XYY’ Calculus from Soil Consolidation to All Dynamics

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Abstract. Terzaghi (1923) brought differential calculus in geotechnical engineering by using heat (diffusion) equation in consolidation. But the complexities of soil settlement phenomenon (combined hydrodynamic + creep) with unknown drainage and loading and its evaluation by usual time (*X*) centric consolidation methods exposed that classical Newton-Leibniz calculus is fundamentally incomplete. It forces to see *3D (XYY')* dynamic reality on *2D (XY & XY')* flat planes ignoring the most vital (*YY'*) phase plane that is the *X* (usually Time) invariant geometric *DNA* of the dynamic system. For example: *DNA* of Terzaghi’s *1D* Eq on semi-log *U-ln(dU/dT)* or *U-lnU’* plot is Symmetrical *S* shaped curve, on *U-U’* plot is straight line for *U=60-100%* and Straight line again on *U-(1/U’)* plot for *U=0-40%* [9-11]. Similarly, *DNA* of *Sine* wave is circle, Exponential function is straight line, Catenary is Hyperbola, Parabola is Parabola etc. As *XY* and *XY'* planes are hinged around *X*-axis, the classical calculus fails when *X* is unknown and that is usual case in the field, forensic geotechnics or sinking infrastructures on soft soils. The *XYY'* calculus with its *YY'* plane analysis was developed in mid 1990s and extensively used to overcome these limitations in soil consolidation. However, it is universally applicable in other sciences also like diffusion, creep, wave, thermodynamics, electro-dynamics, economics, epidemiology, mechanics, oscillations etc Instead of borrowing mathematical laws from physics, heat, hydraulics, mechanics etc Geotechnical Engineering is now in a position to give them back a very useful mathematical tool.

Keywords: *Soil Consolidation, Forensic Geotechnics, Pore Pressure, XYY’ Calculus, Tewatia YY’ Approach, Dynamic Systems, Phase-Space Geometry, Asymptotic Linearity Theorem*

1 Introduction

Traditional geotechnical engineering is anchored by the fundamental one-dimensional theory of consolidation formalized by Terzaghi in 1923, which models the time-dependent expulsion of pore water and the resulting dissipation of excess pore water pressure under applied effective stresses [1].

$$U = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} \exp(-M^2 T_v) \quad (1)$$

Where, U is degree of consolidation, T_v is Time factor and $M = 2m+1$. However, the foundational reliance on Newton-Leibniz chronological timelines ($X - Y$ and $X - Y'$ approaches, where X is usually time) presents a severe limitation when attempting to analyse actively sinking, pre-existing infrastructures where X is unknown. For example; Taylor δ-$\sqrt{t}$ (Y-$\sqrt{X}$), Casagrande *δ-log(t) (Y-log(X))*, Parkin *log(dδ/dt)-log(t)* (*logY'-logX*) methods fail when t is not known, where δ is settlement and t is time [2-4].

2 *XYY'* Calculus & Tewatia's *YY' Plane*

The *X*-axis is the pillar of classical Newton-Leibniz (*XY & XY'*) Calculus, and it fails when X is unknown. It shows a *3D XYY'* trajectory on *2D XY* (Blue) *& XY'* (Green) planes only by missing the critical *YY'* (Orange) plane that is the X (usually time) invariant geometric *DNA* of the dynamic system (Fig 1-2). The complete mechanical state of the dynamic consolidation process at any specific instant is not a simple coordinate pair, but a continuous, dynamic position vector $r(X)$ tracing a definitive curve in three-dimensional space:

$$r(X) = Xi + f(X)j + f'(X)k = Xi + Y(X)j + Y'(X)k$$

Where $i, j,$ and k are the standard unit vectors. Any two-dimensional projection of this specific curve (such as the standard time-settlement plot) represents a partial, fundamentally incomplete view that necessarily discards critical mechanical information regarding the system's internal velocity architecture.

