

# Bearing Information and Geometric Bias in Finite-Gauge Helical DAS Arrays With Calibration Uncertainty

Yuanyuan Cao, Ning Hu, Chengtao Huang, and Kunjie Shi

**Abstract**—directional information. We analyze this loss and geometric bias in helical distributed acoustic sensing arrays under a longitudinal-plane-wave model. A phase-preserving angular filter distinguishes complete response cancellation from nonzero single-mode responses for which an unknown complex source amplitude absorbs the dependence on bearing. Near an integer-turn information null, an endpoint formula gives the coefficient of quadratic information recovery for a finite array. Analytic remainder bounds and a radius-uncertainty exclusion band provide local criteria for gauge selection. Finite sampling can suppress the recovery entirely. Decomposing the radius derivative separates rotation accumulated through registration from other shape changes. In a 60-channel synthetic example, a 2% radius increase produces  $13.287^\circ$  pseudo-true bias at fixed arc coordinates but  $0.00559^\circ$  at fixed material winding coordinates. A continuous-domain comparison confirms that the nominal frequency–gauge choice maximizes worst-case conditional information over the tested bearing range and  $\pm 2\%$  radius interval. In an off-grid, full-circle estimation study, the task selected by signal energy has 3.05 times the mean squared error of the nominal choice. Nominal and interval-aware rules select the same action and have identical errors. Finite calibration changes the predicted risk but leaves this choice unchanged over the tested budgets.

**Index Terms**—distributed acoustic sensing, helical array, bearing estimation, gauge length, Fisher information, model mismatch, calibration

## I. INTRODUCTION

**H**ELICALLY wound fiber gives distributed acoustic sensing (DAS) a directionally varying strain response [1], [2]. A recorded channel coherently averages that response over a finite gauge. Gauge length, cable geometry, and arrival direction therefore affect both the measured signal and the information available for bearing estimation [3]–[5]. The design problem is to choose a frequency and gauge that preserve distinguishable

bearing changes while accounting for uncertain geometry and finite calibration. Signal energy alone cannot answer this question: changes absorbed by an unknown source amplitude do not identify bearing. Choosing a longer gauge for noise reduction can therefore sacrifice bearing information, even when a point-response model predicts improvement.

Näsholm et al. [3] combine cable layout, strain directivity, and gauge averaging in a slowness-space steered response. Lim Chen Ning and Sava [6] use multiple fiber geometries to reconstruct strain tensors and assess conditioning. Whole-turn component cancellation [7] and regular-helix rank deficiency [8] have also been established under approximately constant strain within a gauge. These reconstruction questions differ from bearing estimation with an unknown source amplitude. Xenaki and Gerstoft [9] relate acquisition windows to DAS beampatterns, equivalent aperture, and spatial coherence; Cuny et al. [4] select variable gauges using signal-to-noise ratio and distortion. Practical gauge selection [5] and helical amplitude–frequency response [10] provide additional measurement context.

Projected information and misspecified likelihood are established tools [11]–[14]. Belloni et al. [15] relate Fourier manifold separation, modal truncation, and calibration noise to direction error. Related calibration methods address gain/phase errors [16], sensor-position errors for distributed sources [17], and absolute-reference constraints [18], [19]. Our point-channel preprint [20] analyzes the amplitude–delay crossover, pseudo-true bias, processing-chain compatibility, and local sandwich risk. We use those point-channel results as a baseline for analyzing the phase-preserving finite-gauge operator and its geometric registration.

We study frequency–gauge selection through the bearing changes that remain distinguishable after fitting an unknown complex amplitude. The finite-gauge operator determines where this information vanishes; geometric registration determines the bias caused by radius error. Calibration then propagates uncertainty in these geometric and transfer parameters into the bearing risk used for task selection.

We derive an explicit finite-array endpoint coefficient and a finite remainder bound for recovery near integer-

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turn information nulls. The phase-preserving angular filter distinguishes response cancellation from nonzero single-mode information nulls and identifies sampling degeneracy (Sec. III). Decomposing the radius derivatives separates rotation accumulated through registration from shape changes and gives path-specific local bias criteria (Sec. IV). Continuous-domain bounds support conditional-information comparisons over a prescribed radius interval. Finite-calibration risk connects these results to task selection (Sec. V). Derivations are in Supplement Appendices A–I; numerical settings are in the accompanying code archive’s computational protocol, `computation/COMPUTATIONAL_PROTOCOL.md`.

