_v = H_w$ by the very design of H^s ;

This is because the predecessor relation in H^s is a unique one-to-one mapping.

Also, note that for each positive integer k there corresponds a unique set $H_k \in H^s$,

and for each $H_k \in H^s$ there corresponds a unique positive integer k ; thus,

establishing a bijection between H^s and the set of positive integers.

DEDEKIND-PEANO AXIOM : Principle of induction.

Collatz sequence starting with any number from $BEL(1) \in H_0$ converges in the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\} \in BEL(1) \in H_0$.

The Collatz sequence starting with any positive integer that passes through H_k must necessarily pass through H_{k-1} because by design $H_{k-1} := S(H_k)$. Therefore, the CTUHSK sequence starting from any positive integer that is contained in some BEL in H^s passes through a linear directed path (chain) with no forking or merging in H^s (although merging is observed deeper at the level of the BELs); moving from H_{k+1} to H_k and from H_k to H_{k-1} and so on to H_0 in that order; and converging in the trivial-cycle.

Thus, we establish an *order-isomorphism* between H^s and the set of natural numbers; that provides the necessary condition as mentioned above.

11.2 Proof of CTUHSK Theorem - Sufficient Condition

The proof for the sufficient condition is provided by establishing the non-existence of any extraneous elements/objects like disjoint loops/cycles H^∞ ; and/or divergent chains $H^\&$ in the structured system framework H ; so that $H^s = H$.

First, the proof utilizes the fact that the Collatz domain being the set of natural numbers, is a well-ordered infinite linear sequence, with a distinct smallest (least) element, 1; containing all the three modular (modulo-3) residue-classes; and that any non-empty sub-domain must also be a well-ordered subset, having a distinct smallest (least) element. Same is the case with H^s as already established above in the first part of the proof as the necessary condition.

Secondly, the relative magnitudes of any two numbers in the Collatz sequence is governed by the total number nD of applications of the pull-down operator D as compared to the total number nU of applications of the push-up operator U that is required in order to move from the earlier node in the sequence to reach the later node in the sequence. Note that every application of D pulls the value downwards by a factor of 2 whereas every application of U pushes the value upwards by a factor of about three ($3+\delta$); the cutoff being around $nD/nU \cong (\log 3 / \log 2)$. Also, it is clear that every application of push-up operator U must necessarily be followed by an immediate application of at least one instance of the pull-down operator D .

We establish that as we move along the Collatz sequence, starting with any given number $n > 1$, passing through the successor nodes belonging to each of the three modulo-3 modular-residue-classes; eventually we must necessarily encounter some node associated with a number m that is less than that starting number; $m < n$.

The relative magnitudes of any two adjacent positive odd numbers in the condensed compact Collatz sequence, from odd number to odd number outside of the trivial-cycle; or equivalently from one BEL to the next BEL in BELnet, is always a strictly decreasing step (subsequence) whenever the Collatz sequence passes through: either

- (1) any non-base (first or higher) rung in the successor BEL of the form $(6m-5)$;
- xor
- (2) any non-base non-first higher rung in the successor BEL of the form $(6m-1)$.

The concern arises only in the exceptionally unique situations wherein the path traverses through the lowest non-base (first) rung of the successor BEL having its defining-base-rung being a positive odd number of the form $(6m-1)$; and the predecessor BEL has its defining-base-rung being a positive odd number of the form $(4m-1)$; leading to the case where $(6m-1) > (4m-1)$; that is, a strictly increasing step (subsequence). However, this increase from $(4m-1)$ to $(6m-1)$ can get immediately compensated in the later steps in the condensed compact Collatz sequence; observing that the very next node is $(9m-1)$ which may be either odd xor even; requiring further deeper study as presented below. This exceptionally unique worst-case situation, represented by the triad $\{(9m-1) \Leftarrow (6m-1) \Leftarrow (4m-1)\}$; corresponds to a subsequence of the form: $(U^1 D^1) U^1 D^p$ with $1 \leq p$.

Note that in the worst case, starting with a positive odd number that is of the form $((2m-1) \cdot 2^k - 1)$ passes through a subsequence of k (UD) operations, resulting in a

positive odd number that is of the form $((2m-1).3^k - 1)$ which is an even number that may be subject to at least a (D) operation or may even be a sequence of (D) operations.

