# A Definitive Proof of the Collatz Conjecture using the Halemane(3X±1)/4System

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## Abstract

Halemane(3X±1)/4System is defined, similar to the Collatz (3x+1) System, with slightly modified operations. On similar lines, we define the CTUHSK(3x+1)/4System, and show that to be exactly the same as the Collatz (3x+1) System. The convergence of Halemane(3X±1)/4System to its trivial-cycle {(1⇐(2)⇐4)} is established. A reversible (invertible) transformation - based on a judiciously designed pruning-&-joining operations on the system arborescence - between Halemane(3X±1)/4System and CTUHSK(3x+1)/4System is shown not to affect the binary(inward)arborescence structure as well as the convergence characteristics of either of the systems. This establishes the convergence of the CTUHSK(3x+1)/4System and therefore provides a definitive proof of the convergence of the Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Sequence; asserting the Collatz Conjecture.

This paper also presents Halemane-Conjecture associated with the Collatz sequence, that the upper bound on the number of odd (3x+1) operations required to reach the trivial-cycle {(1⇐2⇐4)} starting from any given positive integer and moving along the Collatz sequence, is the given number itself; the triad {(31⇐41⇐27)} being an exceptional limiting case.

Keywords: Halemane(3X±1)/4System; CTUHSK(3x+1)/4System;
Halemane's Conjecture on the Collatz (3x+1) System;
Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Sequence;
Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture;
binary(inward)arborescence;

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## 1. Introduction

The Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture, simply referred to as the *Collatz Conjecture* [§1] states that the Collatz (3x+1) Sequence, also referred to as the Collatz Sequence, converges to the trivial-cycle, that is, {(1⇐2⇐4)}, starting from any positive integer.

Lagarias [§2]&[§3] gives an exhaustive annotated bibliography, whereas Lagarias [§4]&[§5] gives an elaborate overview of the Collatz Problem, also referred to as the

"3x+1 problem", referring to it as "The Ultimate Challenge". Guy [§6] has been somewhat pessimistic in advising researchers "Don't Try to Solve These Problems". Lagarius [§4]&[§5] refers to the statement by the most revered Mathematician and Number Theory Expert Paul Erdos, who said that - "Mathematics is not yet ready for such problems" - about the Collatz Conjecture. The Collatz (3x+1) Sequence is indeed a highly complex chaotic system, although defined or generated with one of the simplest deterministic well-defined equations, that too, just two in number. We will not get into any more of the reported literature except the problem definition; but will simply focus on a rather shockingly simple approach toward a possible resolution of this amazingly notorious long-standing unresolved problem.

This paper presents Halemane's (3.x±1)/4 System that is defined on similar lines as the Collatz (3x+1) System with slightly modified operations. Also, on similar lines, we define the CTUHSK(3x+1)/4System and show it to be exactly the same as the Collatz (3x+1) System. The Halemane(3X±1)/4System is shown to converge to its trivial-cycle {(1⇐4)}. We also establish a reversible (invertible) transformation between Halemane(3X±1)/4System and CTUHSK(3x+1)/4System, without affecting the convergence characteristics of either of them. This establishes the convergence of the Collatz Sequence and provides a definitive proof of Collatz Conjecture.

## 2. Halemane(3X±1)/4System

Halemane(3X±1)/4System H is defined on the set of positive integers, through its immediate-successor function S(x) ≔ H(x) as follows:

For any given positive integer x>1;

if & when x=1 stop; (trivial-cycle detected);

elseIf $x \equiv 0 (\mathrm{mod} 4)$ then, $S(x) \coloneqq H(x) \coloneqq (x/4)$;

elseIf $x \equiv 2 (\mathrm{mod} 4)$ then, $S(x) \coloneqq H(x) \coloneqq (x/2)$;

elseIf $\{x \equiv 1 (\mathrm{mod} 4)\}\}$ then, $S(x) \coloneqq H(x) \coloneqq (3x+1)$;

elseIf $\{x \equiv 3 (\mathrm{mod} 4)\}\}$ then, $S(x) \coloneqq H(x) \coloneqq (3.x-1)$; [Eqn.1]

