

Supplemental Material for “Bearing Information and Geometric Bias in Finite-Gauge Helical DAS Arrays With Calibration Uncertainty”

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This supplement derives the results and specifies the numerical methods under the companion manuscript’s longitudinal-plane-wave, uniform-medium model and complete rectangular gauges.

APPENDIX A AMPLITUDE ELIMINATION AND PHYSICAL INFORMATION TERMS

For $y_l = \beta_l a(\xi) + n_l$, take known parameter-independent $C \succ 0$, $a \neq 0$, $W = C^{-1/2}$, $b = Wa$, and $D_w = W[a_{,1}, \dots, a_{,P}]$. The real Gaussian mean-information blocks use $2 \operatorname{Re}(v^H w)$. Eliminating the two amplitude directions b, jb removes their complex line:

$$\begin{aligned} J_\xi &= 2 \sum_l |\beta_l|^2 \operatorname{Re}\{D_w^H P_b^\perp D_w\}, \\ P_b^\perp &= I - b(b^H b)^{-1} b^H. \end{aligned} \quad (40)$$

Whitening yields the main-text S_a . A shared coefficient over K snapshots has energy $K|\beta|^2$. Independently unknown coefficients give the same conditional information, but a different finite-sample fitting model.

Differentiating $q^2 e^{-jkz}$ inside the direction-independent gauge integral gives the physical derivative split in the main text, whose projected cross term is signed. The output-polar split is a different decomposition: its envelope/phase cross term vanishes under real diagonal phase-rotated precision and positive envelopes. For parameter-dependent covariance, $\operatorname{tr}(C^{-1} C_{,i} C^{-1} C_{,j})$ must be added before nuisance elimination.

The kinematic transfer assumes the fiber follows prescribed host strain, $\varepsilon_f = t^T \varepsilon_{\text{host}} t$. Mechanical transfer varying within a gauge enters the integral; the post-gauge factor H_i generally cannot represent it. Noise variances refer to averaged strain coefficients. For calibrated linear phase readout $\Delta\varphi = K_{\text{opt}} G \bar{\varepsilon}$ with $G > 0$, K_{opt} has units rad m^{-1} per unit strain. Fixed phase-noise variance gives $\operatorname{var}(\bar{\varepsilon}) = \sigma_\varphi^2 / (K_{\text{opt}} G)^2$. The main-text σ_r^2 uses this normalization and inherits gauge dependence from the optical conversion.

APPENDIX B ANGULAR MODES, FINITE GRAMS, AND THE COMPLETE-TURN PROOF

Let $h = p/(2\pi)$, $\ell = (R^2 + h^2)^{1/2}$, and $\psi(s) = s/\ell + \gamma$. Set $A = R \cos \phi/\ell$, $B = h \sin \phi/\ell$, $\zeta = kR \cos \phi$, and $\nu = kh \sin \phi$. For $\chi = \psi - \theta$, direct substitution into the helical tangent and position yields

$$t^T u = B - A \sin \chi, \quad kr^T u = \zeta \cos \chi + \nu s/\ell. \quad (41)$$

The periodic part has an absolutely convergent Fourier series $g(\chi) = \sum_n c_n e^{jn\chi}$ for $g = (B - A \sin \chi)^2 e^{-j\zeta \cos \chi}$. Integrating each mode over a complete rectangular gauge gives

$$a_{G,i} = H_i e^{-j\nu s_i/\ell} \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}\left(\frac{(n - \nu)G}{2\ell}\right) e^{jn(\psi_i - \theta)}. \quad (42)$$

The axial phase shifts the sinc argument when $\phi \neq 0$. Dropping it would incorrectly extend the horizontal complete-turn conclusion to oblique arrivals.

At $\phi = 0$, $\nu = 0$. For every channel with $G = m(2\pi\ell)$, $m \geq 1$ integer, the sinc in (42) vanishes for every $n \neq 0$. Thus $a_{G,i} = H_i c_0$ for any set of channel centers, and $a_{G,\theta} = 0$. If $c_0 \neq 0$ and some $H_i \neq 0$, response energy is positive but projected bearing information is exactly zero. Otherwise the entire response is zero and (40) is inapplicable. Here

$$c_0 = \frac{A^2}{2} [J_0(\zeta) + J_2(\zeta)], \quad (43)$$

so a nonzero root of $J_1(\zeta)$ is an explicit zero-response example. More generally, let $G = m2\pi\ell$ with $m \geq 1$ and $m\nu \in \mathbb{Z}$. For integer $\nu = q$, all $n \neq q$ modes vanish and $a_i = H_i c_q e^{jq(\gamma - \theta)}$. Thus $a_{,\theta} = -jq a$ and its amplitude-projected information is zero when the response is nonzero. For noninteger ν with integer $m\nu$, every sinc argument is a nonzero integer multiple of π , so the whole response vanishes.

