# System(3X±1)/4 Framework for the Collatz (3x+1) Problem

© Dr.(Prof.) Keshava Prasad Halemane,
Professor - retired (2017JAN31) from
Department of Mathematical And Computational Sciences,
National Institute of Technology Karnataka Surathkal,
Srinivasnagar, Mangaluru - 575025, India.
https://orcid.org/0000-0003-3483-3521

Independent Researcher @ SASHESHA
(no affiliations; no funding; no data used; no conflict of interests)
https://www.linkedin.com/in/keshavaprasadahalemane/
k.prasad.h@gmail.com

## Abstract

In order to gain deeper insight into the Collatz (3x+1) Problem, we design a framework based on System(3X±1)/4; defined similar to the Collatz (3x+1) system, with slightly modified operations. On similar lines, we define the CTUHSK(3x+1)/4 system, and show that to be exactly the same as the Collatz (3x+1) system. The convergence of System(3X±1)/4 to its trivial-cycle {(1⇐(2)⇐4)} is established using the persistent local characteristic property of 'odd-to-odd monotonicity' in its defining function. We also present a meticulously designed reversible (invertible) transformation (reduction); between System(3X±1)/4 and the CTUHSK(3x+1)/4 system; based on pruning-&-joining operations on the system arborescence; which does not affect the binary(inward)arborescence structure as well as the convergence characteristics of either of these systems. This establishes the convergence of the CTUHSK(3x+1)/4 system to its trivial-cycle {(1⇐2⇐4)}.

This paper also presents Halemane-Conjecture associated with the Collatz sequence, that the upper bound on the number of odd (3x+1) operations required to reach its trivial-cycle {(1⇐2⇐4)} starting from any given positive integer and moving along the Collatz sequence, is the given number itself; the triad {(31⇐41⇐27)} being an exceptional limiting case.

Keywords: System(3X±1)/4; CTUHSK(3x+1)/4 system;

Halemane's Conjecture on the Collatz (3x+1) system;

Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Sequence;

Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture;

binary(inward)arborescence;

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## 1. Introduction

The Collatz-Thwaites-Ulam-Hasse-Syracuse-Kakutani (CTUHSK) Conjecture, simply referred to as the *Collatz Conjecture* [§1] states that the Collatz (3x+1) Sequence, also referred to as the Collatz Sequence, converges to its trivial-cycle; that is, {(1⇐2⇐4)}, starting from any positive integer.

Lagarias [§2]&[§3] gives an exhaustive annotated bibliography, whereas Lagarias [§4]&[§5] gives an elaborate overview of the Collatz Problem, also referred to as the "3x+1 problem", referring to it as "The Ultimate Challenge". Guy [§6] has been somewhat pessimistic in advising researchers "Don't Try to Solve These Problems". Lagarius [§4]&[§5] refers to the statement by the most revered Mathematician and Number Theory Expert Paul Erdos, who said that - "Mathematics is not yet ready for such problems" - about the Collatz Conjecture. The Collatz (3x+1) Sequence is indeed a highly complex chaotic system, although defined or generated with one of the simplest deterministic well-defined equations, that too, just two in number. We will not get into any more of the reported literature except the problem definition; but will simply focus on a rather shockingly simple approach toward a possible resolution of this amazingly notorious long-standing unresolved problem.

This paper presents System(3.x±1)/4 that is defined on similar lines as the Collatz (3x+1) system with slightly modified operations. Also, on similar lines, the CTUHSK(3x+1)/4 system is defined and is shown to be exactly the same as the Collatz (3x+1) system. The System(3X±1)/4 is shown to converge to its trivial-cycle {(1⇐(2)⇐4)}; using the local 'odd-to-odd monotonicity' property of the defining function. A meticulously designed reversible (invertible) transformation (reduction); between System(3X±1)/4 and the CTUHSK(3x+1)/4 system; based on pruning-&-joining operations on the system arborescence; is shown not to affect the binary(inward)arborescence structure as well as the convergence characteristics; of either of these systems. It turns out that this establishes the convergence of the CTUHSK(3x+1)/4 system to its trivial-cycle {(1⇐2⇐4)}.