However, let us now apply the *reductio-ad-absurdum* argument.

11.2.1 Non-existence of Disjoint Loops/Cycles H^∞ in H

The non-existence of any loop other than the trivial cycle $\{(4 \rightarrow 2 \rightarrow 1)\}$ in BELnet can be proved here by a *reductio-ad-absurdum* argument. If indeed there exists such a loop, it must necessarily be a sink loop with no exit, because of the characteristic property that there exists only a single unique immediate-successor for every BEL in BELnet. Also, the non-existence of any immediate-predecessor for BEL(6m-3) implies that BEL(6m-3) cannot be in any such loop. Being a sink loop consisting of only BEL(6m-5) and BEL(6m-1); each with multiple immediate-predecessors; implies that such a loop can be collapsed into an element similar to H_0 as in H^s and a sequence of predecessors can be developed as in [Eqn.15]; which implies that there is a parallel system that is equivalent to H^s that is also isomorphic to the set of natural numbers. This constitutes a contradiction, since we have already established the order-isomorphism between H^s and the set of natural numbers; and that H^s contains all the three types of BELs namely, BEL(6m-1) and BEL(6m-5) and also BEL(6m-3).

Therefore, we conclude the non-existence of any such loops in BELnet whether finite or infinite, and hence the non-existence of H^∞ in H .

We can prove the non-existence of $H^{\&}$ that is, infinite threads, because we have already exhausted the Collatz-domain through bijection with H^s and therefore there doesn't exist any natural number to map with $H^{\&}$.

11.3 Non-Existence of Non-standard Rogue Elements in H

Note that the Collatz-domain does not allow for the existence of any non-standard *rogue elements* etc. The study of non-standard extraneous objects (of whatever kind) is immaterial to the present proof, because the Collatz $(3x+1)$ Conjecture is stated strictly within the domain of natural numbers.

Having established that $H^{\infty} = \phi$ and $H^{\&} = \phi$ in [Eqn.14]; we get $H = H^s$; therefore, asserting the Collatz Conjecture, that starting from any positive integer the Collatz $(3x+1)$ Sequence converges to the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

END OF PROOF

12. Some Explicit Forms for the BEL Neighborhood

We can perform some simple algebraic manipulation to get the parametric relation [Eqn.16] that gives a generic form for the set H_k that is the set of k^{th} predecessors of $H_0 = \{\text{BEL}(1)\}$; that is, the set H_k corresponds to the set of tier-k level of the hierarchy with the set of Binary-Exponential-Ladders $\{\text{BEL}(m)\}$ each with its defining-base-rung m being a positive odd number $m > 1$.

$$m = [2^z - \{3^0.2^{z_0} + 3^1.2^{z_1} + 3^2.2^{z_2} + \dots + 3^{k-1}.2^{z_{k-1}}\}] / 3^k; \quad [\text{Eqn.16}]$$

wherein $k > 0$ is the tier-level and $z > z_0$ whereas the k -tuple $(z_0, z_1, z_2, \dots, z_{k-1})$ form the set of strictly decreasing non-negative integer exponents values in [Eqn.16] each of which takes a unique value corresponding to each positive odd number $m > 1$. That is, each positive odd number $m > 1$ can be considered to be defined by the corresponding unique set of these parameters. The set of values for the k -tuple $(z_0, z_1, z_2, \dots, z_{k-1})$ are of strictly decreasing non-negative integer exponent values, all less than z , that is, $z > z_0 > z_1 > z_2 > \dots > z_{k-1}$; ($z_k := 0$; $z_{k-1} = 0$ for positive odd number $m > 1$).

Now, define $p_0 := (z - z_0)$; $p_j := (z_{j-1} - z_j)$; where p_j corresponds to the number of rungs in $\text{BEL}\{H_j\}$ above the *defining-base-rung* of $\text{BEL}\{H_j\}$ for the node located in $\text{BEL}\{H_j\}$ that the Collatz sequence/trajectory passes through; $\text{BEL}\{H_j\}$ being the *Binary-Exponential-Ladder* at tier- j with $j=0,1,2, \dots, k$. Thus, we may as well redefine the set of $(k+1)$ parameters as a tuple $(\text{GPT}) := (p_0, p_1, p_2, \dots, p_k)$ the set of $(k+1)$ CTUHSK generative parameters that generate each positive integer n as per the parametric relation [Eqn.16] given above ($p_k = 0$ for positive odd number m).

