Pair-valued bi-matrix games and lexicographic equilibrium strategy profiles

by

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Abstract

For pair-valued bi-matrix games we show that a necessary and sufficient condition for a strategy profile to be a lexicographic equilibrium strategy profile is that the strategy profile must be an equilibrium strategy profile for the bi-matrix game corresponding to the first coordinates and if the expected value of an action for a player in this bi-matrix game is equal to the expected value of this bi-matrix game for the player, then the expected value of the same action for the same player in the bi-matrix game corresponding to the second coordinates is less than or equal to its expected value in this second bi-matrix game. This result is used in an example of a pair-valued bi-matrix game to show that it has no lexicographic equilibrium strategy profile. Subsequently we prove a necessary and sufficient condition in terms of a bi-linear programming problem with a real-valued objective function and constraints which are also real-valued. We show that a possible resolution of the coordination problem in 2×2 coordination games could be achieved by introducing the concept of a dominant player, thereby embedding the coordination game in a pair-valued bi-matrix game and then choosing the lexicographic equilibrium strategy profile of the pair-valued bi-matrix game.

Keywords: pair-valued bi-matrix games, lexicographic equilibrium strategy profile, bi-linear programming, 2×2 coordination games, dominant player

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1. Introduction: In this paper we are concerned with lexicographic equilibrium strategy profiles of pair-valued bi-matrix games. Bi-matrix games (see Chandrasekaran (nd)) are two-person/player interactive decision-making problems, where each player has a non-empty finite set of actions to choose from. The two players are generally referred to as the row player and the column player. A strategy for a player is a randomization (probability distribution) over its own finite set of actions. In classical bi-matrix game theory, each player receives a one-dimensional pay-off. While Keeney and Raiffa (1993) contains an extensive discussion on multi-attribute decision making, the possibility of bi-matrix games with multi-attribute outcomes is not mentioned in there. To the best of our knowledge, the first formal description of bi-matrix games with outcomes being pairs of real numbers (pair-valued outcomes), can be found in Radhakrishnan and Saikeerthana (2020) followed by Lahiri (2026).

Suppose there are two mutually exclusive and exhaustive states of nature for the day ahead: “rain” and “no-rain”. In each state of nature the reward is either an umbrella or nothing. If the first coordinate of an ordered pair denotes the reward if the realized state of nature is “rain” and the second coordinate of the same pair denotes the reward if the realized state of nature is “no-rain”, then there are four different alternatives: (i) an umbrella for sure, i.e., (umbrella, umbrella), (ii) an umbrella if it rains, nothing otherwise, i.e., (umbrella, nothing), (iii) nothing if it rains, an umbrella otherwise, i.e., (nothing, umbrella) and lastly (iv) nothing for sure, i.e., (nothing, nothing). It is not unreasonable to assume that a person would strictly prefer alternative (i) to alternative (ii), alternative (ii) to alternative (iii) and alternative (iii) to alternative (iv). Such preferences are referred to as “lexicographic preferences” (see Fishburn (1975), Varian (!978)).

In Radhakrishnan and Saikeerthana (2020), the pair-valued outcomes (pairs of real numbers) are ordered lexicographically, and in this context they introduce a solution concept that can be referred to as a “lexicographic equilibrium action profile” (i.e., an action profile where no player with lexicographic preferences over outcomes has any incentive to unilaterally deviate from). In Lahiri (2026), we extend the permissible solutions from “action profiles” to “strategy profiles” and introduce the solution concept- “lexicographic equilibrium strategy profile”. Note 5 of this paper spells out an interesting property of the set of all lexicographic equilibrium strategy profiles of a pair-valued bi-matrix game, namely, the set is invariant with respect to “affine transformations with strictly positive slope” of the pair-valued bi-matrix game.

Our first major result here is a necessary and sufficient condition for a strategy profile to be a lexicographic equilibrium strategy profile in terms of strategy profiles of the two bi-matrix games corresponding to each of the two coordinates of the outcomes. The condition requires that the strategy profile must be an equilibrium strategy profile for the bi-matrix game corresponding to the first coordinates and if the expected value of an action for a player in this bi-matrix game is equal to the expected value of this bi-matrix game for the player, then the expected value of the same action for the same player in the bi-matrix game corresponding to the second coordinates is less than or equal to its expected value in this second bi-matrix game. This result is used in an example of a pair-valued bi-matrix game to show that it has no lexicographic equilibrium strategy profile. In Lahiri (2026), there is an example of a pair-valued bi-matrix game for which the absence of a lexicographic equilibrium strategy profile is established by direct calculation. We confirm the non-existence of lexicographic equilibrium strategy profile for the counter-example that we provide here by direct calculation too.

Our next result (proposition 2) is a modification of the seminal characterization result in Mangasarian and Stone (1964), concerning the set of equilibrium strategy profiles of a bi-matrix game and the set of solutions of a bi-linear programming problem. Although the bi-linear programming problem in Mangasarian and Stone (1964) is a special case of the bi-linear programming we are concerned with, our result *is not* a generalization of the main theorem in Mangasarian and Stone (1964), simply because the possibility of non-existence of lexicographic equilibrium strategy profile, requires us to include an additional condition that the value of the objective function of the bi-linear programming problem at the candidate solution is zero.

The bi-linear programming problem we consider includes constraints of the form: if the expected value of an action for a player in the bi-matrix game arising from the first coordinate of the outcomes is less than the expected value of this bi-matrix game for the player, then the probability assigned by the strategy to this action must be zero. We also show that for a completely mixed strategy profile, such constraints are not required for proving that the strategy profile is a lexicographic equilibrium strategy profile.

Our final result, is a proposition similar to proposition 2, except that the kind of constraints discussed above are omitted and the objective function instead of being real-valued, is pair-valued. The maximization of the pair-valued objective function is with respect to the lexicographic preference relation, which is a linear ordering. This proposition is proposition 3 in Lahiri (2026) and hence we now have a new proof of proposition 3 in Lahiri (2026).

In the final section of this paper, we show that for a bi-matrix game with multiple pure strategy equilibrium profiles, we can introduce the concept of a dominant player by converting the bi-matrix game to a pair-valued bi-matrix game. In this bi-matrix game, the dominant player has the first coordinates equal to 1 for exactly one action and 0 for all other actions. The first coordinates for the other player are 0 for all actions. The second coordinates for both players are as in the bi-matrix game with multiple pure strategy equilibrium profiles. A lexicographic equilibrium strategy profile for the pair-valued bi-matrix game must be one which assigns probability 1 to the action of the dominant player which has non-zero first coordinates. This could be one way of resolving coordination problems in bi-matrix games with multiple pure strategy equilibrium profiles.

