Consolidation of Sinking Infrastructures by Pore Pressure using XYY' Calculus

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Abstract. The consolidation of sinking infrastructures on soft soils is characterized by unknown initial conditions $(t_0, \delta_0, u_0, \sigma_0)$ and ambiguous boundary conditions (type, shape, and size of drainage). Under these forensic conditions, classical Newton-Leibniz chronological frameworks and traditional consolidation methodologies fail, triggering the "Missing Time" problem. This paper introduces the application of the XYY' Calculus framework—originating from complex soil mechanics in the mid-1990s—to model infrastructural consolidation using exclusively localized pore pressure measurements, completely bypassing the need for physical settlement data. By transforming the process from the chronological time domain into a geometric state-space, pore water pressure is mapped directly against its instantaneous dissipation rate. This proves that late-stage consolidation converges into an invariant linear phase-space trajectory known as the Master Decay Asymptote. This linearity encapsulates the universal Central Operator (Λ), allowing for the flawless extraction of the coefficient of consolidation (c_v) irrespective of initial loading times. The technique isolates pure hydrodynamic diffusion from secondary structural creep, rendering it a robust mechanism for 3D underground mapping. Optimal piezometer placement strategies permit the determination of vertical c_v at 30% to 40% consolidation, while radial consolidation parameters can be extracted at just 5% to 10% consolidation. This framework simplifies complex partial differential equations into linear algebraic relationships, providing ideal normalized feature sets for AI and Machine Learning applications in modern computational geomechanics. This is particularly useful for global infrastructure challenges like the Mexico City's subsidence, Kansai Airport, Jakarta City, the MOSE project Venice etc

Keywords: *Soil Consolidation, Pore Pressure, XYY' Calculus, Sinking Infrastructures, Tewatia YY' Approach, Settlement*

1 Introduction

The stability and longevity of global civil infrastructure constructed upon compressible soft soils rely heavily on accurately predicting and managing the continuous process of soil consolidation. Traditional geotechnical engineering is anchored by the fundamental one-dimensional theory of consolidation formalized by Terzaghi in 1923, which models the time-dependent expulsion of pore water and the resulting dissipation of excess pore water pressure under applied effective stresses [1]. However, the foundational reliance on Newton-Leibniz chronological timelines ($X - Y$ and $X - Y'$ approaches, where X is usually time) presents a severe limitation when attempting to analyze actively sinking, pre-existing infrastructures.

When assessing existing structures—whether historical monuments, aging commercial foundations, or environments undergoing staged, undocumented load increments—the precise initial time of loading (t_0), the initial structural settlement (δ_0), and the initial uniform excess pore pressure (u_0) are inherently unknown. The inability to establish chronological zero paralyzes traditional time-centric dynamic models [2-5], a mathematical bottleneck known as the "Missing Time" problem. To overcome these limitations the *XYY'* Calculus was developed in mid 1990s and extensively used in consolidation but in settlement analysis only [6-13].

This paper details the comprehensive implementation of the XYY' Calculus framework in consolidation pore pressure analysis. By shifting dynamic analysis from the chronological domain to a geometric state (u) versus velocity (du/dt)) domain, engineers can evaluate consolidating strata relying entirely on localized pore pressure dissipation, independently of global settlement data (Fig 1).