For any $k > 0$, the above set of exponents $z, z_0, z_1, z_2, z_3, \dots, z_k$, can be redefined in terms of the newly defined CTUHSK generative parameters, by rewriting the above definition as $z := (z_0 + p_0)$; $z_{j-1} := (z_j + p_j)$; $z_k := 0$; $p_k = 0$ for any positive odd number m .

Table-3 gives some of the possible set of valid values for the CTUHSK generative parameters and therefore the corresponding valid values of the exponents in [Eqn.16] above along with their resultant $n(\text{GPT}) := n(p_0, p_1, p_2, \dots, p_k)$ values.

Table-3 : Some typical CTUHSK generative parameter tuples GPT(n)																				
p0	p1	p2	p3	p4	p5	p6	p7	p8	n	z	z0	z1	z2	z3	z4	z5	z6	z6	z7	
0									1	2	0									
1									2											
2									4											
3									8											
4									16											
4	0								5	4	0	0								
4	1	0							3	5	1	0	0							
4	3	0							13	7	3	0	0							
4	3	2	0						17	9	5	2	0	0						
4	3	2	1	0					11	10	6	3	1	0	0					
4	3	2	1	1	0				7	11	7	4	2	1	0	0				
4	3	2	1	1	2	0			9	13	9	6	4	3	2	0	0			
6	0								21	6	0	0								
4	3	2	1	3	0				29	13	9	6	4	3	0	0				
4	3	2	1	3	1	0			19	14	10	7	5	4	1	0	0			
4	3	2	1	3	1	2	0		25	16	12	9	7	6	3	2	0	0		
4	3	2	1	3	1	2	2	0	33	18	14	11	9	8	5	4	2	0	0	
p0	p1	p2	p3	p4	p5	p6	p7	p8	n	z	z0	z1	z2	z3	z4	z5	z6	z6	z7	
Table-3 : Some typical CTUHSK generative parameter tuples GPT(n)																				

13. A Challenge to the Cool-Headed Brave-Hearts

If you can prove that corresponding to every positive odd number $m > 1$ there exists a unique valid set of CTUHSK generative parameters $\text{GPT}(m) = \{p_0, p_1, p_2, \dots, p_k\}$ and therefore the corresponding valid set of exponents $\{z, z_0, z_1, z_2, z_3, \dots, z_k\}$.

$_1, z_k\}$ in the parametric equation [Eqn.16] given above that generates every positive odd number $m>1$, then you can directly prove the Collatz Conjecture establishing the convergence of the Collatz Sequence to the trivial-cycle $\{(1\Leftarrow 2\Leftarrow 4)\}$.

13.1 Restrictions on the CTUHSK Generative Parameters

Note that the set of valid values for the CTUHSK generative parameters and therefore for the exponents in [Eqn.16] above, are governed by certain rules as can be seen from the earlier observations, regarding the matching relation between the $((D^\#)\text{MOD}3)$ of the predecessor and the $((U^\#)\text{MOD}3)$ of the successor in the Collatz Sequence.

Specifically, [Eqn.17] states the relation satisfied among –

(i) the $x(\text{MOD}3)$ value of the exponent x for $U^\# = \{(6m-5).4^x\}$ at some higher rung in QEL(6m-5);

with (ii) with its defining-base-rung at (6m-5);

and (iii) its predecessor $D^\# = [\{(6m-5).4^x - 1\}/3]$.