A generalization of pair-valued bi-matrix game, referred to as "multi-attribute bi-matrix games" and substantial analysis in the context of the latter is available at the following link: https://doi.org/10.31224/7402.

2. The Framework of Pair-valued Bi-matrix Games: For positive integers m, n consider an interactive decision making context (game) with two players/participants/decision-makers who we denote by 'a' and 'b' respectively, and where player 'a' has 'm' actions to choose from and 'b' has 'n' actions to choose from. A **strategy for player 'a'** is a point in $\Delta^{m-1} = \{x \in \mathbb{R}^m_+ | \sum_{i=1}^{m} x_i = 1\}$ and a **strategy for player 'b'** is a point in $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ | \sum_{j=1}^{n} y_j = 1\}$.

A strategy for a player is said to be **completely mixed** if all its coordinates are positive.

The above concept can be found in Kaplansky (1945).

A strategy for player 'a' is a probability distribution on {1, ..., m}, with each coordinate of the strategy denoting the probability with which player 'a' chooses the corresponding action. A strategy for player 'b' is a probability distribution on {1, ..., n}, with each coordinate of the strategy denoting the probability with which player 'b' chooses the corresponding action.

Notation 1: For $i \in \{1, \ldots, m\}$, let $E^{(m,i)}$ be the **m-dimensional i^{th} unit coordinate column vector** (i.e., 1 in its i^{th} coordinate and 0 in all other coordinates) and let $E^{(m)} = \sum_{h=1}^{m} E^{(m,h)}$ be the **m-dimensional sum vector**. For $j \in \{1, \ldots, n\}$, let $E^{(n,j)}$ be the **n-dimensional j^{th} unit coordinate column vector** (i.e., 1 in its j^{th} coordinate and 0 in all other coordinates) and let $E^{(n)} = \sum_{k=1}^{n} E^{(n,k)}$ be the n-**dimensional sum vector**.▌

If a strategy x for player 'a' is such that $x = E^{(m,i)}$ for some $i \in \{1, \ldots, m\}$, then x is said to be the **i^{th} pure strategy for the player 'a'**.

If a strategy y for player 'b' is such that $y = E^{(n,j)}$ for some $j \in \{1, \ldots, n\}$, then y is said to be the **j^{th} pure strategy for the player 'b'**.

A pair (x, y) where x is a strategy for player 'a' and y is a strategy for player 'b' is said to be a **strategy profile**.

As in Kaplansky (1945), a strategy profile is said to be **(a) completely mixed (strategy profile)** if the strategies for both players are completely mixed.

If both x and y are pure strategies then the strategy profile (x, y) is said to be a **pure strategy profile**.

Note 1: If (x, y) is a pure strategy profile, then we may represent it as (i, j) if x is the i^{th} pure strategy for player 'a' and j is the j^{th} pure strategy for player 'b'. $\blacksquare$

Let $\mathbb{R}^{m\times n}$ be the set of all m×n matrices.

Given $C \in \mathbb{R}^{m\times n}$ and $(i, j) \in \{1, \ldots, m\}\times\{1, \ldots, n\}$, let c_{ij} denote the $(i, j)^{th}$ entry of C, C_i denote the i^{th} row of C and C^j denote the j^{th} column of C.

For our purposes here, a member of $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ will be referred to as a **bi-matrix game** and $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ is the **set of all bi-matrix games**.

Given $(A, B) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ and $(i, j) \in \{1, \ldots, m\}\times\{1, \ldots, n\}$, if player 'a' (who may also be referred to as the **row player**) chooses row i and player 'b' (who may also be referred to as the **column player**) chooses column j, the payoff received by the row player is a_{ij} and payoff received by the column player is b_{ij}. A is the **payoff matrix of the row player** and B **is the payoff matrix of the column player**.

Given a bi-matrix game (A, B) a strategy profile (x, y) is said to be an **equilibrium strategy profile for (A, B)**, if for all $(i, j) \in \{1, \ldots, m\}\times\{1, \ldots, n\}$: $A_i y \leq x^T A y$ and $x^T B^j \leq x^T B y$.

An equilibrium strategy profile (x, y) for (A, B) is said to be a **pure strategy equilibrium profile for (A, B)** if (x, y) is a pure strategy profile.

It is quite possible that at an equilibrium strategy profile (x, y) for (A, B), x is a pure strategy, where as y is *not* as the following example reveals.

Example 1: Let $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. (x, y) is an equilibrium strategy profile for (A, B) if and only if $x_1 = 1$ and $y \in \Delta^1$. $\blacksquare$

Let $\mathbb{R}^{(2,\, m\times n)}$ denote the set of all matrices m×n matrices whose entries are pairs of real number.

If $V\epsilon\mathbb{R}^{(2,\, m\times n)}$, then for all $(i, j)\epsilon\{1, \ldots, m\}\times\{1, \ldots, n\}$, the typical (though not invariable) notations are as follows: $(i, j)^{th}$ entry of V is denoted by $(v_{ij}^{(1)}, v_{ij}^{(2)})$, the i^{th} row of V is denoted by V_i and the j^{th} column of V is denoted by V^j.

For our purposes here, a member of $\mathbb{R}^{(2,\, m\times n)} \times \mathbb{R}^{(2,\, m\times n)}$ will be referred to as a **pair-valued bi-matrix game** and $\mathbb{R}^{(2,\, m\times n)} \times \mathbb{R}^{(2,\, m\times n)}$ as the **set of all pair-valued bi-matrix games**.

Let $\succcurlyeq$ be a binary relation on $\mathbb{R}^2$ such that for all $(\gamma_1, \eta_1), (\gamma_2, \eta_2)\epsilon\mathbb{R}^2$: $(\gamma_1, \eta_1) \succcurlyeq (\gamma_2, \eta_2)$ <u>if and only if</u> either $\gamma_1 > \gamma_2$ or $\gamma_1 = \gamma_2$ and $\eta_1 \geq \eta_2$.

Clearly $\succcurlyeq$ is a linear order on $\mathbb{R}^2$. It is generally known as the **lexicographic preference ordering (lexicographic preferences) on** $\mathbb{R}^2$.

Let $\succ$ denote the asymmetric part of $\succcurlyeq$.

For $v\epsilon\mathbb{R}^2$, we shall refer to $\{u\epsilon\mathbb{R}^2 | v \succcurlyeq u\}$ as the "**no-better than**" (**NBT**) **set** of v.