## 2. System(3X±1)/4

System(3X±1)/4 H is defined by its immediate-successor function S(x) for every positive integer x, as follows:

For any given positive integer $x > 1$;

if & when x=1 stop; trivial-cycle {(1⇐(2)⇐4)} detected; else continue;

elseIf $x \equiv 0 (\mathrm{mod} 4)$ then, $S(x) \coloneqq (x/4)$;

elseIf $x \equiv 2 (\mathrm{mod} 4)$ then, $S(x) \coloneqq (x/2)$;

elseIf $\{x \equiv 1 (\mathrm{mod} 4)\}\}$ then, $S(x) \coloneqq (3x+1)$;

elseIf $\{x \equiv 3 (\mathrm{mod} 4)\}\}$ then, $S(x) \coloneqq (3x-1)$; [Eqn.1]

Note that as per the above definition, for every positive integer x; $y \coloneqq S(x)$ is the single unique immediate-successor of x. The 4x4 state transition matrix H(x, y) for System(3X±1)/4 is given in Table-1. Note that $[c]_b$ represents the modular (modulo-b) residue class c that a given positive integer belongs to.

| H(x,y) | $[0]_4$ | $[2]_4$ | $[1]_4$ | $[3]_4$ |
|---|---|---|---|---|
| $[0]_4$ | 1 | 1 | 1 | 1 |
| $[2]_4$ | 0 | 0 | 1 | 1 |
| $[1]_4$ | 1 | 0 | 0 | 0 |
| $[3]_4$ | 1 | 0 | 0 | 0 |

Table-1: System(3X±1)/4

In order to gain better understanding of the structure of System(3X±1)/4; we first construct the inverse function defined by the immediate-predecessor relation P(y) derived from the above definition.

For any given positive integer $y > 1$;
if & when $y=1$ stop; trivial-cycle $\{(1 \Leftarrow (2) \Leftarrow 4)\}$ detected; else continue;
if $y \in \{[0]_4; [2]_4\}; [1]_4; [3]_4\}$; then, $P(y) \coloneqq (4y)$;
if $y \in \{[1]_4; [3]_4\}$; then, $P(y) \coloneqq (2y)$;
if $y \in \{[4]_{12}\}$; then, $P(y) \coloneqq (y-1)/3$;
if $y \in \{[8]_{12}\}$; then, $P(y) \coloneqq (y+1)/3$; [Eqn.2]

Note that for any given positive even number, $y \in \{[0]_4; [2]_4\}$; there exists exactly one *immediate-even-predecessor*, $x \coloneqq 4y$; whereas any given positive odd integer, $y \in \{[1]_4; [3]_4\}$; has exactly two *immediate-even-predecessors*, $\{x\} \coloneqq \{2y; 4y\}$. Any given positive even number $y \in \{[4]_{12}\}$ has exactly one *immediate-odd-predecessor*, $x \coloneqq (y-1)/3$; and any given positive even number $y \in \{[8]_{12}\}$ has exactly one *immediate-odd-predecessor*, $x \coloneqq (y+1)/3$; whereas, any given positive even number $y \in \{[0]_{12}; [2]_{12}; [6]_{12}; [10]_{12}\}$ cannot have any *immediate-odd-predecessor*. These characteristic properties result in a '*binary*(inward)*arborescence*' structure with its trivial-cycle $\{(1 \Leftarrow (2) \Leftarrow 4)\}$ as a root; and every odd multiple of 3 being an 'odd-leaf-node'; that is, none of its even predecessors can have any immediate-odd-predecessor.