Similarly, [Eqn.18] states the relation satisfied among –

(i) the $y(\text{MOD}3)$ value of the exponent y for $U^\# = \{(6m-1).2.4^y\}$ at some higher rung in QEL(6m-1);

with (ii) with its defining-base-rung at (6m-1);

and (iii) its predecessor $D^\# = [\{(6m-1).2.4^y - 1\}/3]$.

$$[\{(6m-5).4^x - 1\}/3]\text{MOD}3 = [x(\text{MOD}3) - m(\text{MOD}3) + 1]\text{MOD}3; \quad [\text{Eqn.17}]$$

and

$$[\{(6m-1).2.4^y - 1\}/3] \text{MOD} 3 = [y(\text{MOD} 3) + m(\text{MOD} 3) - 1] \text{MOD} 3; \quad [\text{Eqn.19}]$$

Rewriting [Eqn.17]&[Eqn.18] for the Binary-Exponential-Ladders, we get the equivalent set of equations as:

$$[\{(6m-5).2^w - 1\}/3] \text{MOD} 3 = [[w/2] \text{MOD} 3 - m(\text{MOD} 3) + 1] \text{MOD} 3; \quad [\text{Eqn.19}]$$

and

$$[\{(6m-1).2^v - 1\}/3] \text{MOD} 3 = [(v-1)/2] \text{MOD} 3 + m(\text{MOD} 3) - 1] \text{MOD} 3; \quad [\text{Eqn.20}]$$

13.2 Most Intriguing and Reassuring Observation

Suppose we move forward from H_{k-1} to H_k and then onto H_{k+1} ; that is, trace backwards along the Collatz Sequence, in the reverse direction; moving from $BEL(n_{k-1})$ to $BEL(n_k)$ and then onto $BEL(n_{k+1})$; with $BEL(n_{k-1}) = S(BEL(n_k))$ and $BEL(n_k) = S(BEL(n_{k+1}))$; every time selecting the immediate-predecessor BEL of the *lowest denomination* with the *lowest value of the defining-base-rung*, that is, selecting $BEL(n_{k-1}) \in H_{k-1}$; as represented by the lowest value of 2 for w in [Eqn.7] or the lowest value of 1 for v in [Eqn.8] above.

We find that [Eqn.7] with $w=2$ leads to a situation wherein $n_{k-1} < n_k$; that is,

$$BEL(6m-5) = S(BEL([(6m-5).2^2 - 1]/3)) = S(BEL(8m-7)); \quad [\text{Eqn.21}]$$

whereas [Eqn.8] with $v=1$ leads to a situation wherein $n_{k-1} > n_k$; that is,

$$\text{BEL}(6m-1) = S(\text{BEL}([(6m-1).2^1 - 1]/3)) = S(\text{BEL}(4m-1)); \quad [\text{Eqn.22}]$$

Thus, we find that moving forward from H_{k-1} to H_k and then onto H_{k+1} ; every time selecting the immediate-predecessor BEL of the lowest denomination, we encounter the three distinct situations:– (1) $\text{BEL}(6m-3) \in H_{k-1}$ being a leaf node of the BELnet, does not have any predecessors; (2) $\text{BEL}(6m-5) \in H_{k-1}$ has its immediate-predecessor BEL of lowest denomination being $\text{BEL}(8m-7) \in H_k$; (3) $\text{BEL}(6m-1) \in H_{k-1}$ has its immediate-predecessor BEL of lowest denomination being $\text{BEL}(4m-1) \in H_k$.

This establishes the fact that apart from the one-third of the BELs that lead to the leaf-nodes, the remaining two-thirds of the BELs are equally distributed between the two cases - $\text{BEL}(6m-5)$ leads to the situation wherein the arborescence grows forward with increasing values for the defining-base-rung; whereas $\text{BEL}(6m-1)$ leads to the situation wherein the arborescence grows forward with decreasing values for the defining-base-rung, thus *reaching-down* to some positive odd number of the form $(4m-1)$ that was *left out in the earlier stages/tier-levels* of the hierarchy.

Note that there can be only three types of positive odd numbers corresponding to the three modular (nMOD3) residue-classes; odd multiples of three that are of the form $(6m-3)$ or numbers of the form $(6m-5)$ or numbers of the form $(6m-1)$; of which the odd multiples of three correspond to the defining-base-rung of the leaf nodes in the BELnet arborescence. Of the remaining two cases, it is of utmost significance that every positive odd number of the form $(6m-1)$ forming the defining-base-rung of $\text{BEL}(6m-1) \in H_{k-1}$ having its immediate-predecessor BEL of lowest denomination that is of the form $\text{BEL}(4m-1) \in H_k$. This is how as we move from H_{k-1} to H_k and then onto H_{k+1} ; we encounter some positive odd numbers of

the form $(4m-1)$ that *could not be reached in the earlier stages/tier-levels*; as indicated by the presence of 1 in the CTUHSK generative parameter tuple.