Note 2: It is easy to see that if $u^{(1)}, u^{(2)}, v\epsilon\mathbb{R}^2$ and $\theta\epsilon[0, 1)$, then $[v \succcurlyeq u^{(1)}, v\succ u^{(2)}]$ implies $[v \succ \theta u^{(1)} + (1-\theta)u^{(2)}]$. Thus, the "no-better than" (NBT) sets for lexicographic preferences on $\mathbb{R}^2$ are *strictly convex*. ▌

Notation 2: Given a pair-valued bi-matrix game (V(a), V(b)), for $r\epsilon\{1, 2\}$ and $\alpha\epsilon\{a, b\}$, let $V^{(r)}(\alpha)$ be the m×n real-valued matrix whose $(i, j)^{th}$ term for $(i, j)\epsilon\{1, \ldots, m\}\times\{1, \ldots, n\}$ is $v_{ij}^{(r)}(\alpha)$. The i^{th} row of $V^{(r)}(\alpha)$ for $i\epsilon\{1, \ldots, m\}$ is denoted by $V_i^{(r)}(\alpha)$ and the j^{th} column of $V^{(r)}(\alpha)$ for $j\epsilon\{1, \ldots, n\}$ is denoted by $V^{(r)j}(\alpha)$. For $(x, y)\epsilon\Delta^{m-1}\times\Delta^{n-1}$ and $\alpha\epsilon\{a, b\}$, $x^TV(\alpha)y = \sum_{i=1}^{m}\sum_{j=1}^{n} x_iy_j(v_{ij}^{(1)}(\alpha), v_{ij}^{(2)}(\alpha)) = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_iy_jv_{ij}^{(1)}(\alpha), x_iy_jv_{ij}^{(2)}(\alpha))$. ▌

Given a pair-valued bi-matrix game (V(a), V(b)), a strategy profile (x, y) is said to be a **lexicographic equilibrium strategy profile** for (V(a), V(b)) if $x^TV(a)y \succcurlyeq V_i(a)y$ for all $i\epsilon\{1, \ldots, m\}$and $x^TV(b)y \succcurlyeq x^TV^j(b)$ for all $j\epsilon\{1, \ldots, n\}$.

Note 3: Since, NBT sets for lexicographic preferences are strictly convex, (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)) <u>if and only if</u> for all $w\epsilon\Delta^{m-1}$ and $z\epsilon\Delta^{n-1}$: $x^TV(a)y \succcurlyeq w^TV(a)y$ and $x^TV(b)y \succcurlyeq x^TV(b)z$. ▌

Note 4: Since, NBT sets for lexicographic preferences are strictly convex, whenever (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)), it must be the case that $[x^TV(a)y \succ V_i(a)y$ implies $x_i = 0$ (and hence $x_i > 0$ implies $x^TV(a)y = V_i(a)y)]$

and [$x^TV(b)y \succ x^TV^j(b)$ implies $y_j = 0$ (and hence $y_j > 0$ implies $x^TV(b)y = x^TV^j(b)$)]. ▮

Note 5: (Invariance of lexicographic equilibrium strategy profile with respect to "affine transformations with strictly positive slope") Given a pair-valued bi-matrix game (V(a), V(b)) let (W(a), W(b))$\epsilon\mathbb{R}^{(2,\, m\times n)} \times \mathbb{R}^{(2,\, m\times n)}$ satisfy the following property: for all (r, α)ϵ{1, 2}×{a, b} and (i, j)ϵ{1, …, m}×{1, ..., n}, the (i, j)th term of $W^{(r)}(\alpha)$, $w_{ij}^{(r)}(\alpha) = w_{11}^{(r)}(\alpha)$. For each (r, α)ϵ{1, 2}×{a, b}, let $d^{(r)}(\alpha)$ be a *strictly positive* real number. Let (U(a), U(b))$\epsilon\mathbb{R}^{(2,\, m\times n)} \times \mathbb{R}^{(2,\, m\times n)}$ be such that for each (r, α)ϵ{1, 2}×{a, b}: $U^{(r)}(\alpha) = d^{(r)}(\alpha)V^{(r)}(\alpha) + W^{(r)}(\alpha)$. Then it is easily verified that a strategy profile (x, y) is a lexicographic equilibrium strategy for (V(a), V(b)) if and only if it is a lexicographic equilibrium strategy profile. ▮

3. A first necessary and sufficient condition for lexicographic equilibrium strategy profile: Given a strategy profile (w, z), let a(w, z) = {iϵ{1, …, m}|$w^TV^{(1)}(a)z = V_i^{(1)}(a)z$} and b(x,y) = {jϵ{1, …, n}|$w^TV^{(1)}(b)z = w^TV^{(1)j}(b)$}.

The following result is an immediate consequence of the definition of lexicographic preferences.

Proposition 1: A strategy profile (x, y) is a lexicographic equilibrium strategy profile for the pair-valued bi-matrix game (V(a), V(b)) if and only if it satisfies the following three conditions:

(i) (x, y) is an equilibrium strategy profile for the bi-matrix game ($V^{(1)}(a)$, $V^{(1)}(b)$).

(ii) $x^TV^{(2)}(a)y \geq V_i^{(2)}(a)y$ for all iϵa(x, y).

(iii) $x^TV^{(2)}(b)y \geq x^TV^{(2)j}(b)$ for all jϵb(x, y).

Proof: Let (x, y) be a strategy profile.

Suppose (x, y) is a lexicographic equilibrium strategy profile. Thus, $x^TV(a)y \succcurlyeq V_i(a)y$ for all iϵ{1, …, m} and $x^TV(b)y \succcurlyeq x^TV^j(b)$ for all jϵ{1, …, n}.

If for some iϵ{1, …, m}, $V_i^{(1)}(a)y > x^TV^{(1)}(a)y$, then for the same i, it must be that $V_i(a)y \succ x^TV(a)y$, contrary to our assumption about (x, y). Thus, $x^TV^{(1)}(a)y \geq V_i^{(1)}(a)y$ for all iϵ{1, …, m}.

If for some jϵ{1, …, n}, $x^TV^{(1)j}(b) > x^TV^{(1)}(b)y$, then for the same j, it must be that $x^TV^j(b) \succ x^TV(b)y$, contrary to our assumption about (x, y). Thus, $x^TV^{(1)}(b)y \geq x^TV^{(1)j}(b)$ for all jϵ{1, …, n}.