Table-2 gives the expanded 12x12 state transition matrix H(x, y) for System(3X±1)/4, that explicitly captures and represents the above-mentioned characteristic properties.

| H(x,y) | $[0]_{12}$ | $[4]_{12}$ | $[8]_{12}$ | $[2]_{12}$ | $[6]_{12}$ | $[10]_{12}$ | $[1]_{12}$ | $[5]_{12}$ | $[9]_{12}$ | $[3]_{12}$ | $[7]_{12}$ | $[11]_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $[0]_{12}$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $[4]_{12}$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| $[8]_{12}$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $[2]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| $[6]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $[10]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $[1]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[5]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[9]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[3]_{12}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[7]_{12}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[11]_{12}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table-2: System(3X±1)/4 - 12x12 State Transition Matrix

## 3. Observations on the Structure of System(3X±1)/4 H

The binary(inward)arborescence structure of the System(3X±1)/4 H has the following structure:

(0) every node has exactly one immediate-successor.

(1) every odd node has exactly two immediate-even-predecessors.

(2) every even node has exactly one immediate-even-predecessor.

(3) every even node $x \in \{[4]_{12}; [8]_{12}\}$ has exactly one immediate-odd-predecessor; such nodes are called ‘active-nodes’.

(4) no even node $x \in \{[0]_{12}; [2]_{12}; [6]_{12}; [10]_{12}\}$ has any immediate-odd-predecessor; such nodes are called ‘inactive-nodes’.

(5) every odd-node x $\in [3]_6 \coloneqq \{[3]_{12}; [9]_{12}\}$;

that is, x being an odd-multiple-of-3; is an 'odd-leaf-node';

none of its even predecessors has any immediate-odd-predecessor.

## 4. Convergence of System(3X±1)/4

Note that every application of the $(3x\pm1)$ operation produces an immediate resultant positive even numbers $x \in \{[0]_4\}$; and therefore every $(3x\pm1)$ operation *must necessarily be followed* by *at least one* application of the $(x/4)$ operation; that may even be followed by further repetitions of it; leading to an *immediate decrease* in the numerical value of the operand. Therefore, the resulting odd-to-odd subsequence is a "*monotonically strictly decreasing odd-to-odd subsequence*" – a characteristic property that we refer to as, a persistent local 'odd-to-odd monotonicity'. This immediately establishes our assertion, that –

ASSERTION-1:

The System(3X±1)/4 converges to its trivial-cycle $\{(1 \Leftarrow (2) \Leftarrow 4)\}$; starting with any positive integer $x > 1$.

This convergence property immediately implies our assertion, that –

ASSERTION-2:

The set {nodes(H)} of the system arborescence H; representing the transitive-closure of the subset {1,5,3}; that is, $[\{1,5,3\}H+]$; under the inverse mapping of System(3X±1)/4; is the entire set of positive integers N.

## 5. Collatz (3x+1) system re-written as CTUHSK(3x+1)/4 system

The Collatz(3x+1)/2System can be defined by its immediate-successor function S(x) for the (classical/canonical) Collatz (3x+1) system that can be written as –

For any given positive integer $x > 1$;
if & when $x=1$ stop; trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$ detected; else continue;
elseIf $x \equiv 0 (\mathrm{mod} 2)$ then, $S(x) \coloneqq (x/2)$;
elseIf $\{(x \equiv 1 (\mathrm{mod} 2)\}$ then, $S(x) \coloneqq (3x+1)$; [Eqn.3]

Note that as per [Eqn.3]; for every positive integer x there exists a single unique immediate-successor y.

Also note that the immediate-successor of any positive odd number x of the form (4m+1) in the Collatz sequence is a positive even number of the form (12m+4); that is, a number $y \in \{[0]_4\}$ that is always divisible by 4 *at least once*, and therefore the combined operation can be denoted by a single ((3x+1)/4) operation; which leads to a clear decrease in the numerical value for the resultant odd number.

## 6. Critical-Issue, Critical-(Decision)-Node, Detour-Link, Detour-Node

Observe that the immediate-successor of any positive odd number x of the form (4m-1) in the Collatz sequence is a positive even number y of the form (12m-2); that is, a number $y \in \{[2]_4\}$ that is always divisible by 2 *exactly once* only. The combined operation can be denoted by a single ((3x+1)/2) operation, with its immediate resultant being a positive odd number z of the form (6m-1); with an increase in the numerical value of the operand. We designate this specific step {(12m-2)⇐(4m-1)} in the Collatz system as a critical *detour-link*. The critical detour-link is the very source for the possible ambiguity (*critical-issue*) regarding the convergence of the Collatz sequence. We designate the initial-node (4m-1) of a detour-link as its *critical-(decision)-node*; and its terminal-node (12m-2) as its *detour-node*. This *critical-issue* is addressed appropriately here in this paper.