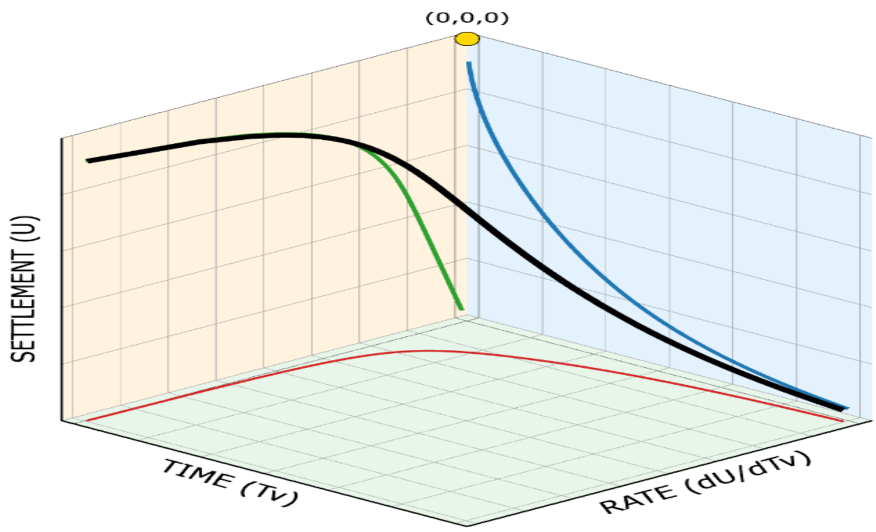

Fig 1: The Real *3D* Path of *1D* Consolidation (*T-U-U'* - Black Curve). Blue Wall, Integral Plane (*X-Y*): Classic Curve (*Time vs U; T-U*). Orange Wall, Tewatia's Differential Plane (*Y-Y'*) *DNA*: *U vs Rate; U-U'*. Green Floor, Newton's Differential Plane (*X-Y'*): *Time vs Rate; T-U'*. Origin (0,0,0), U increases downwards from *0-1*.

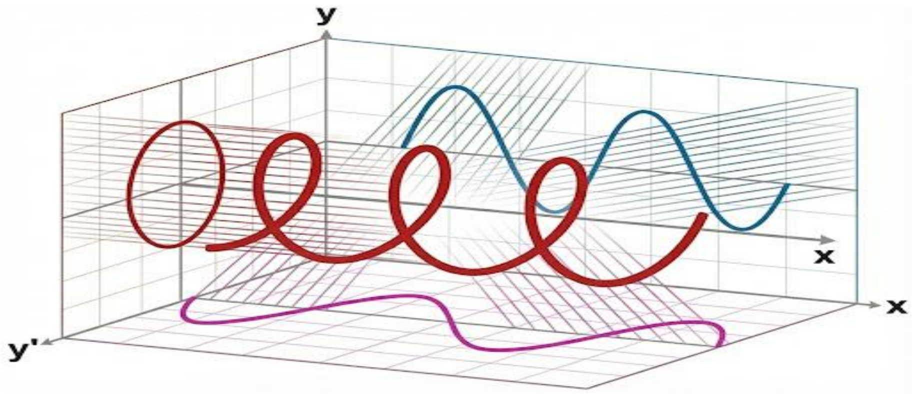

Fig 2. The function $y = sin(x)$ in *3D XYY'* Framework is a Helix in space, spiralling forward through time. Its *DNA* or the geometric signature on *Y-Y'* plane is a circle of unit radius.

The *XYY'* Calculus originated through a series of seminal research papers published in the mid-1990s by Prof. Sudhir K. Tewatia, marking the explicit genesis of the *3D* triad of *Time-Settlement-Slope (X-Y-Y')* Fig 1. Early research showed that mere Time-Settlement (t-δ *or X-Y*) [2,3,5] and Time-Velocity (t-$d\delta/dt$ or *X-Y'*) plots [4] are insufficient for an in-depth understanding of the consolidation behavior of clayey soils [6]. The slope (Y', dY/dt, $d\delta/dt$, dU/dT or $dU/d\sqrt{T}$) is an equally, if not more, important parameter [7-8]. The slope *Y'* (as $dU/d\sqrt{T}$ or $d\delta/d\sqrt{t}$) along with *XY* (as U-$\sqrt{T}$ or δ-$\sqrt{t}$) plot was used for the first time in Geotechnical Engineering in Improved $\sqrt{t}$ method [7-8], adding a new dimension *Y'* in Taylor's (*XY*) δ-$\sqrt{t}$ [3] *2D* graph, thus completing *3D XYY'* triad. *The 3D T-Chart* [8] method is a novel advanced utilizations of the *XYY'* framework. The defining historical breakthrough occurred in June 1998 with landmark publication in the *ASTM* Geotechnical Testing Journal [9]. These foundational papers successfully challenged the century-old reliance on the Integral *XY* and Newton's Differential *XY'* temporal planes by explicitly discovering and introducing the missing *YY'* phase plane to geotechnical engineering [9-10] that encodes other two *XY* and *XY'* planes in it. The complete *YY' plane* trajectory can be obtained from the data snippet without knowing initial and boundary conditions. Ultimately, this mathematical paradigm transcended geotechnical engineering altogether, being formalized as a universal calculus applicable to thermodynamics, radioactive decay, cosmology, and any dissipative physical system [11].