For continuous uniform center sampling, white readout noise, $H_i \equiv 1$, and $W_n = |b_n|^2$ with $b_n = c_n \operatorname{sinc}((n - \nu)G/(2\ell))$, orthogonality gives $\langle a, a \rangle = \sum W_n$, $\langle a, a_{,\theta} \rangle = -j \sum n W_n$, and $\langle a_{,\theta}, a_{,\theta} \rangle =$

$\sum n^2 W_n$. Equation (40) then becomes the weighted modal variance stated in the main text. A single surviving mode changes only a common complex phase, which the unknown source coefficient absorbs.

For a finite set of centers and a retained order $|n| \leq M$, put $V_{in} = H_i e^{-j\nu s_i/\ell} e^{jn(\psi_i - \theta)}$, $d_n = -jnb_n$, and $Q = C^{-1}$. Complex-amplitude elimination gives the exact finite retained-mode information, provided $Vb \neq 0$,

$$F_N^{(M)} = 2E \min_{z \in \mathbb{C}} (d - bz)^H V^H Q V (d - bz). \quad (44)$$

If $Vb = 0$, the regular unknown-amplitude Schur complement is undefined; this truncated zero-response state is reported separately. For white noise and $H_i = 1$, let $\Gamma = V^H V/N$ and let the columns of U form an orthonormal basis of $\text{span}\{b, d\}$. If $\epsilon = \|U^H(\Gamma - I)U\|_2 < 1$, then every $d - bz$ lies in that subspace. Applying $1 - \epsilon \leq x^H \Gamma x / \|x\|^2 \leq 1 + \epsilon$ before minimizing over z proves

$$(1 - \epsilon)F_\infty^{(M)} \leq F_N^{(M)} \leq (1 + \epsilon)F_\infty^{(M)}. \quad (45)$$

Both sides use the same finite modal truncation.

A. Near-turn coefficient and finite remainder

Write $G = G_\star(1 + \delta)$ and use the definitions of Theorem 2 in the main text. At integer $\nu = q_\star$, the point response repeats after a turn, so the two endpoints coincide and $G_\star a_{\cdot G}(G_\star) = a_e - a_\star$. Since $a_{\cdot \theta} + jq_\star a_\star = 0$, differentiation gives $w(0) = 0$, $w'(0) = v_\star$. The smooth nonzero-response projection then yields $P_b^\perp w = \delta P_{b_\star}^\perp v_\star + O(\delta^2)$, proving the coefficient and the $O(\delta^4)$ case when it vanishes.

The projector distance equals the sine of the principal angle between the two one-dimensional subspaces:

$$\|P_b^\perp - P_{b_\star}^\perp\| = \frac{\|P_b^\perp b_\star\|}{\|b_\star\|} \leq \frac{r_a}{\|b_\star\|}. \quad (46)$$

The assumption $r_a < \|b_\star\|$ ensures $b \neq 0$. Applying the triangle and reverse-triangle inequalities to $P_b^\perp(w - \delta v_\star) + \delta(P_b^\perp - P_{b_\star}^\perp)v_\star$ proves the main-text finite information bound.

The remainder envelopes follow from the complete modal series. Set $\tau = \delta/(1 + \delta)$, $h_e = W(a_e - a_\star)$, $H_Q = \sqrt{N}\|W \text{diag}(H)\|_2$, and $D_j = \sum_n |n - q_\star|^j |c_n|$. For $n \neq q_\star$, the exact gauge factor is $\tau(-1)^{m(n-q_\star)} \text{sinc}[m\pi(n - q_\star)\delta]$. Using $|\text{sinc } x - 1| \leq x^2/6$ gives

$$\begin{aligned} r_a &= |\tau| \{ \|h_e\| + (m\pi\delta)^2 H_Q D_2/6 \}, \\ r_w &= |\tau - \delta| \|v_\star\| + |\tau| (m\pi\delta)^2 H_Q D_3/6. \end{aligned} \quad (47)$$

The moments D_j can be replaced by valid upper bounds. For the five coefficients q_l^{pol} defined in Appendix C, let $Z = |\zeta|$ and $d_l = |l - q_\star|$. The Bessel

inequality $|J_n(\zeta)| \leq I_n(Z)$ and generating function $\sum_n I_n(Z) e^{nt} = e^{Z \cosh t}$ imply

$$\begin{aligned} \sum_n |n| I_n(Z) &\leq e^Z \sqrt{Z}, \quad \sum_n n^2 I_n(Z) = e^Z Z, \\ \sum_n |n|^3 I_n(Z) &\leq e^Z Z \sqrt{1 + 3Z}. \end{aligned} \quad (48)$$