As a first step, the Collatz (3x+1) system originally defined by [Eqn.3] can now be re-written as an equivalent CTUHSK(3x+1)/4 system; with a special single application of a (x/2) operation in such specific cases; whenever the operand is a positive even number of the form (12m-2) arising as the immediate resultant of the application of the (3x+1) operation on a positive odd number of the form (4m-1).

The Collatz (3x+1) system; now re-written & renamed as its equivalent; that is, the CTUHSK(3x+1)/4 system C; is defined by its immediate-successor function:

Start with any given positive integer $x > 1$;
if & when $x=1$ stop; trivial-cycle $\{(1\Leftarrow 2\Leftarrow 4)\}$ detected; else continue;
elseIf $x\equiv 0 \pmod 4$ then, $S(x) \coloneqq (x/4)$;
elseIf $x\equiv 2 \pmod 4$ then, $S(x) \coloneqq (x/2)$;
elseIf $\{(x\equiv 1 \pmod 4)) \vee ((x\equiv 3 \pmod 4))\}$ then, $S(x) \coloneqq (3x+1)$; [Eqn.4]

ASSERTION-3:

The CTUHSK(3x+1)/4 system C is exactly identical to the Collatz (3x+1) system; by its very definition.

Note that as per the above definition of CTUHSK(3X+1)/4 system; every positive integer x; has its single unique immediate-successor, y ≔ S(x). The 4x4 state transition matrix C(x, y) for CTUHSK(3X+1)/4 system C is given in Table-3.

| C(x,y) | $[0]_4$ | $[2]_4$ | $[1]_4$ | $[3]_4$ |
|---|---|---|---|---|
| $[0]_4$ | 1 | 1 | 1 | 1 |
| $[2]_4$ | 0 | 0 | 1 | 1 |
| $[1]_4$ | 1 | 0 | 0 | 0 |
| $[3]_4$ | 0 | 1 | 0 | 0 |

Table-3: CTUHSK(3x+1)/4 system

In order to gain better understanding of the structure of CTUHSK(3X+1)/4 system; we construct the inverse function, defined by the immediate-predecessor relation P(y); derived from the above definition.

For any given positive integer $y > 1$;

if & when y=1 stop; trivial-cycle {(1⇐2⇐4)} detected; else continue;

if y ∈ {$[0]_4$; $[2]_4$}; $[1]_4$; $[3]_4$}; then, P(y) ≔ (4y);

if y ∈ {$[1]_4$; $[3]_4$}; then, P(y) ≔ (2y);

if y ∈ {$[4]_{12}$; $[10]_6$}; then, P(y) ≔ (y-1)/3; [Eqn.5]

Note that for any given positive even number, y ∈ {$[0]_4$; $[2]_4$}; there exists exactly one *immediate-even-predecessor*, x ≔ 4y; whereas, any given positive odd integer, y ∈ {$[1]_4$; $[3]_4$}; has exactly two *immediate-even-predecessors*, {x} ≔ {2y; 4y}. A positive even number y ∈ {$[4]_{12}$; $[10]_{12}$} has exactly one *immediate-odd-predecessor*, x ≔ (y-1)/3; whereas, a positive even number y ∈ {$[0]_{12}$; $[2]_{12}$; $[6]_{12}$; $[8]_{12}$} cannot have any *immediate-odd-predecessor*. These characteristic properties result in a '*binary*(inward)*arborescence*' structure with its trivial-cycle {(1⇐2⇐4)} as a root; and every odd multiple of 3 being an 'odd-leaf-node'; that is, none of its even predecessors can have any immediate-odd-predecessor.