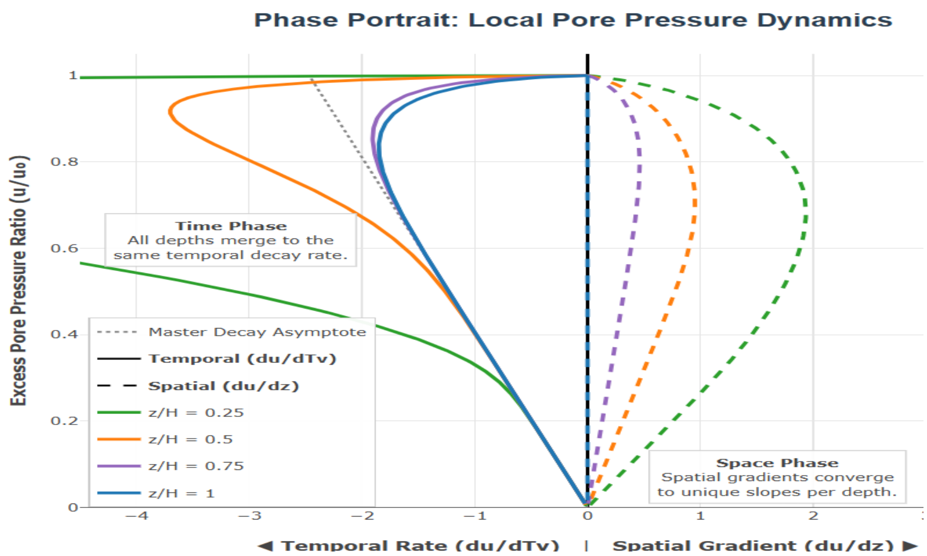

Fig 1: Theoretical pore pressure, (u/u_0) versus (du/dTv) curves for various depths (z) to drainage path (H) ratios in vertical consolidation. Time factor, $Tv = Cv.t/H^2$. Slope of the Theoretical Master Decay Asymptote, Central Operator Λ is $4/\pi^2$.

2 *XYY'* Calculus Framework & Tewatia's *YY' Plane*

The *X*-axis is the pillar of classical Newton-Leibniz (*XY & XY'*) Calculus, and it fails when *X* is unknown. It shows a *3D XYY'* trajectory on *2D XY* (Blue) *& XY'* (Green) planes only by missing the critical *YY'* (Orange) plane that is the *X* (usually time) invariant geometric *DNA* of the dynamic system (Fig 1-2). The complete mechanical state of the dynamic consolidation process at any specific instant is not a simple coordinate (*XY or XY'*) pair, but a continuous, dynamic position vector $r(X)$ tracing a definitive curve in three-dimensional space [7,12]:

$$r(X) = Xi + f(X)j + f'(X)k = Xi + Y(X)j + Y'(X)k$$

Where $i, j,$ and k are the standard unit vectors.

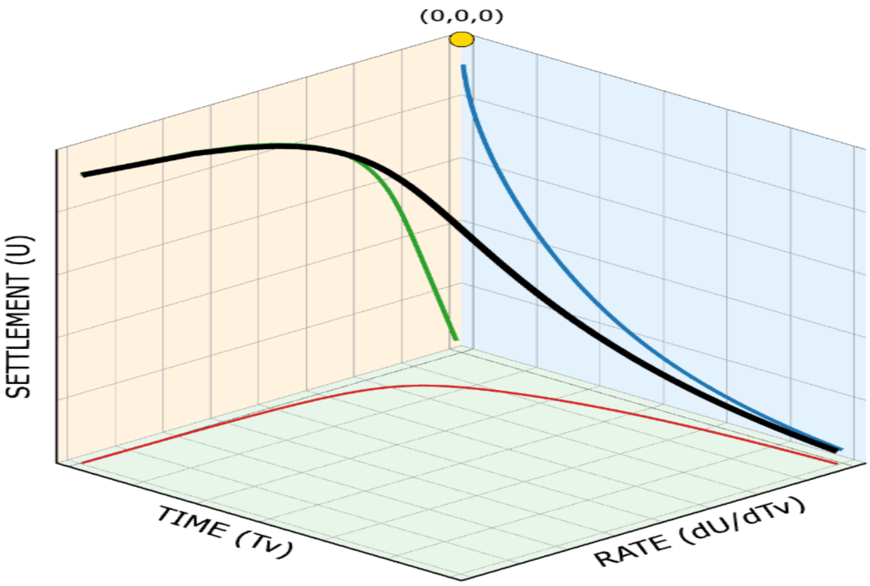

Fig 2: The Real *3D* Path of *1D* Consolidation (*T-U-U'* - Black Curve). Blue Wall, Integral Plane (*X-Y*): Classic Curve (*Time vs U; T-U*). Orange Wall, Tewatia's Differential Plane (*Y-Y'*) *DNA*: *U vs Rate; U-U'*. Green Floor, Newton's Differential Plane (*X-Y'*): *Time vs Rate; T-U'*. Origin (0,0,0), *U* increases downwards from *0-1*.