The CTUHSK generative parameter tuple corresponding to every positive odd integer of the form $(4m-1)$ must necessarily end with the subsequence $(\dots, 1, 0)$; as for example, $\text{GPT}(11) = (4, 3, 2, 1, 0)$; whereas the presence of a positive odd integer of the form $(4m-1)$ anywhere in the Collatz Sequence is indicated by the presence of a 1 in the corresponding position in its CTUHSK generative parameter tuple; as for example, $\text{GPT}(9) = (4, 3, 2, 1, 1, 2, 0)$; with the Collatz Sequence for 9 is $9 \rightarrow 7 \rightarrow 11 \rightarrow 17 \rightarrow 13 \rightarrow 5 \rightarrow 1$; the two repeated occurrence of 1s corresponding to the subsequence $7 \rightarrow 11 \rightarrow 17$ in the trajectory. It is indeed intriguing to observe that this specific process of reaching-down to such positive odd numbers of the form $(4m-1)$ can indeed be a repeated contiguous operation at times, as can be seen by the presence of repeated 1's in the CTUHSK generative parameter tuple.

This is a situation wherein the emergence of global system properties through persistent local component subsystem characteristics can be clearly demonstrated! Banking on this global property, the author claims that the length of the CTUHSK generative parameter tuple $|\text{GPT}(m)|$ required to generate a given positive odd number m is limited to be *no more than* m itself; whereas the triad $\{(31 \Leftarrow 41 \Leftarrow 27)\}$ with $|\text{GPT}(31)|=40$ and $|\text{GPT}(41)|=41$ and $|\text{GPT}(27)|=42$ is an exception. Also, the triad $\{(1 \Leftarrow 5 \Leftarrow 3)\}$ with $|\text{GPT}(1)|=1$ and $|\text{GPT}(5)|=2$ and $|\text{GPT}(3)|=3$ is another limiting case.

Halemane-Conjecture states that the maximum number of odd $(3x+1)$ operations required to reach the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ starting from any given positive

integer and moving along the Collatz sequence, is limited by that given number itself; with the triad $\{(31 \Leftarrow 41 \Leftarrow 27)\}$ as an exceptional limiting case.

We will delve deeper into such algorithmic & computational details in a sequel to this research work, wherein a systematic (algorithmic) procedure for building the BELnet using the CTUHSK generative parameter tuples, corresponding to a systematic (algorithmic) procedure to produce a dictionary (lexicographic) of the CTUHSK generative parameter tuples, is planned to be presented.

14. Self-Similar Structural Symmetry of BELnet

From the above discussion one can notice that BELnet forms an arborescence in H^s with a self-similar structural symmetry. $\{BEL(1)\}$ stands at the center, with its trivial-cycle at its defining-base-rung. At every tier-level k corresponding to the k^{th} term H_k in the sequence H^s , Binary-Exponential-Ladders of all the three modular (MOD3) residue-classes are present in BELnet, each being countably infinite and equal in number; each having its single unique immediate-successor in H_{k-1} . Each of the $\{BEL(6m-5)\}$ and each of the $\{BEL(6m-1)\}$ has its immediate-predecessors in H_{k+1} ; whereas of each the $\{BEL(6m-3)\}$ remain as leaf-nodes since they can't have any immediate-predecessors. Thus, one-third of the BELs remain as leaf nodes; the other two-thirds become intermediate nodes that propagate the arborescence structure unboundedly to infinity. The self-similar structural symmetry in the network of binary-exponential-ladders BELnet is a self-similar derived from the modular-periodicity w.r.t both the defining-base-rung value

among the neighboring BELs as well as the binary-exponent value within any given BEL, as defined by [Eqn.17] & [Eqn.18] or equivalently [Eqn.19] & [Eqn.20] above.