Table-4 gives the expanded 12x12 state transition matrix C(x, y) for CTUHSK(3X+1)/4 system, that explicitly captures and represents the above-mentioned characteristic properties.

| C(x,y) | $[0]_{12}$ | $[4]_{12}$ | $[8]_{12}$ | $[2]_{12}$ | $[6]_{12}$ | $[10]_{12}$ | $[1]_{12}$ | $[5]_{12}$ | $[9]_{12}$ | $[3]_{12}$ | $[7]_{12}$ | $[11]_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $[0]_{12}$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $[4]_{12}$ | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| $[8]_{12}$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $[2]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| $[6]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $[10]_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $[1]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[5]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[9]_{12}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[3]_{12}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[7]_{12}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[11]_{12}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Table-4: CTUHSK(3x+1)/4 system 12x12 State Transition Matrix

The highlighted entries in Table-3 & Table-4 indicate the differences in the CTUHSK(3x+1)/4 system as compared with the corresponding entries in System(3X±1)/4 presented in Table-1 & Table-2.

## 7. Observations on the Structure of CTUHSK(3X+1)/4 system C

The binary(inward)arborescence structure of the CTUHSK(3X+1)/4 system C has the following structure:

(0) every node has exactly one immediate-successor.

(1) every odd node has exactly two immediate-even-predecessors.

(2) every even node has exactly one immediate-even-predecessor.

(3) every even node x $\in \{[4]_{12}; [10]_{12}\}$ has exactly one immediate-odd-predecessor; such nodes are called 'active-nodes'.

(4) no even node x $\in \{[0]_{12}; [2]_{12}; [6]_{12}; [8]_{12}\}$ has any immediate-odd-predecessor; such nodes are called 'inactive-nodes'.

(5) every odd-node x $\in [3]_6 \coloneqq \{[3]_{12}; [9]_{12}\}$;
that is, x being an odd-multiple-of-3; is an 'odd-leaf-node';
none of its even predecessors has any immediate-odd-predecessor.

The system arborescence represents the transitive-closure of the subset {1,5,3}; that is, [{1,5,3}C+]; under the inverse mapping of CTUHSK(3x+1)/4 system. We will show that this is exactly the same as [{1,5,3}H+] which in turn is exactly the entire set of positive integers.

## 8. Observation on the Critical Difference between H and C

The difference between the System(3X±1)/4 H and the CTUHSK(3x+1)/4 system C; as highlighted in the tabular representation of the system state transitions; corresponds to only the critical-(decision)-node (4m-1) and its connecting links.

The critical-(decision)-node (4m-1) ∈ {$[3]_{12}$; $[7]_{12}$; $[11]_{12}$} is linked to the detour-node (12m-2) ∈ {$[10]_{12}$} through the detour-link {(12m-2)⇐(4m-1)} ∈ C in the CTUHSK(3x+1)/4 system; whereas it is linked to the node (12m-4) ∈ {$[8]_{12}$} in the System(3X±1)/4. We designate that node (12m-4) as the *shortcut-node* and that link {(12m-4)⇐(4m-1)} ∈ H as the *shortcut-link*.

## 9. From System(3X±1)/4 to the CTUHSK(3x+1)/4 system

Now we define (1) transformation (reduction) H2T2C from the System(3X±1)/4 H to the CTUHSK(3x+1)/4 system C; and also (2) transformation (reduction) C2T2H from the CTUHSK(3x+1)/4 system C to the System(3X±1)/4 H; passing through a sequence of intermediate transition systems T; using a sequence of pruning-&-joining operations on the system arborescence.

Perform the following pruning-&-joining operation on the System(3X±1)/4, H; recursively/iteratively, to transform it into the CTUHSK(3x+1)/4 system C.

(1) PRUNING OPERATION: Disconnect the critical-decision-node (4m-1) from the shortcut-node (12m-4); by pruning the shortcut-link {(12m-4)⇐(4m-1)} in System(3X±1)/4 arborescence H. This pruning operation yields a free-hanging subtree, with the critical-decision-node (4m-1) as its root; that is denoted by (4m-1)H+ ⊆ H; corresponding to a branch of the original arborescence H.