Thus, (x, y) is an equilibrium strategy profile for the bi-matrix game ($V^{(1)}(a)$, $V^{(1)}(b)$).

For any $i \in a(x, y)$, $V_i^{(2)}(a)y > x^T V^{(2)}(a)y$ is equivalent to the statement $V_i^{(1)}(a)y = x^T V^{(1)}(a)y$ and $V_i^{(2)}(a)y > x^T V^{(2)}(a)y$, thereby implying $V_i(a)y \succ x^T V(a)y$, contrary to what we have assumed about (x, y). Thus, $x^T V^{(2)}(a)y \geq V_i^{(2)}(a)y$ for all $i \in a(x, y)$.
Similarly, $x^T V^{(2)}(b)y \geq x^T V^{(2)j}(b)$ for all $j \in b(x, y)$.

Now suppose (x, y) satisfies (i), (ii) and (iii) in the statement of the proposition.

If $V_i(a)y \succ x^T V(a)y$ for some $i \in \{1, \ldots, m\}$, then <u>either</u> $V_i^{(1)}(a)y > x^T V^{(1)}(a)y$ in which case (x, y) is not an equilibrium strategy profile for the bi-matrix game $(V^{(1)}(a), V^{(1)}(b))$ contradicting (i) <u>or</u> $V_i^{(1)}(a)y = x^T V^{(1)}(a)y$, (i.e., $i \in a(x, y)$) and $V_i^{(2)}(a)y > x^T V^{(2)}(a)y$ contradicting (ii).

Thus, $x^T V(a)y \succcurlyeq V_i(a)y$ for all $i \in \{1, \ldots, m\}$.

Similarly, $x^T V(b)y \succcurlyeq x^T V^j(b)$ for all $j \in \{1, \ldots, n\}$.

Thus, (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)). Q.E.D.

4. The possibility of non-existence of lexicographic equilibrium: In this section we will use proposition 1 to show the possibility of non-existence of a lexicographic equilibrium strategy profile.

For m = n = 2, let (V(a), V(b)) be the pair-valued bi-matrix game such that:

$V(a) = \begin{bmatrix} (1,0) & (-1,0) \\ (-1,1) & (1,1) \end{bmatrix}$, $V^{(1)}(b) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $V^{(2)}(b) \in \mathbb{R}^{2\times 2}$.

The bi-matrix game $(V^{(1)}(a), V^{(1)}(b))$ is well and widely known as "*matching pennies*" (see lesson 1.5 in Spaniel (2011-2013)) and it is also well known that it has a unique equilibrium strategy profile (x^*, y^*), where $x^* = (\frac{1}{2}, \frac{1}{2})$ and $y^* = (\frac{1}{2}, \frac{1}{2})$.

Thus, $a(x^*, y^*) = \{1, 2\} = b(x^*, y^*)$.

Thus, by proposition 1, if (V(a), V(b)) were to have any lexicographic equilibrium strategy profile it would have to be (x^*, y^*).

$x^{*T} V^{(2)}(a)y^* = \frac{1}{2} < 1 = (1, 1)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = V_2^{(2)}(a)y$, contradicting requirement (ii) in the statement of proposition 1.

Thus, by proposition 1, (V(a), V(b)) does not have any lexicographic equilibrium strategy profile. We will now confirm this conclusion by direct calculation.

For $y \in \Delta^1$, $V_1^{(1)}(a)y = y_1 - (1-y_1) = 2y_1 - 1$ and $V_2^{(1)}(a)y = -y_1 + (1-y_1) = 1 - 2y_1$.

$V_1^{(1)}(a)y > V_2^{(1)}(a)y$ if and only if $y_1 \epsilon (\frac{1}{2}, 1]$ and $V_2^{(1)}(a)y > V_1^{(1)}(a)y$ if and only if $y_1 \epsilon [0, \frac{1}{2})$.

Thus, for $y \epsilon \Delta^1$ with $y_1 \epsilon (\frac{1}{2}, 1]$, $E^{(2,1)T}V^{(1)}(a)y > x^T V^{(1)}(a)y$ for all $x \epsilon \Delta^1$ with $x_1 < 1$.

Thus, $E^{(2,1)T}V(a)y \succ x^T V(a)y$ for all $x \epsilon \Delta^1$ with $x_1 < 1$

However, $E^{(2,1)T}V^{(2)}(b)E^{(2,2)} = 1 > E^{(2,1)T}V^{(2)}(b)y = 1 - 2y_1$ for all $y \epsilon \Delta^1$ with $y_1 \epsilon (\frac{1}{2}, 1]$.

Thus, $E^{(2,1)T}V(b)E^{(2,2)} \succ E^{(2,1)T}V(b)y$ for all $y \epsilon \Delta^1$ with $y_1 \epsilon (\frac{1}{2}, 1]$

Hence, a strategy profile (x, y) with $y_1 \epsilon (\frac{1}{2}, 1]$ cannot be a lexicographic equilibrium strategy profile for (V(a), V(b)).

For $y \epsilon \Delta^1$ with $y_1 \epsilon [0, \frac{1}{2})$, $E^{(2,2)T}V^{(1)}(a)y > x^T V^{(1)}(a)y$ for all $x \epsilon \Delta^1$ with $x_1 > 0$.

Thus, for $y \epsilon \Delta^1$ with $y_1 \epsilon [0, \frac{1}{2})$, $E^{(2,2)T}V(a)y \succ x^T V(a)y$ for all $x \epsilon \Delta^1$ with $x_1 > 0$.

However, $E^{(2,2)T}V^{(1)}(b)E^{(2,1)} = 1 > E^{(2,2)T}V^{(1)}(b)y = 2y_1 - 1$ for all $y \epsilon \Delta^1$ with $y_1 \epsilon [0, \frac{1}{2})$.

Thus, $E^{(2,2)T}V(b)E^{(2,1)} \succ E^{(2,2)T}V(b)y$ for all $y \epsilon \Delta^1$ with $y_1 \epsilon [0, \frac{1}{2})$.

Hence, a strategy profile (x, y) with $y_1 \epsilon [0, \frac{1}{2})$ cannot be a lexicographic equilibrium strategy profile for (V(a), V(b)).

For $y \epsilon \Delta^1$ with $y_1 = \frac{1}{2}$ (i.e., $y = (\frac{1}{2}, \frac{1}{2})$), $V_1^{(1)}(a)y = V_2^{(1)}(a)y = 0$.