## II. HELICAL OBSERVATION AND PROJECTED INFORMATION

### A. Averaging before estimation

Let  $R$  be helix radius,  $p$  its axial pitch per turn,  $h = p/(2\pi)$ , and  $\ell = (R^2 + h^2)^{1/2}$ . With physical fiber arc coordinate  $s$ , winding phase  $\psi(s) = s/\ell + \gamma$ , helix position and unit tangent are

$$r(s) = (R \cos \psi, R \sin \psi, hs/\ell)^T, \quad (1)$$

$$t(s) = (-R \sin \psi/\ell, R \cos \psi/\ell, h/\ell)^T. \quad (2)$$

For azimuth  $\theta$ , elevation  $\phi$ , and speed  $c$ , define  $u = (\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi)^T$  and  $k = 2\pi f/c$ . Wave speed is known throughout. Elevation is known except in the E5 joint-angle calculation. We use the kinematic transfer  $\varepsilon_f(s) = t(s)^T \varepsilon_{\text{host}}(r(s))t(s)$  for a prescribed longitudinal plane wave. With  $H_i$  acting after averaging, the strain-coefficient response of channel  $i$  is

$$a_{G,i}(\theta, \eta) = \frac{H_i(\eta)}{G} \int_{s_i-G/2}^{s_i+G/2} [t(s)^T u]^2 \exp[-jk r(s)^T u] ds. \quad (3)$$

The model assumes a longitudinal plane wave in a uniform medium, with displacement parallel to  $u$ . A point channel samples tangent strain at one fiber location without spatial averaging: it is the  $G \rightarrow 0$  limit of (3). Unless specified otherwise, numerical examples use  $N = 60$ ,  $R = 0.25$  m,  $p = 0.02$  m,  $s_i = 0.1i$  m,  $\gamma = 0$ , and  $c = 1500$  m/s. Angles in equations are expressed in radians unless stated otherwise. Figs. 1–2 use a separate illustration geometry with a  $20^\circ$  winding angle, specified in their captions. In Fig. 1, the upper panel measures response energy and the lower panel measures distinguishable bearing changes. The difference at one turn is analyzed in Sec. III.

For independent snapshots  $y_l = \beta_l a_G + n_l$  with proper complex Gaussian noise  $n_l \sim \mathcal{CN}(0, C)$  and known, parameter-independent Hermitian positive-definite covariance  $C$ , let  $D = [a_{\cdot,1}, \dots, a_{\cdot,P}]$  contain response

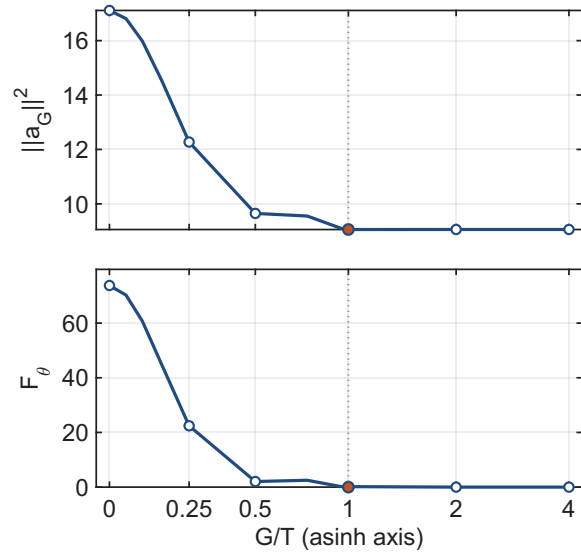


Fig. 1. A complete one-turn gauge retains response energy while eliminating bearing information at horizontal incidence. The synthetic slice uses  $R = 0.25$  m,  $\alpha = \tan^{-1}[p/(2\pi R)] = 20^\circ$ ,  $\theta = \phi = 0$ ,  $kR = 1$ ,  $\rho_E = 1$ ,  $H_i = 1$ , and 60 centers  $s_i = 0.1i$  m. At one turn, response energy is about 9.06 while bearing information is numerically zero. Here  $T = 2\pi\ell$  is one turn; the abscissa uses asinh spacing.

derivatives for real parameters and  $E = \sum_l |\beta_l|^2$ . The responses  $a_G, H_i$  are dimensionless;  $y_l, \beta_l, n_l$  share a strain-coefficient unit, so  $E$  has squared strain-coefficient units. Removing each unknown complex amplitude gives

$$S_a = C^{-1} - C^{-1} a (a^H C^{-1} a)^{-1} a^H C^{-1}, \quad (4)$$

$$J = 2E \operatorname{Re}\{D^H S_a D\}. \quad (5)$$

Amplitude elimination requires  $a^H C^{-1} a > 0$ . Zero responses are recorded separately.

The model’s physical amplitude and propagation derivatives are

$$d_{A,i} = H_i \int w_i(s) 2q(s) q_{,\theta}(s) e^{-jkz(s)} ds, \quad (6)$$

$$d_{\tau,i} = H_i \int w_i(s) (-jkq(s)^2 z_{,\theta}(s)) e^{-jkz(s)} ds, \quad (7)$$

where  $q = t^T u$ ,  $z = r^T u$ , and  $w_i$  is the normalized gauge kernel with units  $\text{m}^{-1}$ . For direction-independent  $H_i$ ,  $a_{,\theta} = d_A + d_\tau$ . Thus

$$F_\theta = F_A + F_\tau + F_\times, \quad F_\times = 4E \operatorname{Re}(d_A^H S_a d_\tau), \quad (8)$$

where  $F_X = 2E \operatorname{Re}(d_X^H S_a d_X)$ . Unlike the special point, diagonal-noise case, coherent averaging generally allows a signed mixed term. Equal  $F_A$  and  $F_\tau$  therefore need not mean that each supplies half of total information. If  $H_i$  depends on  $\theta$ , its derivative requires a separate physical term.