(2) JOINING OPERATION: Connect this critical-decision-node (4m-1) onto the detour-node (12m-2); which is known to be an inactive-node in System(3X±1)/4; by a joining operation; through a newly created detour-link {(12m-2)⇐(4m-1)}. Note that the detour-node (12m-2) is intended to act as an even-predecessor of an odd node (6m-1) in the prospective CTUHSK(3x+1)/4 system C being generated.

Thus, the entire subtree [(4m-1)H+] ⊆ H that was pruned in pruning operation, now gets joined to the detour-node (12m-2) ∈ [(6m-1)T+] as a new branch; where T is an intermediate transition-system; consisting of the newly generated detour-links {(12m-2)⇐(4m-1)} meant to be belonging to system C that is being generated; along with the remaining links of system H.

The above-described pruning-&-joining operation can be performed iteratively; that is, by replacing every link of the form {(12m-4)⇐(4m-1)} ∈ H with a link of the form {(12m-2)⇐(4m-1)} ∈ C; for m ∈ {1, 2, 3, etc.}; without necessarily considering any network-based recursive algorithm to process on the subtrees etc.; although a network-based algorithm may facilitate in a better conceptual understanding of the process involved in the pruning-&-joining operations therein.

ASSERTION-4:

The transformation (reduction) H2T2C does not affect the set of nodes in the system arborescence; but only replaces every shortcut-link by the corresponding detour-link.

This is clear from the above description of the pruning-&-joining operations used in the transformation H2T2C.

ASSERTION-5:

The transformation (reduction) H2T2C does not affect the binary(inward)arborescence structure of the System(3X±1)/4 H.

This is because, in every iteration, the pruned subtree [(4m-1)H+] ⊆ H is joined onto an already existing and hitherto inactive detour-node (12m-2) ∈ H of the System(3X±1)/4 H; or of the intermediate transition system T.

ASSERTION-6:

The transitive-closure of the subset {1,5,3}; that is, [{1,5,3}C+] ⊆ C; under the inverse mapping of the resultant CTUHSK(3x+1)/4 system C; is exactly the same as the transitive-closure of the subset {1,5,3}; that is, [{1,5,3}H+] ⊆ H; under the inverse mapping of the original System(3X±1)/4 H.

This is because, the entire transformation process H2T2C; consisting of a sequence of pruning-&-joining operations; wherein every pruning-&-joining operation preserves the original set of nodes in H while only replacing the shortcut-links {(12m-4)⇐(4m-1)} ∈ H by the detour-links {(12m-2)⇐(4m-1)} ∈ C. Neither any nodes are removed nor any nodes are added during the pruning-&-joining

operations. Also, the connectivity of the system arborescence is unaffected by the exchange of shortcut-links for the detour-links. Therefore, the transitive-closure is unaffected by the transformation process.

## 10. From the CTUHSK(3x+1)/4 system to System(3X±1)/4

A similar pruning-&-joining process can be described to transform CTUHSK(3x+1)/4 system to System(3X±1)/4, that is, by replacing every detour-link of the form {(12m-2)⇐(4m-1)} by the corresponding newly created shortcut-link of the form {(12m-4)⇐(4m-1)}; as described below:

(a) PRUNING OPERATION: Disconnect the critical-decision-node (4m-1) from the detour-node (12m-2); by pruning the detour-link {(12m-2)⇐(4m-1)} in CTUHSK(3x+1)/4 system arborescence C. This pruning operation yields a free-hanging subtree, with the critical-decision-node (4m-1) as its root; denoted by (4m-1)C+ ⊆ C; corresponding to a branch of the original arborescence C.

(b) JOINING OPERATION: Connect this critical-decision-node (4m-1) onto the shortcut-node (12m-4); which is known to be an otherwise inactive-node in CTUHSK(3x+1)/4 system; by a joining operation; through a newly created shortcut-link {(12m-4)⇐(4m-1)}. Note that the shortcut-node (12m-4) is intended to act as an even-predecessor of a node (3m-1) in the prospective System(3X±1)/4 H being generated.