Since, $E^{(2,2)T}V^{(2)}(a)y = 1 > x^T V^{(2)}(a)y$ for all $x \epsilon \Delta^1$ with $x_1 > 0$, we get $E^{(2,2)T}V(a)y \succ x^T V(a)y$ for all $x \epsilon \Delta^1$ with $x_1 > 0$.

However, $E^{(2,2)T}V^{(1)}(b)E^{(2,1)} = 1 > 0 = E^{(2,2)T}V^{(1)}(b)y$ implies $E^{(2,2)T}V(b)E^{(2,1)} \succ E^{(2,2)T}V(b)y$.

Hence, a strategy profile (x, y) with $y_1 = \frac{1}{2}$ cannot be a lexicographic equilibrium strategy profile for (V(a), V(b)).

Thus, *there does not exist* any lexicographic equilibrium strategy profile for (V(a), V(b)).

5. Necessary and sufficient condition for lexicographic equilibrium strategy profile in terms of real-valued bi-linear programming: In this section we provide a necessary and sufficient condition for a strategy profile to be a lexicographic equilibrium strategy profile. This characterization result is closer in spirit to the one for equilibrium strategy profiles in Mangasarian and Stone (1964) than the corresponding result in Lahiri (2026). In this proposition, the objective function is real-valued, and both the left-hand side and the right-hand side of the constraints are real-valued.

Proposition 2: Let (x, y) be a strategy profile. (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)) if and only if there exists real numbers $u^{(r)*}(\alpha)$, for $\alpha\epsilon\{a, b\}$, $r\epsilon\{1, 2\}$ that along with x, y satisfy the following properties:

(i) x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$ solve the bi-linear programming problem:

Maximize $w^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))z - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)}(\alpha)$, subject to $V_i^{(1)}(a)z - u^{(1)}(a) \le 0$ for $i\epsilon\{1, \ldots, m\}$, $V_i^{(2)}(a)z - u^{(2)}(a) \le 0$ for $i\epsilon a(w, z)$, $w_i = 0$ for $i\notin a(w, z)$, $w^TV^{(1)j}(b) - u^{(1)}(b) \le 0$ for $j\epsilon\{1, \ldots, n\}$, $w^TV^{(2)j}(b) - u^{(2)}(b) \le 0$ for $j\epsilon b(w, z)$, $z_j = 0$ for $j\notin b(w, z)$, $w\epsilon\Delta^{m-1}$, $z\epsilon\Delta^{n-1}$, $u^{(r)}(\alpha)\epsilon\mathbb{R}$, for $\alpha\epsilon\{a, b\}$, $r\epsilon\{1, 2\}$.

(ii) $x^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))y - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)*}(\alpha) = 0$.

Proof: Observation: Suppose (w, z) is a strategy profile that satisfies the following constraints: $V_i^{(1)}(a)z - u^{(1)}(a) \le 0$ for $i\epsilon\{1, \ldots, m\}$, $V_i^{(2)}(a)z - u^{(2)}(a) \le 0$ for $i\epsilon a(w, z)$, $w_i = 0$ for $i\notin a(w, z)$, $w^TV^{(1)j}(b) - u^{(1)}(b) \le 0$ for $j\epsilon\{1, \ldots, n\}$, $w^TV^{(2)j}(b) - u^{(2)}(b) \le 0$ for $j\epsilon b(w, z)$, $z_j = 0$ for $j\notin b(w, z)$, $w\epsilon\Delta^{m-1}$, $z\epsilon\Delta^{n-1}$, $u^{(r)}(\alpha)\epsilon\mathbb{R}$, for $\alpha\epsilon\{a, b\}$, $r\epsilon\{1, 2\}$.

Then, $w^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))z - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)}(\alpha) =$

$\sum_{i\epsilon a(w,z)} w_i(\sum_{r\epsilon\{1,2\}}(V_i^{(r)}(a)z - u^{(r)}(a)) + \sum_{j\epsilon b(w,z)}(\sum_{r\epsilon\{1,2\}}(w^TV^{(r)j}(b) - u^{(r)}(b))z_j$

≤ 0. ▌

Suppose (x, y) is a lexicographic equilibrium strategy profile. For $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$, let $u^{(r)*}(\alpha) = x^TV^{(r)}(\alpha)y$.

Then, clearly, x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$ satisfies (ii).

By the definition of $u^{(r)*}(\alpha)$ for $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$, we get $x^T(\sum_{\alpha\epsilon\{a,b\}} V(\alpha))y - \sum_{\alpha\epsilon\{a,b\}}(u^{(1)*}(\alpha), u^{(2)*}(\alpha)) = (0, 0)$.

Further, for all $i\epsilon\{1, \ldots, m\}$, either $u^{(1)*}(a) = x^TV^{(1)}(a)y > V_i^{(1)}(a)y$ or $u^{(1)*}(a) = x^TV^{(1)}(a)y = V_i^{(1)}(a)y$ (i.e., $i\epsilon a(x, y)$) and $u^{(2)*}(a) = x^TV^{(2)}(a)y \ge V_i^{(2)}(a)y$; for all $j\epsilon\{1, \ldots, n\}$, either $u^{(1)*}(b) = x^TV^{(1)}(b)y > x^TV^{(1)j}(b)$ or $u^{(1)*}(b) = x^TV^{(1)}(b)y = x^TV^{(1)j}(b)$ (i.e., $j\epsilon b(x, y)$) and $u^{(2)*}(b) = x^TV^{(2)}(b)y \ge x^TV^{(2)j}(b)$.

Thus, x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$ satisfies all the constraints of the optimization problem in (i).

Combined with the observation right at the beginning of the proof of this proposition and that x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$ satisfies (ii), it follows that x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$ solve the bi-linear programming problem in (i).

Thus, x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$ satisfy both (i) and (ii).

Now suppose x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$satisfy (i) and (ii).

The constraints in (i) imply $V_i^{(1)}(a)$y - $u^{(1)*}(a) \leq 0$ for $i\epsilon\{1, \ldots, m\}$, $V_i^{(2)}(a)$y - $u^{(2)*}(a) \leq 0$ for $i\epsilon a(x, y)$, $x_i = 0$ for $i\notin a(x, y)$, $x^T V^{(1)j}(b) - u^{(1)*}(b) \leq 0$ for $j\epsilon\{1, \ldots, n\}$, $x^T V^{(2)j}(b) - u^{(2)*}(b) \leq 0$ for $j\epsilon b(x, y)$, $y_j = 0$ for $j\notin b(x, y)$, $x\epsilon\Delta^{m-1}$, $y\epsilon\Delta^{n-1}$, $u^{(r)*}(\alpha)\epsilon\mathbb{R}$, for $\alpha\epsilon\{a, b\}$, $r\epsilon\{1, 2\}$.