For an ideal spatially white input noise density  $q_0$  averaged by complete gauges, plus independent post-gauge readout variance  $\sigma_r^2$ , the strain-coefficient covariance is

$$[C_G]_{ij} = q_0 \int w_i(s)w_j(s)ds + \sigma_r^2 \delta_{ij}. \quad (9)$$

Gauge overlap makes this covariance non-diagonal. We also use a fixed post-gauge white-noise model to isolate the analytic mechanism. The density  $q_0$  has units of observation variance times length, so both terms in (9) have covariance units. Readout variance is defined after conversion to the averaged strain coefficient. If the optical conversion depends on gauge length, that dependence must also enter  $C_G$  (Supplement Appendix A).

### B. Acquisition and subsequent spatial averaging

The acquisition gauge and subsequent averaging window are distinct [9]. A realizable digital example applies weights  $(1/4, 1/2, 1/4)$  to adjacent channel coefficients. Let  $T_s \in \mathbb{R}^{58 \times 60}$  implement this valid convolution. Its response and covariance are

$$\tilde{a} = T_s a_G, \quad \tilde{C}_G = T_s C_G T_s^T. \quad (10)$$

For equal channel gains and spacing  $\Delta s$ , the effective nonrectangular kernel is a weighted sum of three shifted acquisition rectangles. Its modal transfer is  $\text{sinc}(\kappa_n G/2) \cos^2(\kappa_n \Delta s/2)$ , where  $\kappa_n = (n - \nu)/\ell$ . For unequal gains, (10) acts on the complete response. With both responses nonzero,  $\tilde{F}_\theta \leq F_\theta$ ; the acquisition's removed modes remain absent (Supplement Appendix I).

## III. ANGULAR MODES, FINITE ARRAYS, AND INFORMATION LOSS

### A. Gauge transfer and modal information

Set  $\chi = \psi - \theta$  and

$$A = R \cos \phi / \ell, \quad B = h \sin \phi / \ell, \quad (11)$$

$$\zeta = kR \cos \phi, \quad \nu = kh \sin \phi. \quad (12)$$

Then  $t^T u = B - A \sin \chi$  and  $kr^T u = \zeta \cos \chi + \nu s / \ell$ . Expand the periodic part as

$$(B - A \sin \chi)^2 e^{-j\zeta \cos \chi} = \sum_{n \in \mathbb{Z}} c_n e^{jn\chi}. \quad (13)$$

For a complete rectangular gauge, direct integration yields

$$a_{G,i} = H_i e^{-j\nu s_i / \ell} \sum_n b_n e^{jn(\psi_i - \theta)}, \quad (14)$$

$$b_n = c_n \text{sinc}\left(\frac{(n - \nu)G}{2\ell}\right), \quad \text{sinc}(x) = \frac{\sin x}{x}. \quad (15)$$

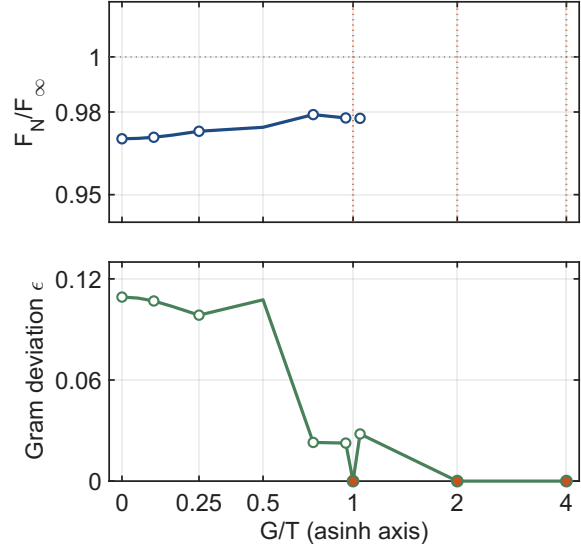


Fig. 2. The continuous modal approximation slightly overestimates finite-array information in the plotted nondegenerate cases. The slice uses  $R = 0.25$  m,  $\alpha = 20^\circ$ ,  $\theta = \phi = 0$ ,  $kR = 1$ ,  $N = 60$ ,  $\rho_E = 1$ , and  $H_i = 1$ . Top: finite/continuous information; bottom: the Gram deviation  $\epsilon$  for modal order  $M = 20$ . Here  $T = 2\pi\ell$  is one turn. The bound applies to this truncated subspace. At one, two, and four turns, the information ratio is undefined because bearing is unidentifiable, so the plotted line breaks.

