Multi-attribute bi-matrix games with Cobb-Douglas preferences

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Abstract

We prove the existence of an equilibrium strategy profile with respect to Cobb-Douglas preferences for multi-attribute bi-matrix games when every coordinate of every attribute-vector under consideration is strictly positive.

Keywords: multi-attribute bi-matrix games, Cobb-Douglas preferences, existence of equilibrium

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1. Introduction: Bi-matrix games (see Chandrasekaran (nd)) are two player/person interactive decision-making problems, with each player having a non-empty finite set of actions to choose from. The existence of equilibrium for bi-matrix games (i.e., the existence of a pair of randomizations, one for each player, where no player has any incentive to unilaterally deviate from) is a well-known result.

In Lahiri (2026), we introduce the concept of a multi-attribute bi-matrix game, which differs from a bi-matrix game only in that except for a real-valued payoff for each player, the outcome for each player is a finite dimensional vector of payoffs of fixed dimension. The simplest way to view such a payoff vector is to assume that there are a fixed and finite number of states of nature exactly one of which will be realized in a future period and each outcome specifies a state-dependent gain (or wealth) vector. We refer to such outcomes as attribute-vectors. As mentioned in Lahiri (2026), it may be worth noting that although Keeney and Raiffa (1993) provides an exhaustive coverage of multi-objective and multi-attribute decision making, including an alternative interpretation of social aggregation of individual preferences, there is no mention in there of its possible application in game theory. We assume that each player has preferences over the finite dimensional Euclidean space of attribute-vectors and allow the preferences of the two players to be different. An equilibrium strategy profile for a multi-attribute bi-matrix game is a strategy profile from which no player, based on its preferences, has any incentive to unilaterally deviate. In Lahiri (2026) we show that there is the possibility of non-existence of any equilibrium strategy profile, if preferences are lexicographic. In the same paper, we extend the seminal result of Mangasarian and Stone (1964) concerning the equivalence of the set of equilibrium strategy profiles of a bi-matrix game and the set of solutions of a bi-linear programming problem, to our framework of multi-attribute bi-matrix games.

In this paper, we provide the simple proof of existence of an equilibrium strategy profile for a multi-attribute bi-matrix game, when the preferences of each player over the set of attribute-vectors can be represented by Cobb-Douglas preferences. Cobb-Douglas preferences (preferences that can be numerically represented by a Cobb-Douglas utility function (see Trockel (1989), Argenziano and Gilboa (2017)), need no introduction in economics, decision theory and game theory. What is new in this paper is its application in the context of multi-attribute bi-matrix games.

In order to facilitate our analysis, we assume that all co-ordinates of any attribute vector under consideration are strictly positive.

2. The model: For positive integers m, n consider an interactive decision making context (game) with two players/participants/decision-makers who we denote by 'a' and 'b' respectively, and where player 'a' has m actions to choose from and 'b' has 'n' actions to choose from.

A **strategy for player 'a'** is a point in $\Delta^{m-1} = \{x \in \mathbb{R}^m_+ | \sum_{i=1}^{m} x_i = 1\}$ and a **strategy for player 'b'** is a point in $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ | \sum_{j=1}^{n} y_j = 1\}$.

A strategy for the player 'a' is a probability distribution on {1, ..., m}, with each coordinate of the strategy denoting the probability with which the player 'a' chooses the corresponding action. A strategy for player 'b' is a probability distribution on {1, ..., n}, with each coordinate of the strategy denoting the probability with which player 'b' chooses the corresponding action.

A pair (x, y) where x is a strategy for player 'a' and y is a strategy for player 'b' is said to be a **strategy profile**.

Let $\mathbb{R}^{m\times n}$ be the set of all m×n matrices.

Given $C \in \mathbb{R}^{m\times n}$ and $(i, j) \in \{1, \ldots, m\}\times\{1, \ldots, n\}$, let c_{ij} denote the $(i, j)^{th}$ entry of C, C_i denote the i^{th} row of C and C^j denote the j^{th} column of C.

For a positive integer p, let $\mathbb{R}^{(p,\, m\times n)}$ denote the set of all m×n matrices whose entries are points in $\mathbb{R}^p$.

If $V \epsilon \mathbb{R}^{(p,\, m\times n)}$, then for all (i, j)ϵ{1, …, m}×{1, ..., n}, the typical (though not invariable) notations are as follows: (i, j)th entry of V will be denoted by $v_{ij} = (v_{ij}^{(1)}, \ldots, v_{ij}^{(p)})$, the ith row of V will be denoted by V_i and the jth column of V will be denoted by V^j.

For our purposes here, a member of $\mathbb{R}^{(p,\, m\times n)} \times \mathbb{R}^{(p\,,\, m\times n)}$ will be referred to as a **p-valued bi-matrix game** and $\mathbb{R}^{(p,\, m\times n)} \times \mathbb{R}^{(p\; m\times n)}$ as the **set of all p-valued bi-matrix games**.