Applying Cauchy-Schwarz to the second and fourth moments gives the odd-moment bounds. The resulting full-series finite envelopes are

$$\begin{aligned} D_2 &\leq e^Z \sum_{l=-2}^2 |q_l^{\text{pol}}| (Z + 2d_l \sqrt{Z} + d_l^2), \\ D_3 &\leq e^Z \sum_{l=-2}^2 |q_l^{\text{pol}}| (Z \sqrt{1 + 3Z} + 3d_l Z \\ &\quad + 3d_l^2 \sqrt{Z} + d_l^3). \end{aligned} \quad (49)$$

These bounds can be conservative at large Z . If C_G varies continuously and remains positive definite, whitening at $G_\star$ preserves the leading coefficient with $o(\delta^2)$ remainder; local Lipschitz continuity gives $O(|\delta|^3)$. The main-text covariance comparison supplies the finite bound for C_G .

At fixed G, p and horizontal incidence, the exclusion-band radius is $R_\star = \sqrt{[G/(2\pi m)]^2 - [p/(2\pi)]^2}$. When it lies in the admitted positive-radius interval, the complete-turn result applies at that radius for any finite center arrangement. The nonzero-response condition distinguishes a regular information null from an entire-response zero.

APPENDIX C

INFINITE-TAIL CONTROL AND NUMERICAL SCOPE

The polynomial factor of g has five dimensionless Fourier coefficients, denoted by q_l^{pol} to distinguish them from the noise density: $q_0^{\text{pol}} = B^2 + A^2/2$, $q_{\pm 1}^{\text{pol}} = \pm jAB$, and $q_{\pm 2}^{\text{pol}} = -A^2/4$. Jacobi-Anger therefore gives

$$c_n = \sum_{l=-2}^2 q_l^{\text{pol}} (-j)^{n-l} J_{n-l}(\zeta). \quad (50)$$

With $x = |\zeta|/2$ and integer $m \geq 0$, the absolute Bessel series yields $|J_m(\zeta)| \leq U_m = x^m e^{x^2/(m+1)}/m!$. The gauge sinc has modulus at most one. Thus the true infinite coefficients satisfy $|b_n| \leq \sum_l |q_l^{\text{pol}}| U_{|n-l|}$. For $Q > M$, sum these envelopes explicitly for $M < |n| \leq Q$. For the remainder let $K_l = Q + 1 - |l|$ and

$r_l = x/(K_l + 1) < 1$. The successive-envelope ratio is at most r_l , giving

$$\begin{aligned} S_{0,l} &\leq \frac{U_{K_l}}{1 - r_l}, \\ S_{1,l} &\leq U_{K_l} \left(\frac{K_l}{1 - r_l} + \frac{r_l}{(1 - r_l)^2} \right). \end{aligned} \quad (51)$$

The full $t_0 = \sum_{|n| > M} |b_n|$ and $t_1 = \sum_{|n| > M} |nb_n|$ are bounded by the explicit finite sums plus $\sum_l 2|q_l^{\text{pol}}|S_{0,l}$ and $\sum_l 2|q_l^{\text{pol}}|(S_{1,l} + |l|S_{0,l})$, respectively. The explicitly summed terms can retain their exact gauge-sinc modulus for a tighter bound. At large $|\zeta|$, increasing the retained order M is also necessary when substantial modal energy lies beyond it; increasing Q alone only refines the tail evaluation. For arbitrary finite center positions and $H_i = 1$,

$$\begin{aligned} \|a - a^{(M)}\|_2 &\leq e_0 = \sqrt{N} t_0, \\ \|a_{,\theta} - a_{,\theta}^{(M)}\|_2 &\leq e_1 = \sqrt{N} t_1. \end{aligned} \quad (52)$$

For fixed nonuniform, direction-independent H_i , replace $\sqrt{N}$ by $\|H\|_2$; colored noise requires corresponding whitened norms.