Thus, the entire subtree [(4m-1)C+] ⊆ C that was pruned in pruning operation, now gets joined to the shortcut-node (12m-4) ∈ (3m-1)T+ as a new branch; where T is an intermediate transition-system; consisting of the newly generated shortcut-links {(12m-4)⇐(4m-1)} meant to be belonging to system H that is being generated, along with the remaining links of system C.

The above-described pruning-&-joining operation can be performed iteratively; that is, by replacing every link of the form {(12m-2)⇐(4m-1)} with a link of the form {(12m-4)⇐(4m-1)}; for m ∈ {1, 2, 3, etc.}; without necessarily considering any network-based recursive algorithm to process on the subtrees etc.; although a network-based algorithm may facilitate in a better conceptual understanding of the process involved in the pruning-&-joining operations therein.

ASSERTION-7:

The transformation (reduction) C2T2H does not affect the set of nodes in the system arborescence; but only replaces every detour-link by the corresponding shortcut-link.

This is clear from the above description of the pruning-&-joining operations used in the transformation C2T2H.

ASSERTION-8:

The transformation (reduction) C2T2H does not affect the binary(inward)arborescence structure of the CTUHSK(3x+1)/4 system C.

This is because, at every iteration, the pruned subtree [(4m-1)C+] ⊆ C is joined onto an already existing and hitherto inactive shortcut-node (12m-4) ∈ C of the CTUHSK(3x+1)/4 system C; or of the intermediate transition system T.

ASSERTION-9:

The transitive-closure of the subset {1,5,3}; that is, [{1,5,3}H+] ⊆ H; under the inverse mapping of the resultant System(3X±1)/4 H; is exactly the same as the transitive-closure of the subset {1,5,3}; that is, [{1,5,3}C+] ⊆ C; under the inverse mapping of the original CTUHSK(3x+1)/4 system C.

This is because, every iteration in the entire transformation process C2T2H; consisting of pruning-&-joining operation; preserves the original set of nodes in C while only replacing the detour-links {(12m-2)⇐(4m-1)} ∈ C by the shortcut-links {(12m-4)⇐(4m-1)} ∈ H. Neither any nodes are removed nor any nodes are added during the pruning-&-joining operations. Also, the connectivity of the system arborescence is unaffected by the exchange of detour-links for the shortcut-links. Therefore, the transitive-closure is unaffected by the transformation process.

## 11. Convergence of the CTUHSK(3x+1)/4 system

Assertion-1 establishes the convergence of the System(3X±1)/4 to its trivial-cycle {(1⇐(2)⇐4)}. Also, the system structure has been shown to be a binary(inward)arborescence with this trivial-cycle as the root and every odd multiple of 3 being an 'odd-leaf-node'. The transitive-closure of the subset {1,5,3}; that is, [{1,5,3}H+]; under the inverse mapping of System(3X±1)/4; is shown to be the entire set of positive integers, in assertion-2. Assertion-3 establishes that the CTUHSK(3x+1)/4 system is exactly identical to the Collatz (3x+1) System. Assertions-4&7 establish that the transformation process does not add or remove any nodes, but only causes an exchange between every shortcut-link and its corresponding detour-link. Assertions-5&8 establish that the transformation process does not affect the binary(inward)arborescence structure of either of these systems. Assertions6&9 establish that the transitive-closure of the subset {1,5,3}; under the inverse mapping of one system; as represented by its binary(inward)arborescence; is exactly the same as that obtained for the other system.

Therefore, these assertions together provide a definitive proof for the convergence of the CTUHSK(3x+1)/4 system to its trivial-cycle $\{(1 \Leftarrow 2 \Leftarrow 4)\}$.

Note that, the above transformation process; from the System(3X±1)/4 to the CTUHSK(3x+1)/4System; also guarantees (through a domain-exhaustion argument); the non-existence of any open chains/loops or open (infinite) threads or any other extraneous elements in the Collatz system.

Also, every pruning-&-joining operation can be considered to be analogous to shifting from a shortcut sub-route to a detour sub-route or from a detour sub-route to a shortcut sub-route; wherein the shifting decision itself is being taken at every critical-(decision)-node.