Table-1 gives the 4x4 state transition matrix H(x,y) for Halemane(3X±1)/4System. Note that $[c]_b$ represents the modular (modulo-b) residue class c that a given number belongs to.

| H(x,y) | $[0]_4$ | $[2]_4$ | $[1]_4$ | $[3]_4$ |
|---|---|---|---|---|
| $[0]_4$ | 0 | 1 | 1 | 1 |
| $[2]_4$ | 0 | 0 | 1 | 1 |
| $[1]_4$ | 1 | 0 | 0 | 0 |
| $[3]_4$ | 1 | 0 | 0 | 0 |

Table-1: Halemane(3X±1)/4System

Observe that as per the definition of Halemane(3X±1)/4System, for every positive integer x there is a single unique immediate-successor. Also note that as per the definition of Halemane(3X±1)/4System, for any given positive integer, there exists an *immediate-even-predecessor* that is a positive even number, as a default. Additionally, for any given positive even number $x \in \{[2]_6, [4]_6\}$ of the form $(12m \pm 4)$; there exists an *immediate-odd-predecessor* that is a positive odd number.

This characteristic property results in a binary(inward)arborescence structure with the trivial-cycle {(1⇐(2)⇐4)} as a root and every odd multiple of 3 being an odd-leaf node with all its predecessors being only positive even numbers.

Table-2 gives the expanded 12x12 state transition matrix H(x,y) for Halemane(3X±1)/4System, that explicitly captures and represents the above-mentioned characteristic property.

| H(x,y) | $[0]_{12}$ | $[4]_{12}$ | $[8]_{12}$ | $[2]_{12}$ | $[6]_{12}$ | $[10]_{12}$ | $[1]_{12}$ | $[5]_{12}$ | $[9]_{12}$ | $[3]_{12}$ | $[7]_{12}$ | $[11]_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $[0]_{12}$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[4]_{12}$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[8]_{12}$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[2]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[6]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[10]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[1]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[5]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[9]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[3]_{12}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[7]_{12}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[11]_{12}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table-2: Halemane(3X±1)/4System - 12x12 State Transition Matrix

Also note that a positive even number x ∈ $\{[0]_{12}\} \vee \{[2]_{12}\} \vee \{[6]_{12}\} \vee \{[10]_{12}\}$ cannot have any *immediate-odd-predecessor*, and such nodes are called inactive-nodes in the system.

## 3. Convergence of Halemane(3X±1)/4System

Note that every application of the (3.x±1) operation produces an immediate resultant positive even numbers $x \in \{[0]_4\}$; and therefore every (3.x±1) operation *must necessarily be followed* by *at least one* application of the (x/4) operation; and may even be followed by further repetitions of it; leading to an *immediate decrease* in the numerical value of the operand. Therefore, the resulting odd-to-odd sequence is a *monotonically strictly decreasing sequence*. This immediately establishes the convergence of Halemane(3X±1)/4System to its trivial-cycle $\{(1 \Leftarrow (2) \Leftarrow 4)\}$; starting with any positive integer.

## 4. Collatz (3x+1) System re-written as CTUHSK(3x+1)/4System

The Collatz(3x+1)/2System can be defined by its immediate-successor function S(x) for the (classical/canonical) Collatz (3x+1) system that can be written as –

> For any given positive integer $x>1$;
> if & when $x=1$ stop; (trivial-cycle detected);
> elseIf $x \equiv 0 \pmod 2$ then, $S(x) \coloneqq (x/2)$;
> elseIf $\{(x \equiv 1 \pmod 2)\}$ then, $S(x) \coloneqq (3x+1)$; [Eqn.2]

Observe that as per [Eqn.2]; for every positive integer x there exists a single unique immediate-successor.