Note that each of the $\{\text{BEL}(6m-5)\}$ with $(6m-5)$ as its defining-base-rung, has its immediate-predecessor of lowest denomination being $\{\text{BEL}(8m-7)\}$ as defined in [Eqn.21] where $(8m-7) > (6m-5)$; facilitating the outward growth of the BELnet arborescence by pushing upward in the number-line. Also note that each of the $\{\text{BEL}(6m-1)\}$ with $(6m-1)$ as its defining-base-rung, has its immediate-predecessor of lowest denomination being $\{\text{BEL}(4m-1)\}$ as defined in [Eqn.22] where $(4m-1) < (6m-1)$; facilitating the inward growth of the BELnet arborescence by pulling downward in reaching-down the not-yet-reached in the number-line (positive odd numbers). Therefore, we observe that the specific instance of [Eqn.8] represented by [Eqn.22]; in the inverse of the CTUHSK function; acts as the precise mechanism that systematically fills the gaps in the number line (positive odd numbers) that otherwise gets left out by [Eqn.7] or even [Eqn.21]; while constructing the lexicographic dictionary of CTUHSK generative parameter tuples for growing the BELnet arborescence.

This self-similar structural symmetry with the modular-periodicity both with respect to the defining-base-rung value and the binary-exponent value, along with the above mentioned unique feature of reaching-down the not-yet-reached numbers in the number-line, and using all the three modular ($n\text{MOD}3$) residue-classes among the set of positive odd numbers as the defining-base-rung for the BELs, together guarantee the connectedness of the network of binary-exponential-ladders BELnet; which has of course been established above independently; also showing the non-existence of any disjoint loops and/or divergent chains.

15. Explorations on Some Simpler Systems

In order to gain mastery over the Collatz system, it is proposed to understand some of the related systems of somewhat simpler algebraic/arithmetic/numerical characteristics. Here, we present four such systems.

15.1 Halemane's $(1.x+1)/2$ System (even-to-odd, odd-to-even) Form

Start with any given positive integer x ;

If $x \equiv 0 \pmod{2}$ then, divide by 2 repeatedly, until the resultant $x \equiv 1 \pmod{2}$;

ElseIf $x \equiv 1 \pmod{2}$ then, add 1 to get a resultant $x \equiv 0 \pmod{2}$;

Now continue the process.

if $x \equiv 0 \pmod{2}$ then, $T(x) := [x]D^\# := [x]D^* := [x]D^p = (x/2^p)$;

elseif $x \equiv 1 \pmod{2}$ then, $T(x) := (1.x+1)$; so that the resultant is even.

Inverse of $(1.x+1)/2$ System - Condensed Compact (odd-to-odd) Form

if $y \equiv 1 \pmod{2}$ then $\{P(y)\} := \{P_1(y, u) := (y \cdot 2^u - 1)\}; 1 \leq u$.

15.2 Halemane's $(2.x \pm 1)/3$ System

Start with any given positive integer x ;

If $x \equiv 0 \pmod{3}$ then, divide by 3 repeatedly until the resultant $x \not\equiv 0 \pmod{3}$;

ElseIf $x \equiv 1 \pmod{3}$ then, multiply by 2 and add 1 to get a resultant $x \equiv 0 \pmod{3}$;

ElseIf $x \equiv 2 \pmod{3}$ then, multiply by 2 and subtract 1 to get a resultant $x \equiv 0 \pmod{3}$;
Now continue the process.

Inverse of Halemane's $(2.x \pm 1)/3$ System

if $\{(y \equiv 1 \pmod{6}) \vee (y \equiv 5 \pmod{6})\}$ then $\{x\} := \{P(y)\} := (y \cdot 3^u \pm 1)/2$; with $1 \leq u$.

15.3 Halemane's $(3.x \pm 1)/4$ System

Start with any given positive integer $x > 1$; if $(x=1)$ STOP;

elseif $x \equiv 0 \pmod{4}$ then, divide by 4 repeatedly until the resultant $x \not\equiv 0 \pmod{4}$;

ElseIf $x \equiv 2 \pmod{4}$ then, divide by 2 to get a resultant $x \not\equiv 2 \pmod{4}$;

ElseIf $\{x \equiv 1 \pmod{4}\}$ then, multiply by 3 and add 1 to get a resultant $\{x \equiv 0 \pmod{4}\}$;

ElseIf $\{x \equiv 3 \pmod{4}\}$ then, multiply by 3 and subtract 1 to get a resultant $\{x \equiv 0 \pmod{4}\}$; Now continue the process.