Let $A_M = \|a^{(M)}\|_2 > e_0$ and $v_M = a_{,\theta}^{(M)}$. For white noise $C = \sigma^2 I$, let $\rho_E = E/\sigma^2$ and use the unwhitened response and tail norms above. Compare the derivative quadratic forms at a fixed projection and then the rank-one projectors at fixed v_M . The projector perturbation norm is at most $2e_0/(A_M - e_0)$, so

$$|F_N - F_N^{(M)}| \leq 2\rho_E \left[(2\|v_M\| + e_1)e_1 + \frac{2\|v_M\|^2 e_0}{A_M - e_0} \right]. \quad (53)$$

When $A_M \leq e_0$, no useful information conclusion follows from the retained response. Equations (45) and (53) control finite sampling and tail truncation, respectively. Writing the right-hand side of (53) as ΔF_M , an analytical sufficient condition for prescribed truncation accuracy is $A_M > e_0$ and

$$\Delta F_M \leq \tau_{\text{abs}} + \tau_{\text{rel}} \max\{F_N^{(M)} - \Delta F_M, 0\}. \quad (54)$$

Here τ_{abs} has information units and τ_{rel} is dimensionless. The positive part is a lower bound on complete information; near a null the absolute tolerance controls acceptance.

Separately, an exact S -node angular DFT has $\tilde{c}_n = \sum_{r \in \mathbb{Z}} c_{n+rS}$. For retained $|n| \leq M < S/2$, all nonzero aliases have $|n + rS| \geq S - M$. Put $K_l = S - M - |l|$ and $r_l = x/(K_l + 1) < 1$. The same Bessel envelope gives the complete coefficient-alias mass bound

$$T_c = \sum_{|m| \geq S-M} |c_m| \leq 2 \sum_{l=-2}^2 |q_l^{\text{pol}}| \frac{U_{K_l}}{1 - r_l}. \quad (55)$$

Since the rectangular-gauge sinc has modulus at most one, the errors *within the retained modes* satisfy

$$\begin{aligned} e_{0,\text{alias}} &= \sqrt{N}(2M + 1)T_c, \\ e_{1,\text{alias}} &= \sqrt{N}M(M + 1)T_c, \end{aligned} \quad (56)$$

$$\begin{aligned} \|a_M^{\text{DFT}} - a^{(M)}\|_2 &\leq e_{0,\text{alias}}, \\ \|a_{,\theta,M}^{\text{DFT}} - a_{,\theta}^{(M)}\|_2 &\leq e_{1,\text{alias}}. \end{aligned} \quad (57)$$

Adding these alias bounds to e_0, e_1 by the triangle inequality bounds the difference between the full and DFT responses. The same FIM projection argument then applies provided $\|a_M^{\text{DFT}}\|_2 > e_0 + e_{0,\text{alias}}$; otherwise the bound remains unresolved. For fixed direction-independent gains, replace $\sqrt{N}$ by $\|H\|_2$. The bound is also unresolved if any $r_l \geq 1$. Responses below the numerical comparison scale are reported as unresolved.

APPENDIX D

ROOT CONTINUATION AND RADIUS REGISTRATION

For a smooth point response on a fixed symmetric rectangular support, Taylor's theorem gives

$$\begin{aligned} a_G(s_i) &= a_0(s_i) + \frac{G^2}{24} a_{0,ss}(s_i) + R_4, \\ |R_4| &\leq \frac{G^4}{1920} \sup_{|\xi| \leq G/2} |a_{0,ssss}(s_i + \xi)|. \end{aligned} \quad (58)$$

The same expansion holds for the physical derivatives. Forming $\mathcal{H}_t$, $t = G^2$, requires differentiating both the response and the amplitude projection; holding the projection fixed omits a term. The implicit-function theorem gives the main-text root continuation at $\mathcal{H}_f \neq 0$, but the continuation fails at tangency. For nonoverlapping gauges with input noise, $C_G = (q_0/G + \sigma_r^2)I$. This common positive scale must be removed before continuing to $G = 0$, because the unnormalized balance tends to zero and does not meet the simple-root premise. The 0.25-m examples use the full integral.

Each of the four radius paths in the main text holds a different set of coordinates fixed. For the fixed-arc path, arc s_i , pitch p , rotation γ , and gauge G are fixed, with $\ell = (R^2 + h^2)^{1/2}$. Let r_R^{mat} denote the point-response derivative at fixed material winding coordinate ψ , pitch, and rotation. Holding s fixed also changes axial phase: $z_{\text{axial}} = hs/\ell$ and $z_{\text{axial},R} = -hsR/\ell^3$. Consequently,

$$r_R^{\text{point}} = r_R^{\text{mat}} + j\nu \frac{sR}{\ell^3} a_0, \quad \nu = kh \sin \phi. \quad (59)$$

This axial term vanishes for horizontal incidence. The full chain rule is

$$\psi_{,R}(s) = -\frac{sR}{\ell^3}, \quad a_{0,R}(s) = \frac{sR}{\ell^3} a_{0,\theta}(s) + r_R^{\text{point}}(s). \quad (60)$$