3 Geotechnical Engineering Functions & Their DNA in *XYY'* Framework:

Parabola $T_v = (\pi/4)\ U^2$: Initial *0-40%U* portion of Terzaghi Eq 1 is approximated with parabola. Differentiating it gives $dT_v/dU = (\pi/2)\ U = 1/U'$; *U* versus Inverse Velocity ($1/U'$) plot is straight line. This property is used to find δ_0 by mapping δ against ($dt/d\delta$). However, in general dynamics parabola $y = x^2$, is parabola itself (though not the same) on *Y-Y'* plot as *dy/dx = 2x = y'* and *y = y'²/4*. But the *DNA* of *y=0.5x²* is exactly, the same parabola as *y' = 0.5y'²* because *y' = dy/dx = x*.

Damped Oscillators: Functions like $y = e^{-ax}\cos(bx)$ used in Earthquake, Soil Dynamics or Machine Foundations form logarithmic spirals converging to the origin in the *yy'* plane. The tightness of the spiral visually encodes the damping ratio (ζ)), instantly identifying system stability without needing to calculate peak values in the time domain (Fig 3). Consolidation is over damped oscillation.

Exponential Decay (Later Part of Terzaghi's 1D & Barron;s Radial Soil Consolidation Eq): The function $y = e^{-x}$ dictates that $y' = -y$. In the $Y - Y'$ phase plane, this is merely a perfect straight line passing through the origin with a slope of -1. This exact linearity forms the mathematical foundation of the *Asymptotic Linearity Theorem* & Master Decay Asymptote.

Sine Wave (Conservative Systems): For a simple harmonic oscillator $y = \sin(x)$, the derivative is $y' = \cos(x)$. In the $Y - Y'$ plane, $y^2 + y'^2 = sin^2(x) + cos^2(x) = 1$ plots as a perfect circle. In the 3D state space, the trajectory is a forward-spiraling helix. The system orbits a fixed attractor indefinitely (Fig 2-3). This immediately proves energy conservation without requiring the integration of complex energy functions in classical time domain analysis.

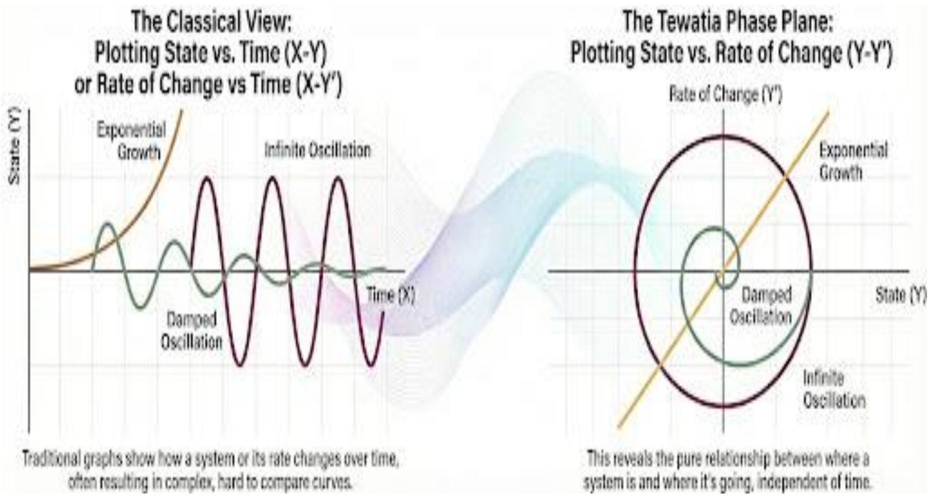

Fig 3: $y = e^x$, $y = e^{-ax}.cos(bx)$ and $y = sin(x)$ functions are shown on *X-Y, X-Y' and Y-Y'* planes as the projections of their respective *3D* curve in *X-Y-Y'* space.