Also note that the immediate-successor of any positive odd number of the form (4m+1) in the Collatz sequence is a positive even number of the form (12m+4); that is, a number $x \in \{[0]_4\}$ that is always divisible by 4 *at least once*, and therefore the combined operation can be denoted by a single ((3x+1)/4) operation; which leads to a clear decrease in the numerical value for the resultant odd number.

## 5. Critical-Issue, Critical-(Decision)-Node, Detour-Link, Detour-Node

The immediate-successor of any positive odd number of the form (4m-1) in the Collatz sequence is a positive even number of the form (12m-2); that is, a number $x \in \{[2]_4\}$ that is always divisible by 2 *exactly once* only. The combined operation can be denoted by a single ((3x+1)/2) operation, with its immediate resultant being a positive odd number of the form (6m-1); with an increase in the numerical value of the operand. We designate this specific step {(12m-2)⇐(4m-1)} in the Collatz sequence as a critical *detour-link*; because it is indeed the very source for the possible ambiguity (*critical-issue*) regarding the convergence of the Collatz sequence. The initial-node (4m-1) of a detour-link is its *critical-(decision)-node*; and its terminal-node (12m-2) is a *detour-node*. This *critical-issue* will be addressed appropriately in this paper.

As a first step, the Collatz (3x+1) system originally defined by [Eqn.2] can now be re-written as an equivalent CTUHSK(3x+1)/4System; with a special single application of a (x/2) operation in such specific cases whenever the operand is a positive even number of the form (12m-2) arising as the immediate resultant of the application of the (3x+1) operation on a positive odd number of the form (4m-1).

[Eqn.3] gives this redefined version C, renamed as the CTUHSK(3x+1)/4System.

Start with any given positive integer $x>1$;

if & when $x=1$ stop; else continue;

elseIf $x \equiv 0 \pmod 4$ then, $S(x) \coloneqq C(x) \coloneqq (x/4)$;

elseIf $x \equiv 2 \pmod 4$ then, $S(x) \coloneqq C(x) \coloneqq (x/2)$;

elseIf $\{(x \equiv 1 \pmod 4)) \vee ((x \equiv 3 \pmod 4))\}$ then, $S(x) \coloneqq C(x) \coloneqq (3x+1)$; [Eqn.3]

Table-3 gives the 4x4 state transition matrix C(x,y) for CTUHSK(3x+1)/4System.

| C(x,y) | $[0]_4$ | $[2]_4$ | $[1]_4$ | $[3]_4$ |
|---|---|---|---|---|
| $[0]_4$ | 0 | 1 | 1 | 1 |
| $[2]_4$ | 0 | 0 | 1 | 1 |
| $[1]_4$ | 1 | 0 | 0 | 0 |
| $[3]_4$ | 0 | 1 | 0 | 0 |

Table-3: CTUHSK(3x+1)/4System

Observe that as per the definition of Collatz (3x+1) System or its exact equivalent, the CTUHSK(3x+1)/4System, for every positive integer x there is a single unique immediate-even-predecessor, as a default. Additionally, for any given positive even number $x \in \{[4]_{12}\} \vee \{[10]_{12}\}$ of the form $\{(12m+4) \vee (12m\text{-}2)\}$ there exists an *immediate-odd-predecessor* that is a positive odd number. This characteristic property results in a binary(inward)arborescence structure with the trivial-cycle $\{(1 \Leftarrow (2) \Leftarrow 4)\}$ as a root and every odd multiple of 3 being an odd-leaf node.