Splitting $s = s_i + \xi$ before integration yields the center-winding, within-gauge-winding, and remaining explicit-radius terms of the main text's radius derivative. For fixed winding coordinates, in contrast, $s'_i = s_i \ell(R)/\ell(R_0)$ and $G' = G \ell(R)/\ell(R_0)$; the gauge must be integrated on the deformed physical arc. Center anchoring instead sets $\gamma(R) = \gamma_0 + \bar{s}(1/\ell_0 - 1/\ell(R))$, where $\bar{s} = (s_0 + s_{N-1})/2$ is the midpoint of the sampled arc (2.95 m in E4). Proportional scaling multiplies R, p, s_i, G together at fixed frequency, so its propagation phase still changes.

Linearizing the amplitude-profiled score at a continuing interior optimum gives the main-text pseudo-true shift. The three radius components enter its numerator linearly.

Reference registration. At the common nominal geometry and fixed H_i , $a_{,\gamma} = -a_{,\theta}$. Differentiating the anchoring rotation therefore gives

$$a_{,R}^{\text{anchor}} = a_{,R}^{\text{arc}} - \frac{R\bar{s}}{\ell^3} a_{,\theta}. \quad (61)$$

If S_R denotes the coefficient of δR in the main-text local shift, then $S_R^{\text{anchor}} = S_R^{\text{arc}} - R\bar{s}/\ell^3$. At nominal geometry, this identity separates imposed reference rotation from remaining radius sensitivity. Finite changes shift the fixed-arc evaluation bearing. The nominal array has winding angle $\tan^{-1}(h/R) = 0.7295^\circ$ and axial span 0.0751 m.

Horizontal path equivalence. Let $R' = \lambda R$, $\lambda > 0$. Fixed winding has $\ell_W = (\lambda^2 R^2 + h^2)^{1/2}$, whereas proportional scaling has $\ell_P = \lambda \ell$. Both preserve channel winding coordinates and the normalized gauge width G/ℓ . At $\phi = 0$, their transverse positions and propagation phases coincide; the tangent projection differs only by a common amplitude. With identical channel factors and rotation,

$$a_W(\theta) = \kappa a_P(\theta), \quad \kappa = \frac{\lambda^2(R^2 + h^2)}{\lambda^2 R^2 + h^2}. \quad (62)$$

For nonzero responses fitted with a common nominal template and covariance, normalized coherence and pseudo-true bearing sets coincide, as in the main text's registration table and Fig. 3(c). At fixed incident coefficient energy and covariance, the common amplitude factor also scales information.

For finite estimates in one fixed angular coordinate, write $b = \theta_{\text{pt}} - \theta_0$ and $e_l = \hat{\theta}_l - \theta_{\text{pt}}$. Then the empirical identity is

$$\begin{aligned} \widehat{\text{MSE}} &= \frac{1}{n} \sum_l e_l^2 + b^2 + 2b\bar{e}, \\ \bar{e} &= \frac{1}{n} \sum_l e_l. \end{aligned} \quad (63)$$

Branch errors or finite-sample estimator bias can produce a nonzero mean cross term. The E4 full-cycle refit uses 721 and 1441 nodes on all 2952 conditions. Best peaks agree within 5.96×10^{-6} degree, but a secondary peak appears only on the finer grid. The twelve sector-boundary optima remain censored.

APPENDIX E

FINITE CALIBRATION AND MISSPECIFIED LOCAL RISK

For the main text's nuisance model, $a_{,b_g} = B \odot a$, $a_{,b_\phi} = jC \odot a$, and $a_{,r} = R_0 a_{,R}$. Applying (40) independently to known-direction records yields J_{cal} . The symmetry $a_{,\theta} + a_{,\gamma} = 0$ explains the unknown-reference null direction.

Numerical recovery. The original bounded L-BFGS-B failures are retained in the archive. Recovery uses a separate objective-only Powell protocol, restricted to finite, rank-four interior endpoints for which both original starts terminate ABNORMAL. The nuisance scales are $(0.01, \pi/180, 0.02, \pi/180)$. Acceptance requires two-start agreement, objective change $\leq 10^{-8}$, scaled-coordinate change and Newton correction $\leq 10^{-6}$, and positive-definite, step-stable Hessians at steps 10^{-3} and 5×10^{-4} . The relative Hessian difference is at most 10^{-6} and each minimum eigenvalue exceeds ten times their norm difference. Accepted task reruns retain the original observations and seeds.