Hyperbola, Sinh & Cosh (Mid Portion of Terzaghi's 1D Soil Consolidation Eq): The projection of $y = \cosh(x)$ on the *yy'* plane yields $y^2 - y'^2 = cosh^2(x) - sinh^2(x) = 1$, which is a strict hyperbola. Mid portion 40-60%*U* of *U-lnU'* plot (Green) is Sinh and its *DNA* (Red Curve) on *U-S* plot (where, *S* is the slope of *U-lnU'* plot. $S = dU/d(lnU') = U'^2/U''$) is a rectangular hyperbola (Fig 4) [9-11].

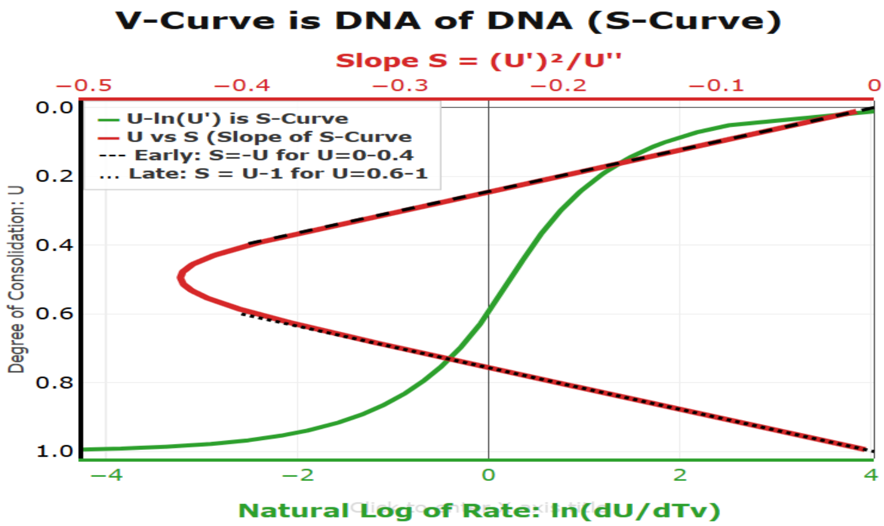

Fig 4. Semi-log plots of *U-logU'* is the *DNA* of Eq 3. The Green S- curve has a slope *0.438* on *ln*, and *1* $(=\delta_{100} - \delta_0)$ *on* log_{10}*, at U=0.5*. The middle portion *0.4<U<0.6* is *Sinh* and its *DNA* (middle portion of red curve is Hyperbola. The Red curve is linear *U-S* plot, where *S* is the slope of *S-curve*; its *DNA of DNA of Eq 3. S = -U for 0<U<0.4 & S = U-1 for 0.6<U<1.*

Logarithmic or Creep of Soil Settlement: $y = a.ln\, x + b$ is a straight line on semi-log *y-y' plot* (Fig 5) [9-11].

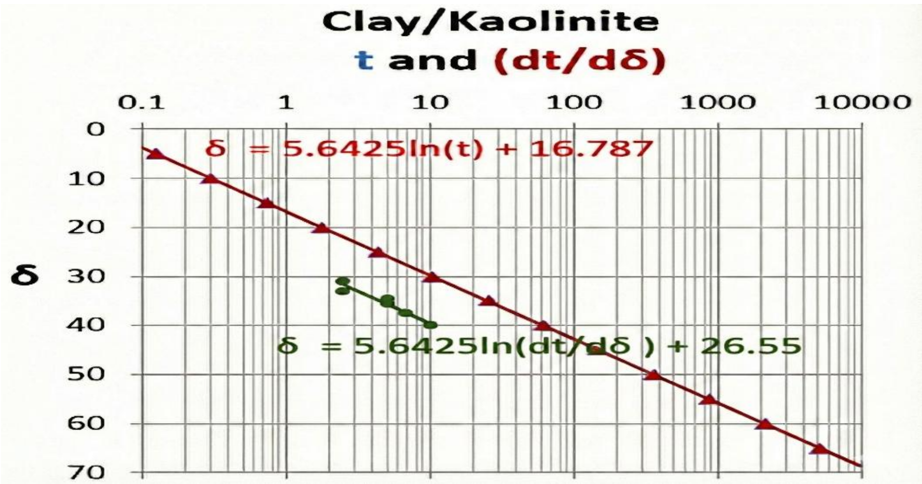

Fig 5. Retrieving the past, present & future *δ-t* data (of old structures roughly like Pisa tower) from the snippet of *δ-(dt/dδ)* plot by simple graphical integration and extracting Cα when initial and boundary conditions are unknown. Unlike, classical calculus, no constant of integration is required in Tewatia's Y-Y' calculus.