The defining historical breakthrough occurred in June 1998 with landmark publication in the *ASTM* Geotechnical Testing Journal [8] and Soils & Foundations of Japanese Geotechnical Society [9] that successfully challenged the century-old reliance on the Integral *XY* and Newton's Differential *XY'* temporal planes by explicitly discovering and introducing the missing *YY'* phase plane to geotechnical engineering that encodes other two *XY* and *XY'* planes in it. The complete *YY' plane* trajectory can be obtained from the data snippet without knowing initial and boundary conditions [10]. Ultimately, this mathematical paradigm transcended geotechnical engineering altogether, being formalized as a universal calculus applicable to thermodynamics, radioactive decay, cosmology, and any dissipative physical system [11,13].

3 Mathematical Limitations of Chronological Frameworks

Eq 1 describes the dissipation of localized excess pore water pressure u dynamically over chronological time t and spatial depth z, heavily regulated by the coefficient of consolidation c_v.

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \tag{1}$$

For a compressible soil layer characterized by a maximum thickness H, subject to double drainage boundaries and an idealized initial uniform excess pore pressure u_0, the analytical solution expands into an infinite Fourier sine series:

$$u(z,t) = 2u_0 \sum_{m=0}^{\infty} \frac{1}{M} \sin\left(\frac{Mz}{H}\right) \exp\left(-M^2 \frac{c_v}{H^2} t\right) \tag{2}$$

In this expansion, the spatial frequency multiplier is strictly defined as $M = \frac{\pi}{2}(2m+1)$. Because the chronological parameter t exists permanently within the argument of the decaying exponential function, the model is entirely dependent upon a known starting point and fails when t is unknown (Fig 3-5).