For calibrate-then-estimate fitting, implicit differentiation gives $\hat{\theta} - \theta^* \simeq u_\theta - L e_\eta$. Here $e_\eta = \hat{\eta} - \eta^*$ and u_θ is the zero-mean local task error with variance $J_{\theta\theta}^{-1}$. Independent task and calibration data therefore give $J_{\theta\theta}^{-1} + L C_{\text{cal}} L^T + (b_{\text{res}} - L m_{\text{cal}})^2$.

For matched efficient calibration, put $C_c = J_{\text{cal}}^{-1} \succ 0$, $j_\theta = J_{\theta\theta} > 0$, $b = J_{\theta\eta}$, and $K_\eta = J_{\eta\eta} - b^T b / j_\theta \succeq 0$. Inverting the joint block information by eliminating θ first gives $V_{\text{joint}} = j_\theta^{-1} + L(C_c^{-1} + K_\eta)^{-1} L^T$, where $L = b/j_\theta$. The difference from $V_{\text{plug}} = j_\theta^{-1} + L C_c L^T$ is

$$L C_c^{1/2} \{I - (I + C_c^{1/2} K_\eta C_c^{1/2})^{-1}\} C_c^{1/2} L^T. \quad (64)$$

It is nonnegative, and is zero exactly when $K_\eta C_c L^T = 0$. Since $a_{,\gamma} = -a_{,\theta}$, the γ row and column of K_η vanish. This is a block-inverse consequence of the model's rotation symmetry; task information about other nuisance directions can still improve the joint prediction.

With unmodeled response, define a pseudo-true point separately for each workflow. For the fixed-calibration profiled task score let A be its expected negative derivative and B its score covariance. Its local asymptotic task variance and nuisance sensitivity are

$$V_{\text{mis}} = A_{\theta\theta}^{-1} B_{\theta\theta} A_{\theta\theta}^{-1}, \quad S_{\text{mis}} = -A_{\theta\theta}^{-1} A_{\theta\eta}. \quad (65)$$

The local plug-in prediction, $V_{\text{mis}} + S_{\text{mis}}C_{\text{cal}}S_{\text{mis}}^T + (b_{\text{res}} + S_{\text{mis}}m_{\text{cal}})^2$, squares first-order linearized bearing error, omitting curvature of $\theta^*(\eta)$ and finite-sample task mean corrections. Even centered calibration error contributes $b_{\text{res}} \text{tr}(TC_{\text{cal}})$ through curvature, where $T = \nabla_{\eta}^2 \theta^*(\eta_c^*)$. For fixed nonzero b_{res} , this term is linear in C_{cal} . Under mismatch, joint fitting uses the complete task–calibration score sandwich with one nuisance amplitude per record.

To assemble the joint sandwich, let ξ contain bearing, the four common nuisance parameters, and the real and imaginary parts of each record’s amplitude. Write $m_j(\xi) = \beta_j a_j$, $e_j = \mu_j - m_j(\xi^*)$, $D_{j,p} = \partial_{\xi_p} m_j$, and $M_{j,pq} = \partial_{\xi_p} \partial_{\xi_q} m_j$, evaluated at the joint pseudo-true point. For independent records with fixed, correctly specified proper-complex noise covariance C_j ,

$$\begin{aligned} A_{pq} &= 2 \sum_j K_j \text{Re} \{ D_{j,p}^H C_j^{-1} D_{j,q} \\ &\quad - M_{j,pq}^H C_j^{-1} e_j \}, \quad (66) \\ B_{pq} &= 2 \sum_j K_j \text{Re} \{ D_{j,p}^H C_j^{-1} D_{j,q} \}. \end{aligned}$$

Calibration records have zero derivatives with respect to task bearing. Amplitude cross derivatives remain in $M_{j,pq}$. The bearing entry of $A^{-1}BA^{-T}$ is the joint local variance; when every $e_j = 0$, $A = B$ and it reduces to the matched information inverse.

APPENDIX F

ONE DEVELOPMENT TRANSFER: LOCAL MISSPECIFIED RISKS

Development transfer ID 0 has unmodeled second- and third-order gain/phase harmonics. The true nuisance vector is $\eta_{\text{true}} = (0.01, \pi/180, 0.02, \pi/180)^T$. The task uses bearing 15° , 800 Hz, $G = 0.25$ m, $\phi = 0$, and $K = 100$; calibration gauges are 0.1 m. Each budget uses 30 independent calibration batches, each with five paired task draws: 300 observations and 600 plug-in/joint fits across both budgets. All fits are finite, with no opposite-half-turn estimates. The table uses 30 independent batch MSEs. The plug-in predictions use the sample mean error about the noiseless calibration target and the full covariance of these same calibration estimates; task noise is independent of calibration noise.