Thus, $x^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))y - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)*}(\alpha) =$
$\sum_{i\epsilon a(x,y)} x_i(\sum_{r\epsilon\{1,2\}} (V_i^{(r)}(a)y - u^{(r)*}(a)) + \sum_{j\epsilon b(x,y)} (\sum_{r\epsilon\{1,2\}} (x^T V^{(r)j}(b) - u^{(r)*}(b))y_j \leq 0$.

If for any $i\epsilon\{1, \ldots, m\}$, $x_i > 0$ and $V_i^{(1)}(a)y - u^{(1)*}(a) < 0$, then
$\sum_{i\epsilon a(x,y)} x_i(\sum_{r\epsilon\{1,2\}} (V_i^{(r)}(a)y - u^{(r)*}(a)) + \sum_{j\epsilon b(x,y)} (\sum_{r\epsilon\{1,2\}} (x^T V^{(r)j}(b) - u^{(r)*}(b))y_j < 0$, contradicting (ii).

Thus, $x_i > 0$ implies $V_i^{(1)}(a)y - u^{(1)*}(a) = 0$.

Thus, $u^{(1)*}(a) = x^T V^{(1)}(a)y$.

Since $x_i > 0$ implies $i\epsilon a(x, y)$, if for $x_i > 0$, $V_i^{(2)}(a)y - u^{(2)*}(a) < 0$, then
$\sum_{i\epsilon a(x,y)} x_i(\sum_{r\epsilon\{1,2\}} (V_i^{(r)}(a)y - u^{(r)*}(a)) + \sum_{j\epsilon b(x,y)} (\sum_{r\epsilon\{1,2\}} (x^T V^{(r)j}(b) - u^{(r)*}(b))y_j < 0$, contradicting (ii).

Thus, $x_i > 0$ implies $V_i^{(2)}(a)y - u^{(2)*}(a) = 0$.

Thus, $u^{(2)*}(a) = x^T V^{(2)}(a)y$.

Thus, $V_i^{(1)}(a)y \leq x^T V^{(1)}(a)y$ for all $i\epsilon\{1, \ldots, m\}$ and $V_i^{(2)}(a)y \leq x^T V^{(2)}(a)y$ for all $i\epsilon a(x, y)$.

Hence, $x^T V(a)y \succcurlyeq V_i(a)y$ for all $i\epsilon\{1, \ldots, m\}$.

A similar argument leads to $u^{(1)*}(b) = x^T V^{(1)}(b)y$ and $u^{(2)*}(b) = x^T V^{(1)}(b)y$.

Thus, $x^T V^{(1)j}(b) \leq x^T V^{(1)}(b)y$ for all $j\epsilon\{1, \ldots, n\}$ and $x^T V^{(2)j}(b) \leq x^T V^{(2)}(b)y$ for all $j\epsilon b(x, y)$.

Thus, $x^T V(b)y \succcurlyeq x^T V^j(b)$ for all $j\epsilon\{1, \ldots, n\}$.

Thus, (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)). Q.E.D.

For "completely mixed" strategy profiles, we have the following result which does not require "$w_i = 0$ for $i \notin a(w, z)$" and "$z_j = 0$ for $j \notin b(w, z)$" in the constraints of the bi-linear programming problem.

Proposition 3: Let (x, y) be a "completely mixed" strategy profile. (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)) <u>if and only if</u> there exists real numbers $u^{(r)*}(\alpha)$, for $\alpha \epsilon \{a, b\}$, $r \epsilon \{1, 2\}$ that along with x, y satisfy the following properties:

(i) x, y, $u^{(r)*}(\alpha)$, $(r, \alpha) \epsilon \{1, 2\} \times \{a, b\}$ solve the bi-linear programming problem:

Maximize $w^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))z - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)}(\alpha)$, subject to $V_i^{(1)}(a)z - u^{(1)}(a) \leq 0$ and $V_i^{(2)}(a)z - u^{(2)}(a) \leq 0$ for $i \epsilon \{1, \ldots, m\}$, $w^TV^{(1)j}(b) - u^{(1)}(b) \leq 0$ and $w^TV^{(2)j}(b) - u^{(2)}(b) \leq 0$ for $j \epsilon \{1, \ldots, n\}$, $w \epsilon \Delta^{m-1}$, $z \epsilon \Delta^{n-1}$, $u^{(r)}(\alpha) \epsilon \mathbb{R}$, for $\alpha \epsilon \{a, b\}$, $r \epsilon \{1, 2\}$.

(ii) $x^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))y - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)*}(\alpha) = 0$.

Proof: If $[V_i^{(1)}(a)z - u^{(1)}(a) \leq 0$ and $V_i^{(2)}(a)z - u^{(2)}(a) \leq 0$ for $i \epsilon \{1, \ldots, m\}$, $w^TV^{(1)j}(b) - u^{(1)}(b) \leq 0$ and $w^TV^{(2)j}(b) - u^{(2)}(b) \leq 0$ for $j \epsilon \{1, \ldots, n\}$, $w \epsilon \Delta^{m-1}$, $z \epsilon \Delta^{n-1}$, $u^{(r)}(\alpha) \epsilon \mathbb{R}$, for $\alpha \epsilon \{a, b\}$, $r \epsilon \{1, 2\}]$, then clearly $[w^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))z - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)}(\alpha) \leq 0]$.

If (x, y) is a lexicographic equilibrium strategy profile then along with $u^{®*}(\alpha) = x^TV^{(r)}(\alpha)y$ for $(r, \alpha) \epsilon \{1, 2\} \times \{a, b\}$, x, y satisfies (i) and (ii) in the statement of this corollary.

Suppose, x, y, $u^{(r)*}(\alpha)$ for $(r, \alpha) \epsilon \{1, 2\} \times \{a, b\}$ satisfy (i) and (ii).

Then, $x^TV^{(r)}(\alpha)y - u^{(r)*}(\alpha) \leq 0$ for $(r, \alpha) \epsilon \{1, 2\} \times \{a, b\}$.

Since x is completely mixed if for any $i \epsilon \{1, \ldots, m\}$, $V_i^{(r)}(a)y - u^{(r)*}(a) < 0$, then $x^TV^{(r)}(a)y - u^{(r)*}(b) < 0$, leading to $x^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))y - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)*}(\alpha) < 0$, thereby contradicting (ii).