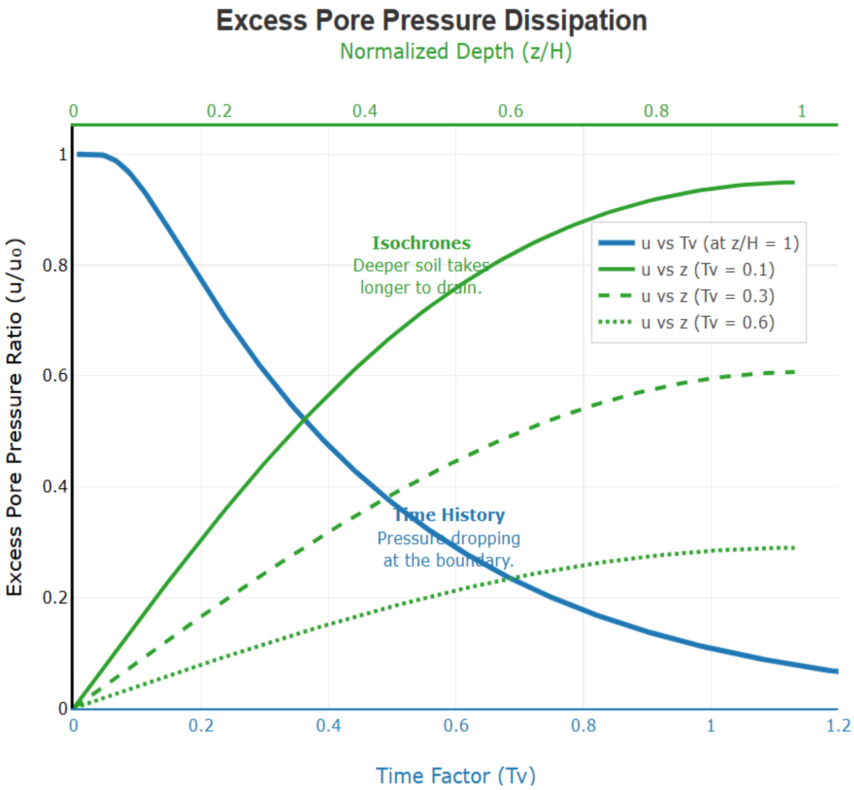

Fig 3: *u/u_0* versus *z/H* for various Time factors, *Tv*

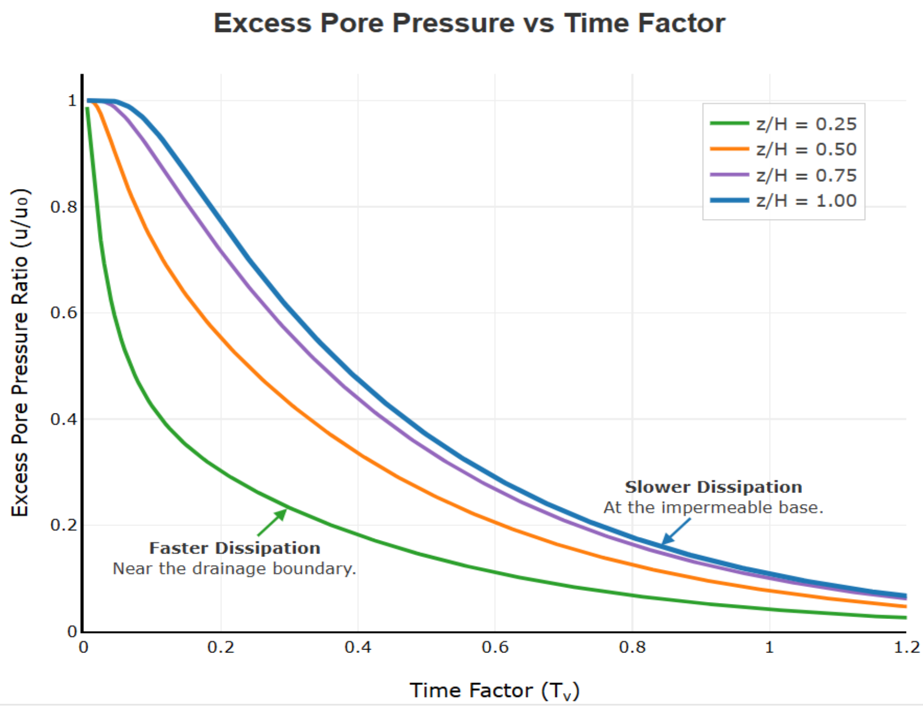

Fig 4: u/u_0 versus Tv for various z/H on linear scale

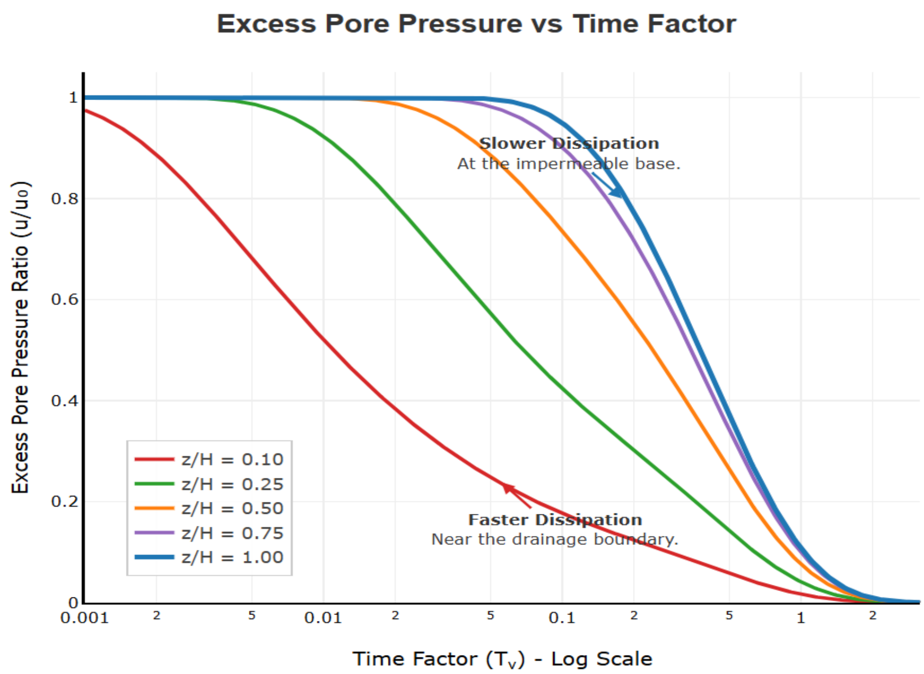

Fig 5: u/u_0 versus Tv for various z/H on log scale

4 *XYY′* Phase Portrait & Evaluation of c_v by Pore pressure only without settlement measurements