The larger budget also increases calibration energy. Interval relative half-widths are 16–21%, exceeding the planned 10%. Zero half-turn events give a batch-level one-sided 95% upper bound of 9.50%. The joint local prediction includes the complete task–calibration sandwich and its own pseudo-true bias. At budget 18, the 43-parameter Hessian is positive definite and full rank. The predicted variance changes by less than 1.2×10^{-12}

TABLE VII
ONE DEVELOPMENT TRANSFER, ONE TASK: OBSERVED
BATCH-LEVEL MSE AND 95% INTERVAL (deg^2), WITH LOCAL
PREDICTIONS AT THE SAME TASK.

Records	Workflow	MC MSE [95% interval]	Local pred.
4	Plug-in	0.005332 [0.004468, 0.006196]	0.005899
4	Joint	0.005354 [0.004476, 0.006232]	0.005863
18	Plug-in	0.004297 [0.003398, 0.005197]	0.004900
18	Joint	0.004299 [0.003400, 0.005197]	0.005075

in relative terms across three derivative steps. The zero-residual sandwich agrees with the matched Schur prediction. Protocol Section Q gives the matrices and derivative-step checks.

For ID 0, differentiation at the archived calibration targets gives $b_{\text{res}} \text{tr}(TC_{\text{cal}}) = -8.85 \times 10^{-8}$ and $-2.36 \times 10^{-8} \text{deg}^2$ at budgets 4 and 18, using the saved calibration covariances. Their magnitudes are below 1.6×10^{-5} of the corresponding linearized predictions; the relative change across three step sizes is less than 1.5×10^{-5} . This local calculation assesses the centered-error curvature term; other finite-sample and higher-order contributions remain outside the linearized approximation.

A separate 50-transfer audit has 97 calculable local plug-in risks: two original calibration failures and one unresolved task score leave three unconditional risks undefined. Its second 25-transfer group was previously screened; the source data and failure labels are retained.

APPENDIX G

E6 REPETITION COUNTS AND UNCERTAINTY INTERVALS

Each of the 72 main cells begins with 2000 independent realizations. If the bootstrap interval for either local variance or sector MSE has full width exceeding 20% of its estimate, sampling is extended once to 5000, retaining the first 2000 draws. Three near-turn cells trigger this rule: $(K, \text{SNR}) = (20, 10 \text{ dB})$, $(100, 0 \text{ dB})$, and $(500, -10 \text{ dB})$. This gives $72(2000) + 3(3000) = 153000$ realizations.

Variance and MSE intervals are the 2.5th and 97.5th percentiles of 1000 independent-realization bootstrap resamples. Exceedance intervals use the two-sided exact binomial construction for nonzero counts and the one-sided upper bound $1 - 0.05^{1/n}$ for zero counts. These are nominal 95% intervals without sequential-coverage adjustment. After extension, sector-MSE relative half-widths in the three cells are 16.63%, 11.87%, and 7.93%; the first two exceed the 10% target and remain marked in main-text Fig. 4.

The two nonzero exceedance rates, 66.96% and 19.75%, have intervals $[65.64, 68.26]\%$ and

TABLE VIII
NOMINAL 95% INTERVALS FOR THE SIX E6 ROWS IN MAIN-TEXT
TABLE IV. BOTH INTERVAL COLUMNS ARE IN deg^2 ; $T = 2\pi\ell$, G IS
IN METERS, AND SNR IS IN DB.

(G, SNR)	n	Local variance	Sector MSE
(0.25, 0)	2000	[0.5781, 0.6465]	[0.5741, 0.6450]
(0.95T, 0)	5000	[18.32, 18.80]	[371.7, 471.4]
(0.25, 10)	2000	[0.05466, 0.06184]	[0.05484, 0.06182]
(0.95T, 10)	2000	[9.711, 10.52]	[14.44, 16.30]
(0.25, 20)	2000	[0.005781, 0.006534]	[0.005763, 0.006542]
(0.95T, 20)	2000	[1.348, 1.523]	[1.350, 1.520]

[18.03, 21.56]%. Each zero-count row has a one-sided 95% upper limit of 0.1497%. The archive's figure_data/e6/cells.csv reports counts and intervals for all 72 cells. Its historical branch_rate field is the 5° exceedance defined in the main text.

