A RECIPROCITY THEOREM FOR BUCKLING STATES OF VAT (STEERED FIBER) COMPOSITE PLATES

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Abstract

In the present paper the flat composite plates in buckling are studied. The plates have a symmetric lay-up and loaded along their piecewise-smooth contour by the in-plane forces. The consideration employs the Classical Lamination Plate Theory. A reciprocity theorem for the buckling states of the plate is derived and analyzed. The theorem may be used as a benchmark for the structural analysis and the optimization software.

Keywords: composite plate, buckling, lay-up, steered fibers, reciprocity

1. INTRODUCTION

Buckling of the anisotropic composite plates is thoroughly examined in many papers (see the books of Lekhnitzky 1956, Turvey and Marshall 1995, Falzon and Aliabadi 2008, Kassapoglou 2010). The theory of the phenomenon is well-developed. Nevertheless, in the theory there are several aspects not explored until now. A study of such an aspect is performed in the present paper.

The paper is devoted to the derivation and analysis of a reciprocity theorem for the buckling states of the anisotropic composite plates.

In Section 2 the theoretical background is presented.

Section 3 deals with the theorem.

The conclusions are presented in Section 4.

2. THEORETICAL BACKGROUND

Below we consider a flat thin composite plate restricted by a piecewise-smooth contour. The mid-plane of the plate coincides the *X*-*Y* plane of the Cartesian coordinate system *XYZ*. The plate has a symmetric lay-up, is restricted by a piecewise-smooth contour and is loaded by the in-plane loads along the contour. The Classical Lamination Plate Theory (CLPT) is used for the description of the plate deflections (see the book of Gibson 1994). The boundary conditions considered are the simple support and/or the clamping (note that the zero deflection boundary condition is a part of the indicated boundary conditions).

Following the book of Gibson (1994), we consider the buckling equilibrium of the infinitely small element of the plate. The equation, describing the force balance at buckling, is written as follows (see also the book of Volmir 1967):

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} = 0 \tag{1}$$

where $M_x, M_y, M_{xy}, Q_x, Q_y$ respectively are the x and y bending moments, the twisting moment, the buckling-induced forces acting normally to x and y edges of the element, with the moments and forces being calculated for the unit length edge.

The Classical Lamination Plate Theory (CLPT), based on linear elasticity relations for every layer and Bernoulli's hypotheses, is used for transition from (1) to an equation for deflections w (see the books of Gibson 1994, Lekhnitzky 1956, Reddy 2004). The equation is written as follows:

$$\begin{aligned} & D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} - \\ & - N_x \frac{\partial^2 w}{\partial x^2} - N_y \frac{\partial^2 w}{\partial y^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0 \end{aligned} \tag{2}$$

where $D_{ij}, i = 1,2,6; j = 1,2,6,$ are the elements of the bending stiffness matrix D, coupling the bending/twisting moments and various second derivatives of the deflection w with respect to x and y. The coupling is written as follows:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{3}$$

The quantities N_x, N_y, N_{xy} in (2) are the corresponding stress flows. Below we will denote the left-hand side of (3) as a vector $\overline{M}$, and the vector at the right-hand side (which is multiplied to D matrix) as $\bar{k}$. Also we use the notation

$$k_x = -\frac{\partial^2 w}{\partial x^2}; k_y = -\frac{\partial^2 w}{\partial y^2}; k_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \tag{4}$$

Then (3) is rewritten in the matrix-vector form as:

$$\overline{M} = D\bar{k} \tag{5}$$

For Q_x, Q_y as the functions of the external loads the following relation is valid (see the books of Mikhlin 1964, Gibson 1994, Volmir 1967).

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = N_x k_x + N_y k_y + 2N_{xy} k_{xy} \tag{6}$$

When we insert (6) and (3) into (1), we obtain (2) resulting from the substitution.

We introduce the quantities φ_x, φ_y and the vector $\overline{\varphi}$, composed of them:

$$\varphi_x = \frac{\partial w}{\partial x}; \varphi_y = \frac{\partial w}{\partial y} \tag{7}$$

$$\overline{\varphi} = \begin{bmatrix} \varphi_x \\ \varphi_y \end{bmatrix} \tag{8}$$

Obviously (Gibson 1994)

$$\overline{Q} = -F\overline{\varphi} \tag{9}$$

where the matrix F depends on x, y and is written as:

$$F = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \tag{10}$$

The total strain potential energy $\widetilde{\Pi}$ and the work of the internal forces W are written respectively as follows:

$$\widetilde{\Pi}(\bar{k}) = \iint dS(\frac{1}{2}D_{11}k_x^2 + D_{12}k_x k_y + \frac{1}{2}D_{22}k_y^2 + 2D_{16}k_x k_{xy} + 2D_{26}k_y k_{xy} + \\ + 2D_{66}k_{xy}^2) == \frac{1}{2}\iint dS\left[\bar{k}^T D\bar{k}\right] \tag{11}$$

$$W(\bar{\varphi}) = -\frac{1}{2}\iint dS\left[N_x\varphi_x^2 + N_y\varphi_y^2 + 2N_{xy}\varphi_x\varphi_y\right] = -\frac{1}{2}\iint dS\left[\bar{\varphi}^T F\bar{\varphi}\right] \tag{12}$$

where the superscript T means matrix transposing.

3. A RECIPROCITY THEOREM

We consider further a reciprocity theorem for two different buckling states. We will denote the values related to the first buckling state by the superscript $'$, and the values related to athe second buckling state by the superscript $''$. The two buckling states correspond to, generally speaking, different loadings.

We consider the following term and implement integration by parts, keeping in mind the simple support or the clamped boundary conditions:

$$\iint dS\overline{M}'^T\bar{k}'' = -\iint dSw''\left[\frac{\partial^2 M'_x}{\partial x^2} + \frac{\partial^2 M'_y}{\partial y^2} + 2\frac{\partial^2 M'_{xy}}{\partial x\partial y}\right] = \\ = -\iint dSw''\left[\frac{\partial Q'_x}{\partial x} + \frac{\partial Q'_y}{\partial y}\right] = \iint dS\overline{Q'}^T\overline{\varphi''} \tag{13}$$

On the other hand

$$\iint dS\overline{M}'^T\bar{k}'' = \iint dS\bar{k}'^T D\bar{k}'' = \iint dS\overline{M}''^T\bar{k}' \tag{14}$$

This leads to the relation

$$\iint dS\overline{Q'}^T\overline{\varphi}'' = \iint dS\overline{Q}''^T\overline{\varphi}' \tag{15}$$

which means, that the work performed by the force factors of the state ′ on the deflections of the state ″, is equal to the work performed by the force factors of the state ″ on the deflections of the state ′. Substituting $\overline{Q}$ via $\overline{\varphi}$ from (9), we obtain:

$$\iint dS\overline{\varphi}'^T F'\overline{\varphi}'' = \iint dS\overline{\varphi}'^T F''\overline{\varphi}'' \tag{16}$$

The obtained theorem is an analogue of the known Betti theorem of the structural mechanics in application to the buckling states of the composite plate. It should be noted that the buckling eigenvalues are located within the F matrices as the multipliers.

It should be also noted that the theorem is obviously valid for the isotropic plates.

4. CONCLUSIONS

- A reciprocity theorem for two buckling states is derived and discussed;
- The theorem may be used as a benchmark for the buckling analysis of the composite plates.

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