## 12. Summary

System(3X±1)/4 has been defined, similar to the Collatz (3x+1) system; with slightly modified operations. We have also defined the CTUHSK(3x+1)/4 system, and shown that to be exactly the same as the Collatz (3x+1) system. The convergence of System(3X±1)/4 to its trivial-cycle {(1⇐(2)⇐4)} is established using the persistent local 'odd-to-odd monotonicity' property of its defining function. A reversible (invertible) transformation between System(3X±1)/4 and CTUHSK(3x+1)/4 system has been designed, based on a recursive/iterative process consisting of pruning-&-joining operations performed on the binary(inward)arborescence structure of the system. This pruning-&-joining operation does not affect the binary(inward)arborescence structure as well as the convergence characteristics of either of these systems; because the pruned subtree is joined onto an already existing inactive-node of the system arborescence; without adding or removing any nodes. Therefore, it establishes the convergence of the CTUHSK(3x+1)/4 system to its trivial-cycle {(1⇐2⇐4)}.

The Halemane-Conjecture on the Collatz (3x+1) sequence states that the maximum limit on number of odd (3x+1) operations required to reach its trivial-cycle {(1⇐2⇐4)}; starting from any given positive integer and moving along the Collatz sequence; is the given number itself; with the triad {(31⇐41⇐27)} as an exceptional limiting case.

## 13. Recommended Reading

[§1]. Wikipedia Page – https://en.wikipedia.org/wiki/Collatz_conjecture

[§2]. Jeffrey C Lagarias;
"The 3x+1 problem: An annotated bibliography (1963--1999) (sorted by author)";
https://arxiv.org/abs/math/0309224

[§3]. Jeffrey C Lagarias;
"The 3x+1 Problem: An Annotated Bibliography, II (2000-2009)";
https://arxiv.org/abs/math/0608208

[§4]. Jeffrey C Lagarias;
"The 3x + 1 Problem: An Overview"
https://arxiv.org/abs/2111.02635

[§5]. Jeffrey C Lagarias;
"The Ultimate Challenge: The 3x+1 Problem";
American Mathematical Society; 2010.

[§6]. Richard K Guy;
"Don't Try to Solve These Problems";
The American Mathematical Monthly: Vol. 90, No. 1, pp. 35-41; 1983.
https://doi.org/10.1080/00029890.1983.11971148

## 14. A Confession and An Assertion

I must necessarily confess that the *core idea behind this proof of the Collatz*

*Conjecture is so stunningly and elusively simple*, that one may simply be taken aback in a profound wonder-struck jaw-drop-silence, possibly with an after-thought: "OMG*, how could it be that it never flashed on me any time earlier*"! However, I assert that क्लिष्टविशयसरलीकरणमेव पाण्डित्यलक्षणम् | न तु सरलविषयसंक्लिष्टीकरणम् || That is, *to make the complex simple is a mark of mastery*; *not to make the simple complex.*

## 15. Declaration Regarding Affiliation and Funding

I, Dr(Prof) Keshava Prasad Halemane, hereby declare that I am a Professor retired as on 2017JAN31 from National Institute of Technology Karnataka Surathkal India, and I am not affiliated to any institution or organization or corporation or any other agency or whatever. This research work has been conducted entirely by me on my own as an Independent Researcher, and that I have not received any funding from any source other than my own savings, and I do not have any obligations or encumbrances of any kind, neither financial nor legal nor of any other kind, regarding the contents of the manuscript - of which I am the original author and creator. Also, I hereby declare there is no conflict of interests.

## 16. Author Rights Retention Notice

## 17. Dedication

To my **ಅಜ್ಜ**(ajja) Karinja Halemane Keshava Bhat & **ಅಜ್ಜಿ**(ajji) Thirumaleshwari, **ಅಪ್ಪ**(appa) Shama Bhat & **ಅಮ್ಮ**(amma) Thirumaleshwari, for their *teachings through love*, that *quality matters more than quantity*; to my wife Vijayalakshmi for her *ever consistent love & support*; to my daughter Sriwidya.Bharati and my twin sons Sriwidya.Ramana & Sriwidya.Prawina for their *love & affection.*