The  $\nu$  shift accounts for the effect of nonzero elevation on the angular-mode filter inside the gauge. The channel factor  $H_i$  acts after this averaging. In Fig. 2, the upper panel tests the continuous modal approximation and the lower panel measures the finite-center correction on the retained subspace.

Under continuous uniform whole-turn center sampling, equal white readout noise,  $H_i \equiv 1$ , and  $S_0 = \sum_n |b_n|^2 > 0$ , modal orthogonality reduces (5) to

$$F_{\theta,\infty} = 2\rho_E N \left[ \sum_n n^2 |b_n|^2 - \frac{(\sum_n n |b_n|^2)^2}{S_0} \right], \quad (16)$$

where  $\rho_E = E/\sigma^2$ . The bracket is  $S_0$  times a weighted variance of mode index. One surviving mode gives no projected local bearing information even when its amplitude is nonzero.

Modal orthogonality does not generally hold for a finite array. For  $|n| \leq M$ , write  $V_{in} = H_i e^{-j\nu s_i / \ell} e^{jn(\psi_i - \theta)}$ ,  $b = (b_n)$ ,  $d = (-jnb_n)$ , and  $Q = C_G^{-1}$ . For  $\|C_G^{-1/2} V b\|_2 > 0$ , its exact truncated information is

$$F_N^{(M)} = 2E \min_{z \in \mathbb{C}} (d - bz)^H V^H Q V (d - bz). \quad (17)$$

A zero truncated response is recorded separately: regular amplitude elimination is then undefined, even if the complete response is nonzero. For white noise and equal channel gains, let  $\Gamma = V^H V / N$  and  $\epsilon = \|U^H(\Gamma - I)U\|_2$  for an orthonormal basis  $U$  of  $\text{span}\{b, d\}$ . When  $\epsilon < 1$ , the truncated continuous quantity obeys

$$(1 - \epsilon)F_\infty^{(M)} \leq F_N^{(M)} \leq (1 + \epsilon)F_\infty^{(M)}. \quad (18)$$

Supplement Appendix C uses the true infinite response and derivative tails to bound the difference between complete and truncated information. It also gives separate bounds on sampled-DFT aliases. E2 compares  $M = 20$  with direct quadrature; Sec. VI-A reports the independent checks based on the tail bounds.

*Theorem 1 (Complete integer-turn gauge):* Suppose  $\phi = 0$ , every channel averages a complete rectangular gauge  $G = m(2\pi\ell)$  for an integer  $m \geq 1$ , and there is no further direction-dependent transfer. Then  $a_{G,\theta} = 0$  at every channel center. If the response is nonzero, its projected bearing information is zero for any finite arrangement of channel centers.

*Proof:* Here  $\nu = 0$ , so  $\text{sinc}(nm\pi) = 0$  for all  $n \neq 0$  in (14). Only  $c_0$  remains, independent of  $\theta$ . The channel response is  $H_i c_0$ , where  $c_0 = (A^2/2)[J_0(\zeta) + J_2(\zeta)]$ . The array response is nonzero when  $c_0 \neq 0$  and some  $H_i \neq 0$ . For example,  $c_0$  vanishes at the first positive zero of  $J_1(\zeta)$ , about 3.831706.

*Oblique amplitude ambiguity.* If  $G = m2\pi\ell$  and  $m\nu \in \mathbb{Z}$ , then integer  $\nu = q$  leaves only mode  $q$ , giving  $a_i = H_i c_q e^{jq(\gamma - \theta)}$ . Its bearing derivative is  $-jq\alpha$ : for  $q \neq 0$  it is nonzero, but is removed by the unknown complex amplitude. If  $\nu$  is noninteger but  $m\nu$  is integer, every mode vanishes and the entire response is zero (Supplement Appendix B).

## B. Information recovery and a geometry exclusion band

*Theorem 2 (Near-turn information recovery):* Fix geometry, centers, direction-independent gains,  $E > 0$ , and known positive-definite, direction-independent covariance  $C$ . Let  $\nu = q_\star \in \mathbb{Z}$ ,  $G_\star = m2\pi\ell$  for integer  $m \geq 1$ , and  $a_\star = a(G_\star) \neq 0$ . Set  $G = G_\star(1 + \delta)$  with  $\delta > -1$  and  $W^H W = C^{-1}$ . Let  $a_e, a_{e,\theta}$  be the point response and derivative, including  $H_i$ , at the endpoints  $s_i + G_\star/2$ . With  $P_b^\perp = I - b(b^H b)^{-1}b^H$ , define  $b_\star = W a_\star$  and  $v_\star = W(a_{e,\theta} + jq_\star a_e)$ . Then

$$K_{\text{turn}} = \|P_{b_\star}^\perp v_\star\|_2^2, \quad (19)$$

$$F_\theta(G) = 2EK_{\text{turn}}\delta^2 + O(|\delta|^3).$$

The coefficient is positive exactly when  $a_{e,\theta} + jq_\star a_e \notin \text{span}_{\mathbb{C}}\{a_\star\}$ .