The union of the sets all p-valued bi-matrix games, allowing p to vary over the set of all natural numbers is what we refer to as **the set of all multi-attribute bi-matrix games** and we may refer to any member of this set as a **multi-attribute bi-matrix game**.

Notation 1: Given a p-valued bi-matrix game (V(a), V(b)), for rϵ{1, …, p} and αϵ{a, b}, let $V^{(r)}(\alpha)$ be the m×n real-valued matrix whose (i, j)th term for (i, j)ϵ{1, …, m}×{1, ..., n} is $v_{ij}^{(r)}(\alpha)$. The ith row of $V^{(r)}(\alpha)$ for iϵ{1, …, m} is denoted by $V_i^{(r)}(\alpha)$ and the jth column of $V^{(r)}(\alpha)$ for jϵ{1, …, n} is denoted by$V^{(r)j}(\alpha)$. For $(x, y)\epsilon\Delta^{m-1}\times\Delta^{n-1}$ and αϵ{a, b}, $x^TV(\alpha)y = \sum_{i=1}^{m}\sum_{j=1}^{n} x_iy_j(v_{ij}^{(1)}(\alpha), \ldots, v_{ij}^{(p)}(\alpha)) = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_iy_jv_{ij}^{(1)}(\alpha), \ldots, x_iy_jv_{ij}^{(p)}(\alpha)) = (x^TV^{(1)}(\alpha)y, \ldots, x^TV^{(p)}(\alpha)y)$. ▌

For αϵ{a, b} let $\succcurlyeq_\alpha$ be the **preference relation** (i.e., complete (or connected), reflexive and transitive binary relation) of α on the set of attribute-vectors, $\mathbb{R}^p$ and let $\succ_\alpha$ denote the asymmetric part of $\succcurlyeq_\alpha$. Let $\sim_\alpha$ denote the symmetric part of $\succcurlyeq_\alpha$.

Given a p-valued bi-matrix game (V(a), V(b)), a strategy profile (x, y) is said to be an **equilibrium strategy profile (with respect to ($\succcurlyeq_a$, $\succcurlyeq_b$))** for (V(a), V(b)) if $x^TV(a)y \succcurlyeq_a w^TV_i(a)y$ for all $w\epsilon\Delta^{m-1}$ and $x^TV(b)y \succcurlyeq_b x^TV(b)z$ for all $z\epsilon\Delta^{n-1}$.

3. Cobb-Douglas preferences on attribute-vectors: Suppose that <a_r| r = 1, .. , p> and <b_r| r = 1, .. , p> are arrays of <u>*strictly positive real numbers*</u> satisfying $\sum_{r=1}^{p} a_r = \sum_{r=1}^{p} b_r = 1$.

For u, vϵ$\mathbb{R}^p$: (i) If u, v $\epsilon\mathbb{R}_+^p$, then let u $\succcurlyeq_a$ v if and only if $\prod_{r=1}^{p} u_r^{a_r} \geq \prod_{r=1}^{p} v_r^{a_r}$ & u $\succcurlyeq_b$ v if and only if $\prod_{r=1}^{p} u_r^{b_r} \geq \prod_{r=1}^{p} v_r^{b_r}$. (ii) If u $\epsilon\mathbb{R}_+^p$ and vϵ$\mathbb{R}^p \setminus \mathbb{R}_+^p$, then let u $\succ_\alpha$ v for α ϵ{a, b}. (iii) If u, vϵ$\mathbb{R}^p \setminus \mathbb{R}_+^p$, then let u $\sim_\alpha$ v for αϵ{a, b}.

$\succcurlyeq_\alpha$ for α ϵ{a, b} is said to be a **Cobb-Douglas preference (relation) on (the set of attribute-vectors)** $\mathbb{R}^p$.

For what follows we shall make an assumption that facilitates the analysis.

Assumption on the structure of (V(a), V(b)): For $\alpha \in \{a, b\}$, $r \in \{1, \ldots, p\}$ and $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$, $v_{ij}^{(r)}(\alpha) > 0$.

Thus, for all $\alpha \in \{a, b\}$, $r \in \{1, \ldots, p\}$ and strategy profile (x, y), it must be the case that $x^T V^{(r)}(\alpha) y > 0$.

The real-valued functions $(x, y) \mapsto \prod_{r=1}^{p} (x^T V^{(r)}(a) y)^{a_r}$ and $(x, y) \mapsto \prod_{r=1}^{p} (x^T V^{(r)}(b) y)^{b_r}$ on $\Delta^{m-1} \times \Delta^{n-1}$ are continuous and *strictly positive-valued.*

4. Existence of equilibrium with respect to Cobb-Douglas Preferences: Under the assumptions introduced in the previous section an equilibrium strategy profile with respect to the preferences exist. That is the content of the following proposition.