Partial Differential Eq 1 is directly plotted in Butterfly Phase Portrait in Fig 6, where ratio of abscissas on both sides of *Y axis* at any u encodes c_v. By this method c_v can be determined at just 5-10% *U* even in case of vertical consolidation that too without solving PDE 1.

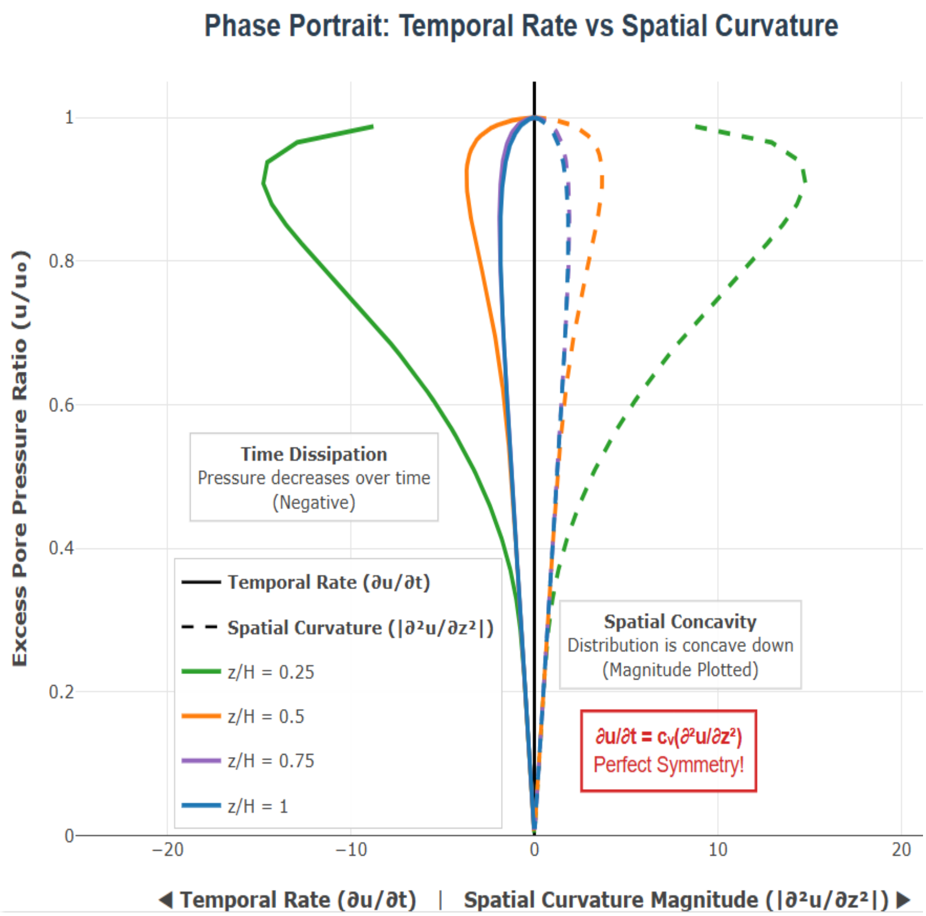

Fig 6: Direct extraction of c_v without solving basic PDE 1; $\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2}$