APPENDIX H PROCESSING DEFINITIONS AND POINT-CHANNEL CHECKS

E6 evaluates profiled likelihood; the following phase-to-bearing checks evaluate MUSIC. They report physical-bearing error and the additional processing offset from the array pseudo-true target. The readout uses one complex coefficient per channel from the nearest nominal-frequency DFT bin of each nonoverlapping Hann-windowed segment. Writing these columns as $Y = [y_1, \dots, y_M]$, it forms $\hat{C} = YY^H/M$. For one source, E_n contains the $N - 1$ smallest-eigenvalue eigenvectors of $\hat{C}$, and the bearing minimizes $\|E_n^H a_{\text{nom}}(\theta)\|_2^2 / \|a_{\text{nom}}(\theta)\|_2^2$. E8 searches $[-90^\circ, 90^\circ]$ on a 0.25° grid and refines an interior minimum by three-point parabolic interpolation. The nominal MUSIC check in the 12-condition E9 field-response study uses a 1° grid.

The 12-condition E8 point baseline uses 200 repeats per condition at 30° , 800 Hz, and 20-dB time-domain SNR. Matched and half-bin RMSEs are 0.0141° and 0.0168° . Protocol Section H retains the transfer and noise conditions. E9 independently differentiates a 3-D displacement wave, projects tangent strain, and integrates the gauge. Three refinements over 12 conditions give maximum finest-grid response difference 1.50×10^{-7} ; the integer-turn zero is roundoff-limited. Its nominal MUSIC estimate is 29.999584°.

APPENDIX I CONTINUOUS DESIGN BOUNDS AND SPATIAL PROCESSING

For any z , inverse Loewner ordering bounds $(a_\theta - az)^H C_G^{-1}(a_\theta - az)$ between its C_{ref}^{-1} counterpart divided

by λ_+ and λ_- . Minimizing over z proves the covariance transfer. For spatial processing, the matrix

$$Q_s = C_G^{1/2} T_s^T (T_s C_G T_s^T)^{-1} T_s C_G^{1/2} \quad (67)$$

is an orthogonal projector. Applying $Q_s \preceq I$ to the whitened residual before amplitude minimization proves $\tilde{F}_\theta \leq F_\theta$. With uniform gains, convolution of the acquisition rectangle and the three-point weights gives the stated sinc-cosine-squared transfer.

For a continuous cell, the rank-one projector inequality used in Appendix B gives $\|P_b^\perp - P_b^\perp\| \leq A_B / \|\bar{b}\|$. The triangle inequality then proves the cell bound. For half-widths h_θ (rad), h_R (m), take

$$\begin{aligned} A_B &= e_a + B_\theta h_\theta + B_R h_R, \\ D_B &= e_\theta + B_{\theta\theta} h_\theta + B_{\theta R} h_R, \end{aligned} \quad (68)$$

where B_x, B_{xy} bound whitened derivatives throughout the cell. The center errors e_a, e_θ include the complete infinite Fourier tail from Appendix C. Minimizing cell lower bounds over a cover gives $\underline{m}_d$; an evaluated point gives an upper bound on $\inf_\Omega F_d$, which proves the design comparison.

For the horizontal model, write $g = A^2 \sin^2 \chi e^{-jZ \cos \chi}$ with $A = R/\ell$ and $Z = kR$. On a radius cell, set $A_+ = R_+/\ell_+$, $a_1 = h^2/\ell_-^3$, $Z_+ = kR_+$, and $U_i = \sup |sR/\ell^3|$ over channel i 's gauge and radius cell. The angular envelopes use

$$\begin{aligned} T_1^2 &= \max_{0 \leq x \leq 1} \{4x(1-x) + Z_+^2 x^3\}, \\ T_2 &= 2 + Z_+^2. \end{aligned} \quad (69)$$

Here $T_1 = Z_+$ for $Z_+^2 \geq 4/3$; otherwise evaluate the polynomial at $x = 8/(8 + \sqrt{64 - 48Z_+^2})$. With $c_2 = 2/(3\sqrt{3})$ and $c_3 = 3\sqrt{3}/16$, componentwise bounds are

$$\begin{aligned} M_\theta &= A_+^2 T_1, & M_{\theta\theta} &= A_+^2 T_2, \\ M_{R,i} &= 2A_+ a_1 + c_2 k A_+^2 + U_i M_\theta, \\ M_{\theta R,i} &= 2A_+ a_1 T_1 + A_+^2 k \sqrt{1 + c_3^2 Z_+^2} \\ &\quad + U_i M_{\theta\theta}. \end{aligned} \quad (70)$$

Fixed-arc differentiation gives these bounds; rectangular averaging preserves them. For fixed gains, $B_x = \|W \text{diag}(H)\|_2 \|(M_{x,i})_i\|_2$, with $W = C_G^{-1/2}$ for raw channels or $W = L_s^{-1} T_s$ after processing. Here L_s is lower triangular and $L_s L_s^H = T_s C_G T_s^T$. Protocol P records the 240-by-100 cells, modal order 16, witness points, and double-precision evaluations.

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