Supplement Appendix B derives this endpoint coefficient and a finite-detuning bound. If  $K_{\text{turn}} = 0$ , the

information is  $O(\delta^4)$  and may vanish identically: centers separated by whole turns at horizontal incidence all sample one winding phase. For  $b = W a(G)$  and  $w = W(a_{e,\theta} + jq_\star a_e)$ , let analytic bounds obey  $\|b - b_\star\| \leq r_a < \|b_\star\|$  and  $\|w - \delta v_\star\| \leq r_w$ . Set  $\Delta = r_w + |\delta| \|v_\star\| r_a / \|b_\star\|$ . Then

$$2E[|\delta|\sqrt{K_{\text{turn}}} - \Delta]_+^2 \leq F_\theta(G) \leq 2E[|\delta|\sqrt{K_{\text{turn}}} + \Delta]^2. \quad (20)$$

Here  $[x]_+ = \max(x, 0)$ ; Supplement Appendix B supplies full-series moment bounds for  $r_a, r_w$ . A positive lower bound establishes local bearing information.

The fixed-covariance bound extends to the design-dependent  $C_G$ . If  $\lambda_- C_{\text{ref}} \preceq C_G \preceq \lambda_+ C_{\text{ref}}$  with  $\lambda_- > 0$ , inverse ordering and amplitude minimization give

$$\frac{F_{C_{\text{ref}}}}{\lambda_+} \leq F_{C_G} \leq \frac{F_{C_{\text{ref}}}}{\lambda_-}. \quad (21)$$

The extreme generalized eigenvalues of  $(C_G, C_{\text{ref}})$  provide these factors. Applying them to (20) gives a finite bound under the actual noise covariance. Here  $G$  indexes a known design, rather than an estimated parameter.

At horizontal incidence, radius uncertainty gives a separate exact exclusion. With fixed physical  $G$  and pitch  $p$ , put  $T(R, p) = \sqrt{(2\pi R)^2 + p^2}$ . If  $R \in [R_-, R_+]$  and

$$mT(R_-, p) \leq G \leq mT(R_+, p), \quad (22)$$

some allowed radius makes  $G$  an integer-turn gauge. Its nonzero response has zero bearing information; a zero response instead falls outside regular amplitude elimination. Avoiding these bands is necessary, not sufficient, for regular positive information throughout the radius interval.

## C. Crossover under gauge averaging

We use the amplitude-delay crossover to diagnose changes in the information components. Task comparisons use total projected information and finite-calibration risk. Define the physical balance function  $\mathcal{H}(f, G, \eta) = F_A(f, G, \eta) - F_\tau(f, G, \eta)$ . A crossover is a root of  $\mathcal{H} = 0$  with positive bearing information. With fixed post-gauge white covariance and a smooth point response,  $a_G = a_0 + (G^2/24)a_{0,ss} + O(G^4)$ . A simple point root  $f_0$  with  $\mathcal{H}_f \neq 0$  continues in  $t = G^2$  as

$$f_\times(t) = f_0 - \frac{\mathcal{H}_t}{\mathcal{H}_f} t + O(t^2). \quad (23)$$

The derivative includes the changing amplitude projection. Supplement Appendix D gives the response remainder; Sec. VI-A checks the local range. For the nominal one-turn gauge at  $\theta = 30^\circ$ , the oblique roots at  $\phi =$

$15^\circ, 30^\circ$  have only  $F_\theta = 1.2475 \times 10^{-4}, 3.7649 \times 10^{-4}$  rad $^{-2}$  at  $\rho_E = 1$ . A balance root need not provide useful total information.

#### D. Continuous-domain design comparison

For worst-case information design [21], the helical operator provides computable bounds over  $\Omega = \Theta \times [R_-, R_+]$ . For a candidate  $d = (f, G)$  and a cell  $\mathcal{B} \subset \Omega$ , write  $b = C_G^{-1/2}a$  and  $v = C_G^{-1/2}a_{,\theta}$ . At its center, let  $\bar{b}, \bar{v}$  have error-inclusive envelopes  $\|b - \bar{b}\| \leq A_{\mathcal{B}} < \|\bar{b}\|$  and  $\|v - \bar{v}\| \leq D_{\mathcal{B}}$  throughout the cell. Set  $q_{\mathcal{B}} = \|P_b^\perp \bar{v}\|$  and  $\Delta_{\mathcal{B}} = D_{\mathcal{B}} + \|\bar{v}\|A_{\mathcal{B}}/\|\bar{b}\|$ . Then

$$2E[q_{\mathcal{B}} - \Delta_{\mathcal{B}}]_+^2 \leq F_d(\theta, R) \leq 2E[q_{\mathcal{B}} + \Delta_{\mathcal{B}}]^2. \quad (24)$$

Supplement Appendix I derives analytic helical derivative envelopes and includes the infinite modal tail in the center error.