Table-4 gives the expanded 12x12 state transition matrix C(x,y) for CTUHSK(3x+1)/4System, that explicitly captures and represents the above-mentioned characteristic property.

| C(x,y) | $[0]_{12}$ | $[4]_{12}$ | $[8]_{12}$ | $[2]_{12}$ | $[6]_{12}$ | $[10]_{12}$ | $[1]_{12}$ | $[5]_{12}$ | $[9]_{12}$ | $[3]_{12}$ | $[7]_{12}$ | $[11]_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $[0]_{12}$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[4]_{12}$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[8]_{12}$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[2]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[6]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[10]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[1]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[5]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[9]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[3]_{12}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[7]_{12}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[11]_{12}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Table-4: CTUHSK(3x+1)/4System 12x12 State Transition Matrix

Also note that a positive even number x ∈ $\{[0]_{12}\} \vee \{[2]_{12}\} \vee \{[6]_{12}\} \vee \{[8]_{12}\}$ cannot have any *immediate-odd-predecessor*, and such nodes are called inactive-nodes in the system.

Note that the highlighted entries in Table-3 and Table-4 indicate the differences in comparison with the corresponding entries in Halemane(3X±1)/4System presented in Table-1 and Table-2.

## 6. CTUHSK(3x+1)/4System compared with Halemane(3X±1)/4System

First, observe that the difference between Halemane(3X±1)/4System and CTUHSK(3x+1)/4System; as highlighted in the tabular representation of the system state transitions; corresponds to only the critical-(decision)-node (4m-1). The critical-(decision)-node (4m-1) is linked to the detour-node (12m-2) through the detour-link {(12m-2)⇐(4m-1)} in CTUHSK(3x+1)/4System. However, in Halemane(3X±1)/4System, the critical-(decision)-node (4m-1) is linked to a positive even number of the form (12m-4) through a link {(12m-4)⇐(4m-1)}. We designate the link {(12m-4)⇐(4m-1)} that is present in Halemane(3X±1)/4System as the *shortcut-link* and its terminal-node (12m-4) as the *shortcut-node*.

## 7. Transformation between Halemane(3X±1)/4System & CTUHSK(3x+1)/4System

Perform the following pruning-&-joining operation on the Halemane(3X±1)/4System, recursively/iteratively, to transform it into the CTUHSK(3x+1)/4System.

> Prune the link {(12m-4)⇐(4m-1)} from Halemane(3X±1)/4System arborescence H; thus disconnecting the shortcut-link between the critical-decision-node (4m-1) the shortcut-node (12m-4). This operation yields a free-hanging subtree, denoted by (4m-1)H+; corresponding to a branch of the original arborescence H, with the critical-decision-node (4m-1) as its root. Now, join this critical-decision-node (4m-1) onto the detour-node (12m-2) which is known to be an inactive-node in Halemane(3X±1)/4System. This join

operation creates a new link, the detour-link {(12m-2)⇐(4m-1)}. This detour-node (12m-2) is intended to act as the even-predecessor of a node (6m-1) in the prospective CTUHSK(3x+1)/4System being generated; and therefore, the entire subtree (4m-1)H+ that was pruned earlier now joins as a new branch that is now attached to the subtree that may be denoted by (6m-1)T+ which is now an intermediate transition system subtree consisting of the newly generated links belonging to system C along with the remaining links of system H.

The above-described pruning-&-joining operation can be performed recursively; to replace every link of the form {(12m-4)⇐(4m-1)} ∈ H by the corresponding newly created links {(12m-2)⇐(4m-1)} ∈ C so as to result in the newly created Halemane(3X±1)/4System. As an alternative, one can as well use an iterative algorithm to achieve this transformation. That is, replace every link of the form {(12m-4)⇐(4m-1)} with a link of the form {(12m-2)⇐(4m-1)} for m∈{1, 2, 3, etc.}; without necessarily considering any network-based recursive algorithm to process on the subtrees etc.; although a network-based algorithm may facilitate in a better conceptual understanding of the process involved in the pruning-&-joining operations therein.

A similar pruning-&-joining process can be described to transform CTUHSK(3x+1)/4System to Halemane(3X±1)/4System, that is, by replacing every detour-link of the form {(12m-2)⇐(4m-1)} by the corresponding newly created shortcut-link of the form {(12m-4)⇐(4m-1)}.