Thus, it must be the case that $V_i^{(r)}(a)y = u^{(r)*}(a)$ for all $i \epsilon \{1, \ldots, m\}$.

Since y is completely mixed if for any $j \epsilon \{1, \ldots, m\}$, $x^TV^{(r)j}(b) - u^{(r)*}(b) < 0$, then $x^TV^{(r)}(b)y - u^{(r)*}(b) < 0$, leading to $x^T(\sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} V^{(r)}(\alpha))y - \sum_{(r,\alpha)\epsilon\{1,2\}\times\{a,b\}} u^{(r)*}(\alpha) < 0$, thereby contradicting (ii).

Thus, it must be the case that $x^TV^{(r)j}(b) = u^{(r)*}(b)$ for all $j \epsilon \{1, \ldots, n\}$.

Thus, $u^{(r)*}(\alpha) = x^TV^{(r)}(\alpha)y$ for $(r, \alpha) \epsilon \{1, 2\} \times \{a, b\}$.

Thus, $V_i^{(r)}(a)$y = $x^TV^{(r)}(a)y$ for i$\in${1, …, m} and r$\in${1, 2} and $x^TV^{(r)j}(b) = x^TV^{(r)}(b)y$ for all j$\in${1, …, n} and r$\in${1, 2}.

Thus, (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)). Q.E.D.

6. Necessary and sufficient condition for lexicographic equilibrium strategy profile in terms of pair-valued bi-linear programming: What would be the consequence of dropping the conditions "$w_i = 0$ for i$\notin$a(w, z)" and "$z_j = 0$ for j$\notin$b(w, z)" from the constraints in (i) of proposition 2?

If for some i$\notin$a(w, z) it is the case that $w_i > 0$, then for that i, $w_iV_i^{(1)}(a)z - w_i u^{(1)}(a) < 0$. This combined with the other inequalities would lead to $w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(1)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(1)}(\alpha) < 0$, so that $(0, 0) \succ (w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(1)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(1)}(\alpha)$, $w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(2)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(2)}(\alpha))$.

Similarly, if for some j$\notin$b(w, z) it is the case that $z_j > 0$, then for that j, $w^TV^{(1)j}(b)z_j - u^{(1)}(b)z_j < 0$. This combined with the other inequalities would once again lead to $w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(1)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(1)}(\alpha) < 0$, so that $(0, 0) \succ (w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(1)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(1)}(\alpha)$, $w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(2)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(2)}(\alpha))$.

Thus, if we dropped the conditions "$w_i = 0$ for i$\notin$a(w, z)" and "$z_j = 0$ for j$\notin$b(w, z)", then: [$V_i^{(1)}(a)z - u^{(1)}(a) \leq 0$ for i$\in${1, …, m}, $V_i^{(2)}(a)z - u^{(2)}(a) \leq 0$ for i$\in$a(w, z), $w^TV^{(1)j}(b) - u^{(1)}(b) \leq 0$ for j$\in${1, …, n}, $w^TV^{(2)j}(b) - u^{(2)}(b) \leq 0$ for j$\in$b(w, z), w$\in\Delta^{m-1}$, z$\in\Delta^{n-1}$, $u^{(r)}(\alpha)\in\mathbb{R}$, for $\alpha\in${a, b}, r$\in${1, 2}] *implies* [$(0, 0) \succcurlyeq (w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(1)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(1)}(\alpha)$, $w^T(\sum_{\alpha\epsilon\{a,b\}} V^{(2)}(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} u^{(2)}(\alpha))$].

Thus we have the following proposition which characterizes lexicographic equilibrium strategy profiles using "pair-valued bi-linear programming problem".

Proposition 4: Let (x, y) be a strategy profile. (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)) <u>if and only if</u> there exists real numbers $u^{(r)*}(\alpha)$, for $\alpha\in${a, b}, r$\in${1, 2} that along with x, y satisfy the following properties:

(i^*) x, y, $u^{(r)*}(\alpha)$, (r, α)$\in${1, 2}×{a, b} solve the *pair-valued bi-linear programming problem*: Maximize (with respect to $\succcurlyeq$) $w^T(\sum_{\alpha\epsilon\{a,b\}} V(\alpha))z - \sum_{\alpha\epsilon\{a,b\}} (u^{(1)}(\alpha), u^{(2)}(\alpha))$, subject to $V_i^{(1)}(a)z - u^{(1)}(a) \leq 0$ for i$\in${1, …, m}, $V_i^{(2)}(a)z - u^{(2)}(a) \leq 0$ for i$\in$a(w, z), $w^TV^{(1)j}(b) - u^{(1)}(b) \leq 0$ for j$\in${1, …, n}, $w^TV^{(2)j}(b) - u^{(2)}(b) \leq 0$ for j$\in$b(w, z), w$\in\Delta^{m-1}$, z$\in\Delta^{n-1}$, $u^{(r)}(\alpha)\in\mathbb{R}$, for $\alpha\in${a, b}, r$\in${1, 2}.

(ii^*) $x^T(\sum_{\alpha\epsilon\{a,b\}} V(\alpha))y - \sum_{\alpha\epsilon\{a,b\}} (u^{(1)*}(\alpha), u^{(2)*}(\alpha)) = (0, 0)$.

Proof: In view of the discussion preceding the statement of the proposition we know that if (w, z) is a strategy profile satisfying the constraints of the pair-valued bi-linear programming problem in (i^*), then $(0, 0) \succcurlyeq w^T(\sum_{\alpha\epsilon\{a,b\}} V(\alpha))z -$

$\sum_{\alpha\epsilon\{a,b\}} (u^{(1)}(\alpha), u^{(2)}(\alpha))$.

The proof that if (x, y) is a lexicographic equilibrium strategy profile, then it satisfies (i^*) and (ii^*) is exactly the same as the proof that if (x, y) is a lexicographic equilibrium strategy profile, then it satisfies (i) and (ii) of proposition 2.

Now suppose x, y, $u^{(r)*}(\alpha)$, $(r, \alpha)\epsilon\{1, 2\}\times\{a, b\}$satisfy ($i^*$) and ($ii^*$).

$V_i^{(1)}(a)y - u^{(1)*}(a) \leq 0$ for $i\epsilon\{1, \ldots, m\}$ implies $x^TV^{(1)}(a)y - u^{(1)*}(a) \leq 0$.

$x^TV^{(1)j}(b) - u^{(1)*}(b) \leq 0$ for $j\epsilon\{1, \ldots, n\}$ implies $x^TV^{(1)}(b)y - u^{(1)*}(b) \leq 0$.