*Theorem 3 (Interval design comparison):* Suppose each candidate has nonzero response on  $\Omega$ , and let cells cover  $\Omega$ . For each candidate, let  $\underline{m}_d$  be the smallest cell lower bound in (24), and  $\overline{m}_d$  the smallest information upper bound at any evaluated point in  $\Omega$ . Then  $\underline{m}_d \leq \inf_{\Omega} F_d \leq \overline{m}_d$ . A candidate  $d_*$  uniquely maximizes worst-case conditional information if

$$\underline{m}_{d_*} > \max_{d \neq d_*} \overline{m}_d. \quad (25)$$

Together, the cell bounds cover the continuous domain, whereas a single evaluated point gives an upper bound on each competitor's minimum. For a nominal choice  $d_0$ , finding  $\underline{m}_{d_0} > \overline{m}_{d_0}$  proves that the choice must change. Conversely, (25) with  $d_* = d_0$  establishes its persistence over the prescribed uncertainty interval. A nominal choice inside the horizontal radius-exclusion band has  $\inf_{\Omega} F_d = 0$  when its crossing response is nonzero. Any candidate with  $\underline{m}_d > 0$  then forces a different conditional-information choice.

#### IV. PHYSICAL GEOMETRY ERROR AND PSEUDO-TRUE BEARING

The preceding information criteria condition on the geometry. A geometry error can shift the fitted bearing even when information is positive. Its bias depends on how the physical array changes.

##### A. Four different radius changes

A radius change must be specified together with the quantities held fixed. We consider four paths from the same nominal geometry. The fixed-arc path keeps  $s_i, p, \gamma, G$  fixed. The fixed-winding radial path keeps center and material-gauge winding coordinates fixed, so  $s_i$  and  $G$  scale with  $\ell(R)$ . The center-anchored path

keeps  $s_i, p, G$  fixed and adjusts  $\gamma$  to preserve  $\psi(\bar{s})$  at the array's arc midpoint,  $\bar{s} = (s_0 + s_{N-1})/2$  (2.95 m in the nominal array). Proportional scaling multiplies  $R, p, s_i, G$  by the same factor at fixed frequency. For the fixed-winding path, with pitch fixed and  $\ell' = [(R')^2 + h^2]^{1/2}$ , the material map is

$$(s'_i, G') = (\ell'/\ell)(s_i, G). \quad (26)$$

Optical range addresses are calibrated to physical arc length. Fixed arc compares equal addresses, fixed winding follows marked material locations and endpoints, center anchoring preserves the midpoint fiducial, and proportional scaling represents uniform dimensional scaling. The installed cable's displacement map determines the relevant path.

For the fixed-arc path,

$$\psi_{,R}(s) = -\frac{sR}{\ell^3}. \quad (27)$$

Let  $a_0(s)$  be the point response before multiplication by  $H_i$ , and  $r_R^{\text{mat}}$  its radius derivative at fixed material winding coordinate, pitch, and rotation. At fixed arc coordinate, axial contraction contributes as well. Define

$$r_R^{\text{point}}(s) = r_R^{\text{mat}}(s) + j\nu \frac{sR}{\ell^3} a_0(s). \quad (28)$$

The added term vanishes at horizontal incidence. The chain rule gives

$$a_{0,R}(s) = \frac{sR}{\ell^3} a_{0,\theta}(s) + r_R^{\text{point}}(s). \quad (29)$$

With a fixed rectangular gauge and direction-independent fixed  $H_i$ , this separates into

$$\begin{aligned} a_{G,i,R} &= \frac{Rs_i}{\ell^3} a_{G,i,\theta} + \frac{R}{\ell^3} H_i \int_{-G/2}^{G/2} \frac{\xi}{G} a_{0,\theta}(s_i + \xi) d\xi \\ &\quad + H_i \int_{-G/2}^{G/2} \frac{1}{G} r_R^{\text{point}}(s_i + \xi) d\xi. \end{aligned} \quad (30)$$

The three terms describe accumulated winding changes at the centers, winding changes within each gauge, and the remaining explicit dependence on radius, including axial phase at oblique incidence. For relative radius  $r = (R - R_0)/R_0$ ,  $a_{,r} = R_0 a_{,R}$ . For variable-gauge paths, derivatives of the kernel and endpoints add terms to (30). Center anchoring instead changes the fixed-arc derivative by  $-(R\bar{s}/\ell^3) a_{,\theta}$ , since  $a_{,\gamma} = -a_{,\theta}$  (Supplement Appendix D).

For actual array mean  $\mu$ , nominal response  $a_{\text{nom}}(\theta)$ , and positive-definite task covariance  $C$ , define  $d(\theta) = \min_{\alpha \in \mathbb{C}} \|\mu - \alpha a_{\text{nom}}(\theta)\|_{C^{-1}}^2$ . An array pseudo-true bearing is any  $\theta^* \in \arg \min_{\theta \in \Theta} d(\theta)$ : the bearing whose

nominal response best fits the actual mean after adjusting the source amplitude. Equivalently, it maximizes covariance-whitened normalized coherence  $c(\theta)$ .