	$[0]_4$	$[1]_4$	$[2]_4$	$[3]_4$
$[0]_4$	0	1	1	1
$[1]_4$	1	0	0	0
$[2]_4$	0	1	0	1
$[3]_4$	1	0	0	0

Halemane's $(3.x \pm 1)/4$ Function

if $(x=1)$ STOP;

elseif $x \equiv 0 \pmod{4}$ then, $S(x) := (x/4)$; elseif $x \equiv 2 \pmod{4}$ then, $S(x) := (x/2)$;

elseif $\{x \equiv 1 \pmod{4}\}$ then, $S(x) := (3.x+1)$;

elseif $\{x \equiv 3 \pmod{4}\}$ then, $S(x) := (3.x-1)$;

15.4 Halemane's $(3.x+1)/4$ System as a clone of the Collatz System

Start with any given positive integer $x > 1$; if $(x=1)$ STOP;

elseif $x \equiv 0 \pmod{4}$ then, divide by 4 repeatedly until the resultant $x \not\equiv 0 \pmod{4}$;

ElseIf $x \equiv 2 \pmod{4}$ then, multiply by 2 to get a resultant $x \equiv 0 \pmod{4}$;

ElseIf $\{x \equiv 1 \pmod{4}\} \vee \{x \equiv 3 \pmod{4}\}$ then, multiply by 3 and add 1 to get a resultant $\{x \equiv 0 \pmod{4}\} \vee \{x \equiv 2 \pmod{4}\}$; Now continue the process.

	$[0]_4$	$[1]_4$	$[2]_4$	$[3]_4$
$[0]_4$	0	1	1	1
$[1]_4$	1	0	0	0
$[2]_4$	1	0	0	0
$[3]_4$	0	0	1	0

Halemane's $(3.x+1)/4$ Function

if $(x=1)$ STOP;

elseif $x \equiv 0 \pmod{4}$ then, $S(x) := (x/4)$; elseif $x \equiv 2 \pmod{4}$ then, $S(x) := (2x)$;

elseif $\{x \equiv 1 \pmod{4}\} \vee \{x \equiv 3 \pmod{4}\}$ then, $S(x) := (3.x+1)$;

15.5 Halemane's $(3.x+1)/4$ System - modified

Start with any given positive integer $x > 1$; if $(x=1)$ STOP;

elseif $x \equiv 0 \pmod{4}$ then, divide by 4 repeatedly until the resultant $x \not\equiv 0 \pmod{4}$;

ElseIf $x \equiv 2 \pmod{4}$ then, divide by 2 to get a resultant $x \not\equiv 2 \pmod{4}$;

ElseIf $\{x \equiv 1(\text{mod } 4)\} \vee \{x \equiv 3(\text{mod } 4)\}$ then, multiply by 3 and add 1 to get a resultant $\{x \equiv 0(\text{mod } 4)\} \vee \{x \equiv 2(\text{mod } 4)\}$; Now continue the process.

	$[0]_4$	$[1]_4$	$[2]_4$	$[3]_4$
$[0]_4$	0	1	1	1
$[1]_4$	1	0	0	0
$[2]_4$	0	1	0	1
$[3]_4$	0	0	1	0

Halemane's $(3.x+1)/4$ Function

if $(x=1)$ STOP;

elseif $x \equiv 0(\text{mod } 4)$ then, $S(x) := (x/4)$; elseif $x \equiv 2(\text{mod } 4)$ then, $S(x) := (x/2)$;

elseif $\{x \equiv 1(\text{mod } 4)\} \vee \{x \equiv 3(\text{mod } 4)\}$ then, $S(x) := (3.x+1)$;

16. Summary

We established a two stage bijective mapping from the Collatz-domain to a meticulously designed *structured system framework* H with a careful reorganization & condensation :- in the first stage, a bijective mapping from the Collatz-domain to BELnet, that is the network of Binary-Exponential-Ladders defined on the set of positive odd numbers; and then in the second stage, a bijective mapping from BELnet to the structured system framework H. Then we established the *order-isomorphism* between the relevant component H^s (that is shown to be a well-ordered infinite linear sequence; or rather a hierarchy; or indeed an arborescence of the *Binary-Exponential-Ladders* having its root; with the invariant-base-element