If $x_i > 0$ for some i satisfying $V_i^{(1)}(a)y - u^{(1)*}(a) < 0$, then $x^T(V^{(1)}(a) + V^{(1)}(b))y - u^{(1)*}(a) - u^{(1)*}(b) < 0$, contrary to our assumption $x^T(\sum_{\alpha\epsilon\{a,b\}} V(\alpha))y -$

$\sum_{\alpha\epsilon\{a,b\}} (u^{(1)*}(\alpha), u^{(2)*}(\alpha)) = (0, 0)$.

Thus, if $i\notin a(x, y)$, then it must be the case that $x_i = 0$.

If $y_j > 0$ for some j satisfying $x^TV^{(1)j} - u^{(1)*}(b) < 0$, then $x^T(V^{(1)}(a) + V^{(1)}(b))y - u^{(1)*}(a) - u^{(1)*}(b) < 0$, contrary to our assumption $x^T(\sum_{\alpha\epsilon\{a,b\}} V(\alpha))y -$

$\sum_{\alpha\epsilon\{a,b\}} (u^{(1)*}(\alpha), u^{(2)*}(\alpha)) = (0, 0)$.

Thus, if $j\notin b(x, y)$, then it must be the case that $y_j = 0$.

Thus, it follows from proposition 2, that (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)). Q.E.D.

Note 6: An alternative proof of the second half of proposition 4, could be the following. Having established $x_i = 0$ whenever $i\notin a(x, y)$ and $y_j = 0$ whenever $j\notin b(x, y)$, we get that $x^TV^{(1)}(a)y = u^{(1)*}(a)$ and $x^TV^{(1)}(b)y = u^{(1)*}(b)$. Thus, the constraints in (i^*) imply (x, y) is an equilibrium strategy profile for $(V^{(1)}(a), V^{(1)}(b))$, $V_i^{(2)}(a)y - u^{(2)*}(a) \leq 0$ for $i\epsilon a(x, y)$ and $x^TV^{(2)j}(b) - u^{(2)*}(b) \leq 0$ for $j\epsilon b(x, y)$. By proposition 1, (x, y) is a lexicographic equilibrium strategy profile for (V(a), V(b)). ▌

Note 7: The statement [$V_i^{(1)}(a)z - u^{(1)}(a) \leq 0$ for $i\epsilon\{1, \ldots, m\}$, $V_i^{(2)}(a)z - u^{(2)}(a) \leq 0$ for $i\epsilon a(w, z)$, $w^TV^{(1)j}(b) - u^{(1)}(b) \leq 0$ for $j\epsilon\{1, \ldots, n\}$, $w^TV^{(2)j}(b) - u^{(2)}(b) \leq 0$ for $j\epsilon b(w, z)$, $w\epsilon\Delta^{m-1}$, $z\epsilon\Delta^{n-1}$, $u^{(r)}(\alpha)\epsilon\mathbb{R}$, for $\alpha\epsilon\{a, b\}$, $r\epsilon\{1, 2\}$] is *equivalent* to the statement [$(0, 0)\succcurlyeq V_i(a)z - (u^{(1)}(a), u^{(2)}(a))$ for $i\epsilon\{1, \ldots, m\}$, $(0, 0) \succcurlyeq w^TV^j(b) - (u^{(1)}(b), u^{(2)}(b))$ for $j\epsilon\{1, \ldots, n\}$, $w\epsilon\Delta^{m-1}$, $z\epsilon\Delta^{n-1}$, $u^{(r)}(\alpha)\epsilon\mathbb{R}$, for $\alpha\epsilon\{a, b\}$, $r\epsilon\{1, 2\}$]. ▌

In view of note 7, proposition 4 in this paper is proposition 3 of Lahiri (2026), which is stated as follows.

Proposition 5: $(x, y) \in \Delta^{m-1} \times \Delta^{n-1}$ is a lexicographic equilibrium strategy profile for the pair-valued bi-matrix game (V(a), V(b)) if and only if for $\alpha \in \{a, b\}$, there exists $u^*(\alpha) = (u^{*(1)}(\alpha), u^{*(1)}(\alpha)) \in \mathbb{R}^2$ that along with x, y satisfy the following two conditions:

(1) x, y, $u^*(a)$, $u^*(b)$ solve the problem: Maximize (with respect to $\succcurlyeq$) $w^T(V(a) + V(b))z - u(a) - u(b)$, subject to $u(a) \succcurlyeq V_i(a)z$ for all $i \in \{1, \ldots, m\}$, $u(b) \succcurlyeq w^T V^{(j)}(b)$ for all $j \in \{1, \ldots, n\}$, $w \in \Delta^{m-1}$, $z \in \Delta^{n-1}$, $u(a), u(b) \in \mathbb{R}^2$.

(2) $x^T(V(a) + V(b))y - u^*(a) - u^*(b) = (0, 0)$.

7. Using lexicographic equilibrium to resolve coordination problems: Consider a 2 ×2 bi-matrix (V(a), V(b)).

We may refer to the row player as **the dominant player** if $V^{(1)}(b)$ is the 2 ×2 zero matrix and $V^{(1)}(a) \in \{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\}$.

Similarly, we may refer to the column player as **the dominant player** if $V^{(1)}(a)$ is the 2 ×2 zero matrix and $V^{(1)}(b) \in \{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\}$.

Suppose the bi-matrix game $(V^{(2)}(a), V^{(2)}(b))$ has two-pure strategy equilibria (1, j), (2, 3-j).

Bi-matrix games of this type are generally referred to as 2 ×2 *co-ordination games*.

If the row player is the dominant player and $V^{(1)}(a) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then the unique lexicographic equilibrium for (V(a), V(b)) is (1, j).

If the row player is the dominant player and $V^{(1)}(a) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, then the unique lexicographic equilibrium for (V(a), V(b)) is (2, 3-j).

If the column player is the dominant player and column j of $V^{(1)}(b)$ is $E^{(2)}$, then the unique lexicographic equilibrium for (V(a), V(b)) is (1, j).

If the column player is the dominant player and column 3-j of $V^{(1)}(b)$ is $E^{(2)}$, then the unique lexicographic equilibrium for (V(a), V(b)) is (2, 3 - j).

It is natural to expect that a dominant player will assign the 2-dimensional sum vector to the action corresponding to the equilibrium strategy profile of $(V^{(2)}(a), V^{(2)}(b))$ where it gets a higher pay-off. Thus, using the lexicographic equilibrium along with the concept of a dominant player, may resolve the coordination problem in such coordination games.

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