The physical bearing remains  $\theta_0$ . On a compact search domain, a global sample minimizer is consistent for the minimizer set of  $d$  if the criterion converges uniformly in probability and that set is well separated. A nonunique target requires a branch rule. Finite-sample errors also include fluctuations and competing-peak selection; Table I separates array bias, fit residual, and processing offset.

Let  $a$  denote the nominal response at true bearing  $\theta_0$ ,  $\delta a$  the response perturbation with the common source amplitude factored out, and  $I_\theta = \text{Re}(a_\theta^H S_a a_\theta)$ . For  $I_\theta > 0$  and an unchanged smooth optimum branch, the amplitude-profiled local pseudo-true shift is

$$\delta\theta_{\text{pt}} = \frac{\text{Re}(a_\theta^H S_a \delta a)}{I_\theta} + O(\|\delta a\|^2). \quad (31)$$

Perturbations orthogonal to the bearing tangent contribute to the fit residual but not to the first-order shift. Searches of the full objective refine detected peaks and record the gap to the runner-up. In 18 conditions used to examine the fixed-arc mechanism, the three derivative components sum to the first-order shift within  $6.76 \times 10^{-7}$  degree. At  $30^\circ$ , 800 Hz, and  $G = 0.25$  m, a 2% radius step gives  $13.67658^\circ$  from center winding,  $0.00272^\circ$  from within-gauge winding, and  $0.00561^\circ$  from the remaining shape changes. The sum,  $13.68491^\circ$ , differs from the finite-change result of  $13.28707^\circ$ ; the 2% step lies outside the prespecified local domain.

At horizontal incidence, fixed-winding and proportional scaling give responses differing only by a common amplitude factor. Their pseudo-true bearing sets therefore coincide for the same fitting covariance and channel transfer; the last-digit differences in Table I are numerical (Supplement Appendix D).

### B. Applicability of the first-order bias approximation

We apply (31) at interior optima satisfying  $-c''(\theta^*) > 0$ , a resolvable gap to the next detected maximum, and continuation of the same best peak as radius changes. The E4 geometry scan uses the  $[-90^\circ, 90^\circ]$  sector. Its boundary optima remain flagged in sector statistics, not assumed to be full-cycle estimates.

Of 2952 E4 configurations, 504 satisfy the stated local checks. The full-cycle refit identifies 12 sector-censored optima at  $90.45^\circ$ – $92.46^\circ$ ; the remaining 2940 agree with the sector results within  $10^{-4}$  degree. Eighteen fixed-arc groups exhibit approximately second-order small-step residuals. The fixed-winding +2% example uses the deformed material gauge  $G' \simeq 0.255$  m and gives

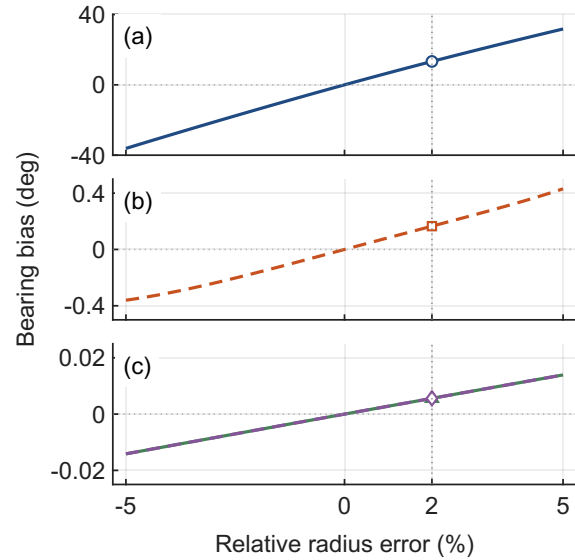


Fig. 3. The same radius error gives markedly different bearing bias under different registration paths. At +2%, fixed-arc and fixed-winding biases are  $13.287^\circ$  and  $0.00559^\circ$ , respectively. For  $\theta_0 = 30^\circ$ , 800 Hz, and nominal  $G = 0.25$  m, (a) shows fixed arc, (b) anchoring at  $\bar{s} = 2.95$  m, and (c) fixed winding (solid) and proportional scaling (dashed). Vertical scales differ across panels; (c) uses a common ordinate for its two paths. The dotted guide marks +2%; all 164 shown optima are interior to the specified sector.