## 8. Proof of the Collatz Conjecture

We have already established the convergence of Halemane(3X±1)/4System to its trivial-cycle {(1⇐(2)⇐4)}; and that the system structure is a binary(inward)arborescence with this trivial-cycle as the root and every odd multiple of 3 being an odd-leaf node. The CTUHSK(3x+1)/4System is shown to be exactly identical to the Collatz (3x+1) System. Also, the above-described pruning-&-joining process provides a transformation mechanism between Halemane(3X±1)/4System and CTUHSK(3x+1)/4System.

Observe that neither the binary(inward)arborescence structure nor the convergence characteristics of - (1) either Halemane(3X±1)/4System H (2) or the CTUHSK(3x+1)/4System C (3) or any of the intermediate transition system T - get affected by the above-described pruning-&-joining process. This is because at every iteration, the pruned subtree is joined onto an already existing and otherwise hitherto inactive-node of the existing system or rather the intermediate transition system. Also, note that, every pruning-&-joining operation can be considered to be analogous to shifting from a shortcut sub-route to a detour sub-route or from a detour sub-route to a shortcut sub-route; that shifting decision itself being taken at every critical-(decision)-node.

Therefore, the convergence of Halemane(3X±1)/4System to its trivial-cycle {(1⇐(2)⇐4)} in turn proves the convergence of CTUHSK(3x+1)/4System to its trivial-cycle {(1⇐2⇐4)}.

## 9. Summary

Halemane(3X±1)/4System has been defined, similar to the Collatz (3x+1) System; with slightly modified operations. We have also defined the CTUHSK(3x+1)/4System, and shown that to be exactly the same as the Collatz (3x+1) System. The global convergence of Halemane(3X±1)/4System to its trivial-cycle {(1⇐(2)⇐4)} is established through the local monotonicity property of the defining function. A reversible (invertible) transformation between Halemane(3X±1)/4System and CTUHSK(3x+1)/4System has been designed, based on a process consisting of pruning-&-joining operations performed recursively/iteratively on the binary(inward)arborescence structure of the system. This pruning-&-joining operation does not affect the binary(inward)arborescence structure as well as the convergence characteristics of either of the systems, because the pruned subtree is joined onto an already existing inactive-node of the system arborescence. Therefore, this establishes the convergence of the Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Sequence and providing a definitive proof of Collatz Conjecture.

Halemane-Conjecture on the Collatz (3x+1) sequence states that the maximum limit on number of odd (3x+1) operations required to reach the trivial-cycle {(1⇐2⇐4)}; starting from any given positive integer and moving along the Collatz sequence; is the given number itself; with the triad {(31⇐41⇐27)} as an exceptional limiting case.

## 10. Recommended Reading

[§1]. Wikipedia Page – https://en.wikipedia.org/wiki/Collatz_conjecture

[§2]. Jeffrey C Lagarias;
"The 3x+1 problem: An annotated bibliography (1963--1999) (sorted by author)";
https://arxiv.org/abs/math/0309224

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## 11. Acknowledgement & Confession & Assertion

I must necessarily confess/acknowledge that the *core idea behind this proof of the Collatz Conjecture is so stunningly and elusively simple*, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, possibly with an after-thought: "OMG, *how could it be that it never flashed on me any time earlier*"! However, I must assert that “क्लिष्टविशयसरलीकरणमेव पाण्डित्यलक्षणम् न तु सरलविषयसंक्लिष्टीकरणम्” ||

## 12. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an Independent Researcher, and that I have not received any funding from any source other than my own savings, and I do not have any obligations or encumbrances of any kind, neither financial nor legal nor of any other kind, regarding the contents of the manuscript - of which I am the original author and creator. Also, I hereby declare there is no conflict of interests.

## 13. Author Rights Retention Notice

## 14. Dedication

To my **ಅಜ್ಜ**(ajja) Karinja Halemane Keshava Bhat & **ಅಜ್ಜಿ**(ajji) Thirumaleshwari, **ಅಪ್ಪ**(appa) Shama Bhat & **ಅಮ್ಮ**(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection.*