Proposition 1: Suppose $\succcurlyeq_a$ and $\succcurlyeq_b$ are Cobb-Douglas preferences and the "Assumption on the structure of (V(a), V(b))" is satisfied. Then, there exists an equilibrium strategy profile with respect to $(\succcurlyeq_a, \succcurlyeq_b)$ for (V(a), V(b)).

Proof: For strategy profile (x, y) the real-valued functions $w \mapsto \prod_{r=1}^{p} (w^T V^{(r)}(a) y)^{a_r}$ on Δ^{m-1} and $z \mapsto \prod_{r=1}^{p} (x^T V^{(r)}(b) z)^{b_r}$ on Δ^{n-1} are strictly positive-valued, continuous and strictly concave.

Thus, for strategy profile (x, y), $\underset{w \in \Delta^{m-1}}{\operatorname{argmax}} \prod_{r=1}^{p} (w^T V^{(r)}(a) y)^{a_r}$ is a "*singleton*" and $\underset{z \in \Delta^{n-1}}{\operatorname{argmax}} \prod_{r=1}^{p} (x^T V^{(r)}(a) z)^{a_r}$ is a "*singleton*".

Let $F: \Delta^{n-1} \to \Delta^{m-1}$ be such that for all $y \in \Delta^{n-1}$, $\underset{w \in \Delta^{m-1}}{\operatorname{argmax}} \prod_{r=1}^{p} (w^T V^{(r)}(a) y)^{a_r} = \{F(y)\}$ and $G: \Delta^{m-1} \to \Delta^{n-1}$ be such that for all $x \in \Delta^{m-1}$, $\underset{z \in \Delta^{n-1}}{\operatorname{argmax}} \prod_{r=1}^{p} (x^T V^{(r)}(a) z)^{a_r} = \{G(x)\}$.

F is the best response function of (the row player) 'a' and G is the best response function of (the column player) 'b'.

Let $\langle y^{(k)} | k \in \mathbb{N} \rangle$ be a sequence in Δ^{n-1} converging to $y \in \Delta^{n-1}$.

Suppose, there exists $\varepsilon > 0$, such that $\|F(y^{(k)}) - F(y)\| > \varepsilon$ infinitely often. Note that for all $k \in \mathbb{N}$ and $x \in \Delta^{m-1}$, it must be the case that $\prod_{r=1}^{p} (F(y^{(k)})^T V^{(r)}(a) y^{(k)})^{a_r} \geq \prod_{r=1}^{p} (x^T V^{(r)}(a) y^{(k)})^{a_r}$. Without loss of generality suppose there exists $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq K$ it is the case that $\|F(y^{(k)}) - F(y)\| > \varepsilon$. Since $\langle F(y^{(k)}) | k \in \mathbb{N} \rangle$ lies in the closed and bounded set Δ^{m-1}, by the Bolzano-Weirstrass's theorem it has a convergent subsequence $\langle F(y^{N(k)}) | k \in \mathbb{N} \rangle$ converging to some $x^\# \in \Delta^{m-1}$. Clearly, $\|x^\# - F(y)\| \geq \varepsilon$ and hence $x^\# \neq F(y)$. Thus, $\langle \prod_{r=1}^{p} (F(y^{N(k)})^T V^{(r)}(a) y^{N(k)})^{a_r} | k \in \mathbb{N} \rangle$

converges to $\prod_{r=1}^{p}(x^{\#T}V^{(r)}(a)y)^{a_r}$ and since for all $x \epsilon \Delta^{m-1}$,

$\prod_{r=1}^{p}(F(y^{N(k)})^T V^{(r)}(a)y^{N(k)})^{a_r} \geq \prod_{r=1}^{p}(x^T V^{(r)}(a)y^{N(k)})^{a_r}$, it must be the case that for all $x \epsilon \Delta^{m-1}$, $\prod_{r=1}^{p}(x^{\#T}V^{(r)}(a)y)^{a_r} \geq \prod_{r=1}^{p}(x^T V^{(r)}(a)y)^{a_r}$, i.e., $x^{\#} \epsilon$ $\underset{w \epsilon \Delta^{m-1}}{\operatorname{argmax}} \prod_{r=1}^{p}(w^T V^{(r)}(a)y)^{a_r}$.

Since $x^{\#} \neq F(y)$, this contradicts the definition of F.

Thus, it must be the case that $<F(y^{(k)})|k \epsilon \mathbb{N}>$ converges to F(y) and hence F is continuous on Δ^{n-1}.

A similar argument shows that G is continuous on Δ^{m-1}.

Consider the function $(x, y) \mapsto (F(y), G(x))$ from $\Delta^{m-1} \times \Delta^{n-1}$ to itself.

Since the function is continuous, by Brouwer's Fixed Point Theorem, there exists $(x^*, y^*) \epsilon \Delta^{m-1} \times \Delta^{n-1}$ such that $(x^*, y^*) = (F(y^*), G(x^*))$.

Clearly (x^*, y^*) is an equilibrium strategy profile with respect to $(\succcurlyeq_a, \succcurlyeq_b)$ for (V(a), V(b)). Q.E.D.

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