However, c_v can be obtained in an easier way by differentiating the classical state Eq 2 with respect to chronological time,

$$\frac{\partial u}{\partial t} = -2u_0 \frac{c_v}{H^2} \sum_{m=0}^{\infty} M \sin\left(\frac{Mz}{H}\right) \exp\left(-M^2 \frac{c_v}{H^2} t\right) \tag{3}$$

In the localized state-space domain—termed the Phase Portrait of *XYY'* framework—the observable physical state (excess pore water pressure u) is plotted directly against the absolute magnitude of its instantaneous dissipation rate ($\left|\frac{\partial u}{\partial t}\right|$). During the early stages of structural consolidation (typically where the dimensionless time factor $T_v <$ 0.2), higher-order Fourier terms heavily influence the spatial distribution, causing distinct, highly non-linear sweeping trajectories in the state-space depending on the exact relative depth z/H of the piezometer Fig 1.

At any fixed depth, z: once these higher-order wave interferences diminish, the series truncates strictly to the fundamental mode ($m = 0$):

$$u(z,t) \approx \frac{4u_0}{\pi} \sin\left(\frac{\pi z}{2H}\right) \exp\left(\frac{-\pi^2 c_v t}{4H^2}\right) \quad (4)$$

$$\frac{\partial u}{\partial t} \approx -\left(\frac{\pi^2 c_v}{4H^2}\right) \cdot u(z,t) = -\mathrm{u}/\Lambda \quad (5)$$

This linear terminal geometric trajectory is formally designated as the Master Decay Asymptote. By calculating the physical geometric slope ($Central\ Operator\ \Lambda = \frac{u}{u'}$) of the continuous master line spanning from the origin through empirical data points, the coefficient of consolidation c_v is flawlessly derived by inverting the slope equation:

$$c_v = \frac{4H^2}{\pi^2 \cdot \Lambda} \quad (6)$$

This pure geometric evaluation completely circumvents chronological initial conditions. Neither the unknown initial structural load time t_0, nor the ambiguous initial pore pressure u_0, nor the exact spatial installation depth z of the monitoring sensor factors into the final geometric slope calculation.

When a piezometer is deliberately inserted at the spatial coordinate $z/H = 0.667$, the positive and negative trajectory deviations observed at $z/H = 0.50$ and $z/H = 0.75$ are permanently neutralized Fig 1. This strategic placement strategy facilitates the fastest known method of vertical consolidation analysis, permitting engineers to secure a perfectly accurate c_v evaluation at just 30% to 40% absolute physical consolidation settlement. This is doubly faster than the fastest method of vertical consolidation by settlement analysis that determines c_v at 50% U [10].

Suppose: u_1 and u_2 are pore pressures at times t_1 and t_2 in short interval; then, $u = u_{avg} = 0.5(u_1 + u_2)$ and $(du/dt) = (u_2 - u_1)/(t_2 - t_1)$. However, there are more precise ways to find u and (du/dt) by polynomial fit etc.

5 Simple Numerical Example

A 1.0 m thick saturated clay layer with unknown t_0, δ_0, u_0 & σ_0 sandwiched between an impermeable rock base (bottom) and a pervious sand layer (top) was found to be consolidating in the field. The piezometer inserted at some depth at some time in arbitrary intervals gave the following *u-u'* data in Table 1. Find c_v.