The integration of classical derivatives into phase-space geometry relies on the chain rule to bridge acceleration, velocity, and geometric slope:

$$Y'' = Y' \cdot \frac{dY\prime}{dY} \tag{2}$$

This fundamental transformation proves that the second derivative is strictly equal to the first derivative multiplied by the geometric slope of the $Y - Y'$ phase trajectory, simplifying complex second-order autonomous differential equations into first-order algebraic forms. This was used to derive Barron's Eq of Radial Consolidation without using partial differential equations [10-11].

3 Resolution of Terzaghi's Eq 1 in 3 parts, The Transition Phase (40%-60% U)

Historically, classical mathematical models attempted to bridge the consolidation curve by jumping directly from an early parabolic approximation to a late-stage exponential approximation at exactly 60% U. The introduction of the $Y - Y'$ plane discovered a previously undocumented transition zone occupying the middle 20% of the process.

$$\ln(U') = 0.4766 \sinh(-4.7870U + 2.3727) + 0.2380 \tag{3}$$

The semi-logarithmic representation of the full trajectory reveals a perfectly symmetrical S-shaped curve with a distinct pivot at 50% consolidation (Fig 4). A Truth Check (μ) parameter mathematically quantifies consistency to ideal Terzaghian behavior:

$$\mu = \frac{t_{50} \cdot v_{50}}{0.245 \cdot s_{50}} \tag{4}$$

In the early parabolic phase ($U < 0.4$), the relationship between settlement and the inverse velocity ($dt/d\delta$) is strictly linear to extract δ_0:

$$\delta = k\left(\frac{dt}{d\delta}\right) + \delta_0 \tag{5}$$

For large time factors, retaining the fundamental mode yields a strict linear relationship between settlement (δ) and its rate ($d\delta/dt$). The geometric slope of this line is the Central Operator (Λ), encoding the underlying diffusion physics:

$$\delta = -\Lambda\frac{d\delta}{dt} + \delta_{100} = -\frac{4H^2}{\pi^2 c_v}\frac{d\delta}{dt} + \delta_{100} \tag{6}$$

The intercept on the δ-axis where the velocity of settlement mathematically reaches zero provides the ultimate settlement limit, δ_{100}.

4 The Asymptotic Linearity Theorem (ALT) & The Master Decay Asymptote

The XYY' framework mathematically proved that complex, multi-dimensional consolidation trajectories naturally converge into invariant straight lines in the phase plane [9-11] (Fig 6-7). For example, radial consolidation exhibits perfect asymptotic linearity as early as 5% to 10% consolidation and vertical consolidation becomes linear by 60%U. Any combination of hydrodynamic consolidation of Vertical, Radial, Cylindrical, Cubical, Spherical with inward or outward flow or any arbitrary shape, type, size of any mixed strata ultimately (in the last) joins master decay asymptote on linear scale. Similarly, any combination of Creep settlements ultimately joins Master Decay Asymptote on Semi-log plot. This master Decay Asymptote manifests itself in Pore Pressure Analysis also in Tewatia's *YY'* plane [10].

