Supplementary Material for “Exact Amplitude–Delay Information Crossover for an Idealized Helical DAS Array”

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I. Proof of Theorem 1

Write the manifold as $\mathbf{a} = \mathbf{D}\mathbf{g}$, where $\mathbf{g} = q^2 \in \mathbb{R}^N$ and $\mathbf{D} = \mathrm{diag}\{\exp(-\mathrm{j}k\mathbf{r}_i^\mathsf{T}\mathbf{u})\}$ is unitary. With $\mathbf{x} = 2q \odot q_\theta$ and $\mathbf{z} = q^2 \odot h$, $h_i = \mathbf{r}_i^\mathsf{T}\mathbf{u}_\theta$,

$$\mathbf{a}_\theta = \mathbf{D}(\mathbf{x} - \mathrm{j}k\mathbf{z}). \tag{11}$$

The nuisance projector satisfies

$$\mathbf{P}_\mathbf{a}^\perp = \mathbf{D}\mathbf{P}_\mathbf{g}^\perp\mathbf{D}^\mathsf{H}, \qquad \mathbf{P}_\mathbf{g}^\perp = \mathbf{I} - \frac{\mathbf{g}\mathbf{g}^\mathsf{T}}{\mathbf{g}^\mathsf{T}\mathbf{g}}. \tag{12}$$

Substitution in the projected Fisher information gives

$$\begin{aligned} F_{\theta\theta} &= 2\rho_E\,\mathrm{Re}\{(\mathbf{x}+\mathrm{j}k\mathbf{z})^\mathsf{T}\mathbf{P}_\mathbf{g}^\perp(\mathbf{x}-\mathrm{j}k\mathbf{z})\} \\ &= 2\rho_E\mathbf{x}^\mathsf{T}\mathbf{P}_\mathbf{g}^\perp\mathbf{x} + 2\rho_E k^2\mathbf{z}^\mathsf{T}\mathbf{P}_\mathbf{g}^\perp\mathbf{z}. \end{aligned} \tag{13}$$

The two cross products are oppositely imaginary because $\mathbf{x}, \mathbf{z}, \mathbf{P}_\mathbf{g}^\perp$ are real; hence their real sum is zero. Define $U = \mathbf{x}^\mathsf{T}\mathbf{P}_\mathbf{g}^\perp\mathbf{x} \geq 0$ and $V = \mathbf{z}^\mathsf{T}\mathbf{P}_\mathbf{g}^\perp\mathbf{z} \geq 0$. For $U, V > 0$, equality of the two terms is $U = (2\pi f/c)^2 V$, whose unique positive solution is

$$f_\times = \frac{c}{2\pi}\sqrt{U/V}. \tag{14}$$

If $U = 0, V > 0$, equality occurs only at $f = 0$; if $U > 0, V = 0$, no finite equality exists; if $U = V = 0$, the projected azimuth derivative vanishes. This also proves independence from ρ_E.

II. Continuous Full-Turn Corollary

Let the helix phase be ψ, with winding angle α measured from the transverse plane:

$$\mathbf{r}(\psi) = [R\cos\psi,\ R\sin\psi,\ p\psi/(2\pi)]^\mathsf{T}, \tag{15}$$

$$\mathbf{t}(\psi) = [-\cos\alpha\sin\psi,\ \cos\alpha\cos\psi,\ \sin\alpha]^\mathsf{T}, \tag{16}$$

where $p = 2\pi R\tan\alpha$. Arc length satisfies $ds/d\psi = R/\cos\alpha$, so uniform arc-length sampling with spacing Δs gives $\Delta\psi = \Delta s\cos\alpha/R$. For a fiber span L, define the effective turn count

$$N_{\mathrm{eff}} = \frac{L\cos\alpha}{2\pi R}. \tag{17}$$

The continuous result uses uniform arc-length averaging over an integer number of complete turns; because $ds/d\psi$ is constant, this is equivalent to uniform averaging in ψ. Set

$$\begin{aligned} A &= \cos\alpha\cos\phi, \\ B &= \sin\alpha\sin\phi, \\ q &= A\sin(\psi-\theta) + B. \end{aligned}$$

Uniform complete-turn averaging uses $\langle\sin^{2m+1}\psi\rangle = 0$, $\langle\sin^2\psi\rangle = 1/2$, $\langle\sin^4\psi\rangle = 3/8$, and $\langle\sin^6\psi\rangle = 5/16$. Projecting $2qq_\theta$ against q^2 gives

$$Q_A = \frac{A^2(A^2+4B^2)}{2}. \tag{18}$$

For the propagation derivative, the transverse position term contributes a factor $R\cos\phi$. Removing its component parallel to q^2 and collecting the even trigonometric moments yields

$$Q_{D0} = \frac{15A^8 - 60A^6B^2 + 160A^4B^4 - 32A^2B^6 + 64B^8}{16(3A^4 + 24A^2B^2 + 8B^4)}. \tag{19}$$

Consequently,

$$f_{\times,\infty} = \frac{c}{2\pi R\cos\phi}\sqrt{\frac{Q_A}{Q_{D0}}}. \tag{20}$$

This expression applies for $R > 0$, $|\phi| < \pi/2$, and $Q_A, Q_{D0} > 0$. At $|\phi| = \pi/2$, azimuth is locally unidentifiable; if either quadratic form vanishes, the finite-array degeneracy classification applies. The axial term in $\mathbf{r}^\mathsf{T}\mathbf{u}_\theta$ vanishes because the axial component of $\mathbf{u}_\theta$ is zero; pitch enters the azimuth crossover through $\mathbf{t}$ and α. For $\phi = 0$, $B = 0$, $Q_A = A^4/2$, and $Q_{D0} = 5A^4/16$, giving $f_{\times,\infty} = c\sqrt{8/5}/(2\pi R)$. The explicit R^{-1} factor proves $\partial\ln f_\times/\partial\ln R = -1$ at fixed α. Finally, $p = 2\pi R\tan\alpha$ implies

$$\frac{\partial\ln f_\times}{\partial\ln p} = \sin\alpha\cos\alpha\frac{\partial\ln f_\times}{\partial\alpha}$$

at fixed R, completing the stated sensitivity relation.