Ans: The Table 1 presents the pore water pressure u and the rate of dissipation $\dot{u} = \frac{\Delta u}{\Delta t}$ at the specified depth. The data stabilizes to a constant ratio $\frac{\dot{u}}{u} \approx -2.66 \times 10^{-6}\ \mathrm{min}^{-1}$ $or\ \Lambda = \frac{u}{u'} = \frac{1}{2.66} \times 10^6\ min$. Using Eq 6

$$c_v = \frac{4H^2}{\Lambda\pi^2} = \frac{4 \times 1^2}{9.8696} \times (2.66 \times 10^{-6}) \approx 1.07 \times 10^{-6}\ \mathrm{m}^2/\mathrm{min}$$

Table 1: Piezometer Data and Dissipation Velocity Analysis

u (kPa)	$u^{\cdot}$ in 10^{-5} (kPa/min)	$(u^{\cdot}/u)$ in 10^{-6}/min
9.75	−2.60	2.67
9.51	−2.53	2.66
9.28	−2.47	2.66
9.05	−2.41	2.66

6 Ambiguous Drainage in Sinking Infrastructures by Pore Pressure

The *XYY'* framework mathematically proved that complex, multi-dimensional consolidation trajectories naturally converge into invariant straight lines in the phase plane [10-13] (Fig 1 & 6]. For example, radial consolidation exhibits perfect asymptotic linearity as early as 5% to 10% consolidation and vertical consolidation becomes linear by 60%U. Any combination of hydrodynamic consolidation of Vertical, Radial, Cylindrical, Cubical, Spherical with inward or outward flow or any arbitrary shape, type, size of any mixed strata ultimately (in the last) joins master decay asymptote on linear scale. Similarly, any combination of Creep settlements ultimately joins Master Decay Asymptote on Semi-log plot. This master Decay Asymptote manifests itself in Pore Pressure Analysis also in Tewatia's *YY'* plane.

There is no need to take the sample from field to laboratory, engineers can extract the fundamental "Central Operator" (Λ) strictly from the geometric slope of field data, allowing for the instant determination of the coefficient of consolidation c_v, c_r, λ_{cH} etc and ultimate settlement. A generalized factor, $\lambda_{cH} = c_v/H^2$, defines the property of in-situ clay under arbitrary drainage conditions that encodes the combined effect of c_v and complicated drainage boundaries. This can be determined in the field simply by observing the rate of dissipation of pore pressure over a small interval, entirely independent of loading history, geometry, or boundary conditions. Primary consolidation theoretically ends where $u = du/dt = 0$. This is particularly useful for global infrastructure challenges like the Mexico City's subsidence, Kansai Airport, Jakarta City, the MOSE project Venice etc. This makes the *XYY'* Calculus not just the simplest but the Only ultimate tool for sinking infrastructures.

7 Radial Consolidation in the Phase Plane by Pore Pressure

Based upon Barron's [5] equal strain theory, Eq 7 is differentiated wrt t

$$u = u_0 \exp\left(\frac{-8c_r t}{d_e^2 F(n)}\right) \quad (7)$$

$$\frac{\partial u}{\partial t} = -\left(\frac{8c_r}{d_e^2 F(n)}\right) u = -\Lambda_r \, . u \quad (8)$$

An immediate and perfect linear relationship forms originating exactly from $t = 0$. By extracting the geometric slope Λ_r of this newly defined master line, the critical radial consolidation coefficient c_r is determined flawlessly:

$$c_r = \frac{d_e^2 F(n)}{8\Lambda_r} \quad (9)$$

This formulation guarantees that robust evaluations of c_r can be rigorously determined at just 5% to 10% structural consolidation, allowing for extremely rapid in-situ hydrodynamic validations.

8 Isolation from Creep, 3D Mapping, and AI Integration

Global physical settlement monitoring inherently captures both active primary hydrodynamic diffusion and secondary structural consolidation (creep) simultaneously, mudding analytical boundaries. By actively mapping purely the internal pore pressure u against the localized hydrodynamic dissipation rate, the XYY' method perfectly isolates fluid diffusion from the macroscopic physical movement of the structure above.