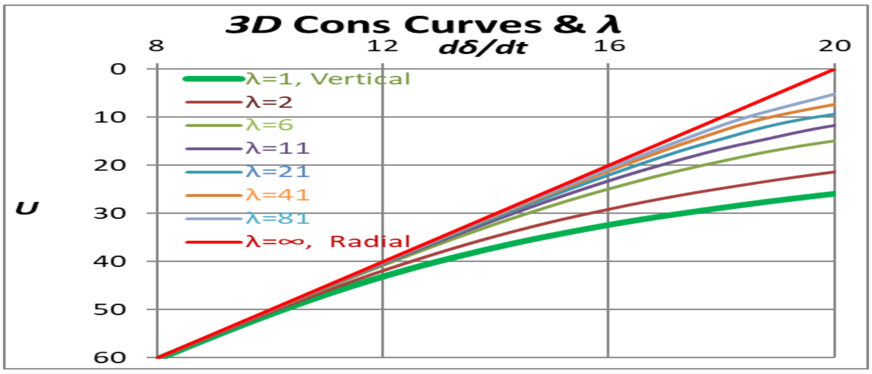

Fig 6. Asymptotic Linearity Theorem: Radial consolidation on Tewatia's Y-Y' plane is a straight line right since beginning, while Vertical consolidation becomes linear at 60% U. In combined vertical and radial (3D) consolidation, λ (varies 1-∞) determines the dominance of radial consolidation over vertical consolidation. But whatever is λ, any combination of consolidation becomes linear at the latest by 60% U or earlier.

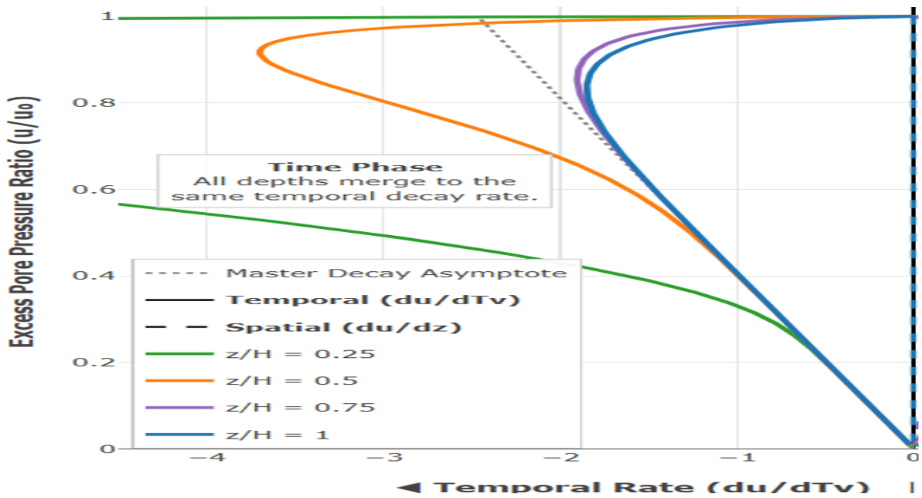

Fig 7: Theoretical pore pressure, (u/u_0) versus (du/dTv) curves for various depths (z) to drainage path (H) ratios in vertical consolidation. Time factor, $Tv = Cv.t/H^2$. Slope of the Theoretical Master Decay Asymptote, Central Operator Λ is $4/\pi^2$.

There is no need to take the sample from field to laboratory, engineers can extract the fundamental "Central Operator" (Λ) strictly from the geometric slope of field data, allowing for the instant determination of the coefficient of consolidation c_v, c_r, λ_{cH} etc and ultimate settlement. The factor λ_{cH} encodes the combined effect of c_v and complicated drainage conditions. This makes the *XYY'* Calculus not just the simplest but the Only ultimate tool for sinking infrastructures.

5 Universal Applicability in All Dynamics

The most profound realization of the XYY' calculus framework extends far beyond the traditional borders of geotechnical engineering. During Terzaghi's formative era in 1923, soil mechanics was primarily a consumer of broader mathematical laws from Heat, Mechanics, Hydraulics, Physics etc. The XYY' calculus fundamentally and permanently reverses this historical trend by showing that the accepted "laws of mechanics and hydraulics" are merely localized subsets of a much larger, universal "Law of Dynamic Dissipative Systems" (Table 1).

Table 1: The Universality of the Diffusivity Coefficient (L^2/T) in Scientific & Other Dynamics

Scientific Field	**Dynamic State Variable (Y)**	**Diffusivity Coefficient Extracted via Slope (Λ)**
Soil Mechanics	Pore Water Pressure (u)	c_v (Coefficient of Consolidation)
Thermodynamics	Temperature (T)	α (Thermal Diffusivity)
Fluid Dynamics	Velocity (v)	ν (Kinematic Viscosity)
Electricity	Voltage (V)	D_e (RC Circuit Diffusivity)
Economics / Finance	Option Price (P)	$\frac{1}{2}\sigma^2$ (Volatility in Black-Scholes Model)
Epidemiology	Infection Count (I)	β (Transmission Rate)

In financial economics, for instance, the famous Black-Scholes model—which earned the Nobel Prize in Economics in 1997—utilizes a parabolic partial differential equation that is functionally identical to the heat equation to price financial options over time. Market volatility (σ^2), which acts as the explicit diffusion rate of the asset's price, is traditionally highly complex to extract chronometrically due to the stochastic nature of market timelines. By mapping the observable option price (Y) directly against its instantaneous rate of change (Y') using the XYY' phase plane framework, market volatility can be treated purely as a geometric Central Operator, completely immune to chronometric start times or temporal noise.