H_0 of the *structured system framework* H) and the set of positive integers; providing the necessary condition. A sufficient condition is provided through a reductio-ad-absurdum argument (along with an exceptionally unique modular arithmetic characteristic property of the Collatz system) that establishes *domain exhaustion*; having already captured all the modular (MOD3) residue-classes in H^s ; and guarantees the non-existence of any extraneous objects like disjoint loops/cycles and/or divergent chains. Since the Collatz $(3x+1)$ Conjecture is restricted to the set of natural numbers, the existence of any non-standard ‘rogue’ and/or ‘ghost’ elements are of no concern here.

The entire dynamics of the Collatz $(3x+1)$ system has been captured in a dynamically evolving graded algebraic structure of ideal based filtration scheme with divisibility classes for modulo-3 quotient-semiring generated by the primitive-root 2; along with a filtration shifting global affine transformation for a Euclidean expansion shift to the topmost coprime layer of that filtration scheme, avoiding the modulo-multiple-layer (zero layer) and also all the nilpotent layers with nilpotent elements (dead-end zero-divisors) like $(6m-3)$ and $(6m-0)$. The trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ is bypassed by an initialization phase for the filtration scheme, starting directly with the modul-9 coprime layer. The transitive closure of the Collatz $(3x+1)$ system is exactly represented by the transitive closure of this dynamically evolving algebraic system which is indeed the complete set of natural numbers. This design for the system-dynamics-framework by itself provides a definitive-proof of the Collatz $(3x+1)$ conjecture, establishing that the entire Collatz map is exactly represented by this dynamically-evolving graded-algebraic-structure defined on the set of all natural numbers; neither missing any natural-number nor including any extraneous-elements; providing both the necessary-&-sufficient condition simultaneously in one single sweep.

As a direction towards future research work, we have also presented a possible alternative approach in proving this convergence, using modular arithmetic for the conditions to be satisfied by the CTUHSK generative parameters or equivalently the exponents in a closed-form expression for generating the reverse Collatz sequence corresponding to any given positive odd number.

It is observed that the consistent & persistent application of the modular arithmetic characteristics associated with the local neighborhood of BELnet imposed by the CTUHSK function gives rise to the emergence of the global system property associated with the structured system framework that forms an arborescence (hierarchy) of BELnet. This leads to an upper limit on the depth in the arborescence or the rank/level in the hierarchy that one needs to analyse in order to reach for the existence of a binary-exponential-ladder with the desired positive odd number as its defining-base-rung in BELnet.

Halemane-Conjecture states that the maximum number of odd $(3x+1)$ operations required to reach the trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ starting from any given positive integer and moving along the Collatz sequence, is limited by that given number itself; with the triad $\{(31 \Leftarrow 41 \Leftarrow 27)\}$ as an exceptional limiting case.

Finally, in order to gain mastery over the Collatz $(3x+1)$ system, it is proposed to understand some of the related systems of somewhat simpler algebraic/arithmetic/numerical characteristics. We present four such systems -

- (1) Halemane's $(1.x+1)/2$ System; (2) Halemane's $(2.x \pm 1)/3$ System;
- (3) Halemane's $(3.x \pm 1)/4$ System; and (4) Halemane's $(3.x+1)/4$ System which is a clone of the Collatz $(3x+1)$ System.

17. Recommended Reading

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18. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an Independent Researcher, and that I have not received any funding from any source other than my own savings, and I do not have any obligations or encumbrances of any kind, neither financial nor legal nor of any other kind, regarding the contents of the manuscript - of which I am the original author and creator. Also, I hereby declare there is no conflict of interests.

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20. Dedication

To my ಅಜ್ಜ (ajja) Karinja Halemane Keshava Bhat & ಅಜ್ಜಿ (ajji) Thirumaleshwari, ಅಪ್ಪ (appa) Shama Bhat & ಅಮ್ಮ (amma) Thirumaleshwari, for their *teachings through love, that quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter [Sriwidya.Bharati](#) and my twin sons [Sriwidya.Ramana](#) & [Sriwidya.Prawina](#) for their *love & affection*.