Achieving absolute immunity to secondary compression guarantees that computed phase plane slopes remain pristine. Deploying a dense spatial array of piezometers beneath a subsiding foundational footprint allows engineers to instantly compute highly localized consolidation parameters, generating robust 3D tensor maps highlighting underground hydrodynamic anisotropies (Fig 3). This is particularly useful for global infrastructure challenges like the Mexico City's subsidence, Kansai Airport, Jakarta City, the MOSE project Venice etc

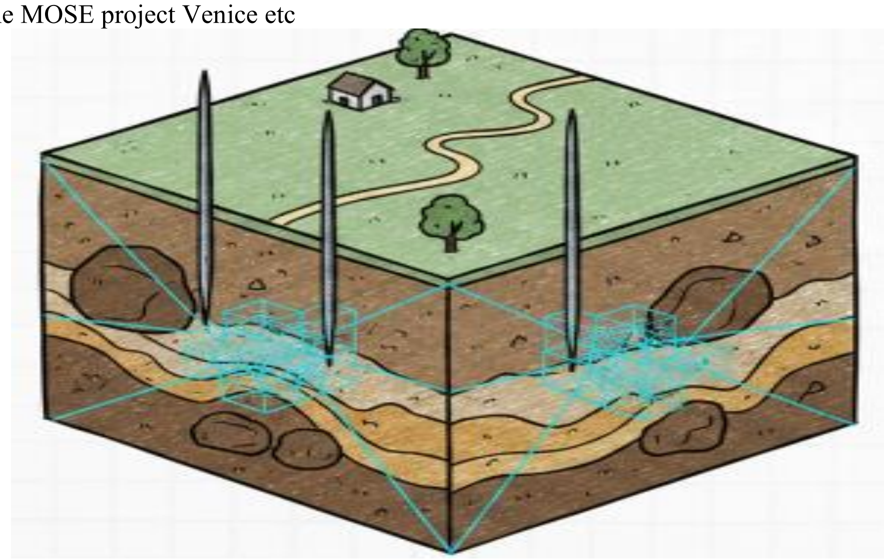

Fig 7: Localized c_v determination & *3D XYZ* Tensor Mapping of Underground Hydrodynamic Anisotropies by piezometers

Furthermore, as the geotechnical community incorporates advanced predictive AI models, the pristine linearity of the state-space Central Operator, Λ simplifies complex 3D geomechanics into robust, time-independent algebraic weights, drastically reducing computational overhead.

9 Conclusion

The stabilization of actively sinking infrastructure fundamentally requires advanced mathematical models capable of operating without initial chronological or any inputs. The *XYY′* Calculus framework—by transitioning multi-dimensional structural behavior into a strictly geometric state-space domain of pore pressure—cleanly bypasses the constraints of the Newton-Leibniz chronological missing time paradigm. The Asymptotic Linearity Theorem proves that spatial diffusion profiles ultimately condense onto a perfectly linear Master Decay Asymptote, allowing for the flawless extraction of precise consolidation coefficients (c_v and c_r) strictly via geometric slope analysis. By optimizing piezometer depth to $z/H = 0.667$ for vertical flow and exploiting the inherent singular mode of radial drainage, geotechnical engineers can evaluate active foundational subsidence with unparalleled speed and mathematical clarity, facilitating precision 3D subsurface mapping and future AI-driven computational geomechanics. The pore pressure consolidation analysis gives true c_v as it completely bypasses settlement measurements and is free from secondary consolidation/settlements effects.

Acknowledgement: Authors are thankful to Dr K Terzaghi for bringing differential calculus into soil mechanics from thermodynamics that led to the development of *XYY'* calculus universally applicable in all sciences**.**

Disclosure of Interests. The authors have no competing interests to declare that are relevant to the content of this article.

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