Similarly, in epidemiology, modeling infection decay trajectories without knowing the exact "Day Zero" of the outbreak paralyzes traditional SIR temporal models. The phase plane framework extracts the transmission decay rate strictly from the geometry of current infection decline. Therefore, geotechnical engineering is now positioned to donate a vastly superior, highly robust, time-independent mathematical tool back to theoretical physics, fluid dynamics, econometrics, and computational biology.

By collapsing complex, second-order dynamic variables into simple, time-invariant geometric signatures, the XYY' framework proves that the fundamental language of change—whether modelling microscopic quantum wave equations, earthly fluid mechanics, or macroscopic cosmological expansion—is intrinsically very simple & geometric when freed from the arbitrary constraints of an observer's clock.

7 Conclusion

The XYY' Calculus elevates geotechnical engineering, providing a universal, time-independent solver applicable to dynamics across all sciences. It definitively resolves the "Missing Time" problem in forensic geotechnics and dynamically dissipative systems by shifting from a time-centric domain to a phase-space geometric domain. By mapping the observable State (Y) against its measurable Rate (Y'), time becomes a recoverable consequence (without integration constants) rather than a mandatory initial input constraint. Leveraging the Asymptotic Linearity Theorem and the Central Operator (Λ), accurate parameters are extracted flawlessly from linear data trajectories. This geometric approach offers unparalleled predictive capabilities in complex soil drainages and historical reconstruction for sinking infrastructure.

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References

1. Terzaghi, K., 1923, Mathematisch-naturewissenschaftliche Klasse, Akademie der Wissenschaften in Wien, Sitzungsberichte, Part Iia, Vol. 132, No. 3/4, pp.125-138.
2. Casagrande, A., 1939, "Notes on Soil Testing for Engineering Purposes," Soil Mech. Series No. 8, Publication from Graduate School of Engineering, No. 268, Howard University, Cambridge, Mass.
3. D. W. Taylor, Fundamentals of Soil Mechanics. New York: John Wiley and Sons Inc., 1948.
4. Parkin, A. K., 1978, "Coefficient of Consolidation by the Velocity Method," Geotechnique, Vol. 28, No. 4, pp. 472-474.

5. Barron, R. A., "Consolidation of Fine-Grained Soils by Drain wells," Transactions of ASCE, Vol.113, 1948, pp. 718–724.
6. Tewatia, S. K., and Venkatachalam, K., Discussion on 'Consolidation behaviour of clayey soils' by Sridharan et al., ASTM Geotechnical Testing Journal, Vol. 20, No. 4, 1997, pp. 484–487.
7. Tewatia, S. K. and Venkatachalam, K., Improved Root t Method to Evaluate Consolidation Test Results, ASTM Geotechnical Testing Journal, Vol. 20, No. 1, 1997, pp. 121–125.
8. Tewatia, S.K., Venkatachalam, K., and Sridharan, A., T-Chart to evaluate consolidation test results, ASTM Geotechnical Testing Journal, Vol. 21, No. 3, 1998, pp. 270–274.
9. Tewatia, S. K., Evaluation of true *Cv* and instantaneous *Cv*, and isolation of secondary consolidation, ASTM Geotechnical Testing Journal, Vol. 21, No. 2, 1998, pp. 102–108.
10. Tewatia, S. K., and Tewatia, K., Consolidation of Soils (book): Rate of Settlement Approach, 1st ed., Lambert Academic Publishing, 2019.
11. Tewatia, S. K., Tewatia, K., and Tewatia, A., "Y-Y' Calculus from Soil Consolidation to other Physical Sciences," in Smart Geotechnics for Smart Societies, Taylor & Francis / CRC Press, 2023, pp. 88–105.