TABLE I  
FOUR RADIUS-REGISTRATION PATHS FOR A +2% RADIUS CHANGE,  $\theta_0 = 30^\circ$ ,  $\phi = 0$ , 800 HZ, AND NOMINAL  $G = 0.25$  M. THE TABLE REPORTS ARRAY BIAS, NORMALIZED SQUARED FIT RESIDUAL  $r_{\text{fit}}$ , AND GAP, DEFINED AS THE NOISELESS MUSIC ESTIMATE MINUS THE ARRAY PSEUDO-TRUE BEARING.

| Path            | Array bias ( $^\circ$ ) | $r_{\text{fit}}$      | Gap ( $^\circ$ )       |
|-----------------|-------------------------|-----------------------|------------------------|
| Fixed arc       | 13.287072               | 0.026531              | $+9.35 \times 10^{-6}$ |
| Fixed winding   | 0.005591                | $3.72 \times 10^{-5}$ | $-5.82 \times 10^{-6}$ |
| Center anchored | 0.165487                | 0.026142              | $+4.29 \times 10^{-6}$ |
| Proportional    | 0.005592                | $3.72 \times 10^{-5}$ | $-7.01 \times 10^{-6}$ |

$0.005591^\circ$  bias, independently reproduced in MATLAB (computational protocol, Section J).

Under the same condition, each path's array response generates the phase sequence for a noiseless Hann–DFT–MUSIC calculation. Table I reports array bias, the offset of the processing target, and the normalized squared residual  $r_{\text{fit}} = d(\theta^*)/(\mu^H C^{-1} \mu) = 1 - c(\theta^*)$ . Here  $C \propto I$ ; Sec. VI-C gives noisy processing checks.

## V. FINITE CALIBRATION AND TASK CHOICE

### A. Common nuisance with per-record amplitudes

The fixed-arc radius and rotation are shared unknowns, together with one smooth gain and one phase mode.

These four parameters relate the geometric sensitivities of Sec. IV to calibration uncertainty:

$$\eta = (r, \gamma, b_g, b_\varphi)^T, \quad H_i = \exp\{b_g B_i + j b_\varphi C_i\}, \quad (32)$$

where  $B_i = \cos(2\pi i/N)$  and  $C_i = \sin(2\pi i/N)$ . Here  $r, b_g$  are dimensionless and  $\gamma, b_\varphi$  are in radians. Within each known-direction calibration record, an unknown complex amplitude is shared across  $K_j$  snapshots;  $\eta$  is shared across all records. Summing the amplitude-eliminated information over records gives

$$J_{\text{cal}} = \sum_j 2K_j |\beta_j|^2 \text{Re}(D_{\eta,j}^H S_{a_j} D_{\eta,j}). \quad (33)$$

Records are independent. The derivative  $a_{,r} = R_0 a_{,R}$  includes the gauge-interior term; nuisance coordinates are scaled before numerical rank assessment.

In the axisymmetric ideal model,  $a(\theta + \Delta, \gamma + \Delta) = a(\theta, \gamma)$ ; equivalently,  $a_{,\theta} + a_{,\gamma} = 0$ . A task with both unknowns therefore has a rotation null direction. Known-direction calibration can break this absolute-reference ambiguity, subject to its own rank and global search. A complete horizontal integer-turn task gauge still has zero bearing information even if calibration determines  $\gamma$ .

### B. Two estimators, two local predictions

On a regular branch with  $J_{\theta\theta} > 0$ , independent matched task and calibration data allow joint fitting to eliminate the nuisance block of the combined information.

The calibrate-then-estimate workflow instead fits  $\hat{\eta}$  using independent calibration data and holds it fixed for the task. Its first-order matched variance is

$$V_{\text{plug}} \simeq J_{\theta\theta}^{-1} + L C_{\text{cal}} L^T, \quad L = J_{\theta\theta}^{-1} J_{\theta\eta}, \quad (34)$$

where  $C_{\text{cal}}$  includes all four-parameter cross-covariances. Calibration bias and residual mean mismatch add a squared mean term.

For efficient calibration,  $C_{\text{cal}} = J_{\text{cal}}^{-1}$  with  $J_{\text{cal}} \succ 0$ . The matched local variances satisfy

$$\begin{aligned} J_{\eta|\theta} &= J_{\eta\eta} - J_{\eta\theta} J_{\theta\theta}^{-1} J_{\theta\eta}, \\ V_{\text{joint}} &= J_{\theta\theta}^{-1} + L(J_{\text{cal}} + J_{\eta|\theta})^{-1} L^T. \end{aligned} \quad (35)$$

Here  $J_{\eta|\theta} \succeq 0$  is task nuisance information after bearing elimination. Thus  $V_{\text{joint}} \leq V_{\text{plug}}$ , with equality exactly when  $J_{\eta|\theta} J_{\text{cal}}^{-1} L^T = 0$ . Rotation symmetry makes its  $\gamma$  row and column zero. A useful gain requires information along nuisance directions that contribute to bearing uncertainty (Supplement Appendix E). Under mismatch, each workflow has its own pseudo-true target. Supplement Appendix E gives the task sandwich variance  $V_{\text{mis}}$  and sensitivity  $S_{\text{mis}} = d\theta^*/d\eta_{\text{fit}}$ .

Let  $m_{\text{cal}}, C_{\text{cal}}$  be calibration-error moments about its noiseless target, and  $b_{\text{res}}$  the corresponding task-bearing offset. Linearizing bearing error gives

$$\begin{aligned} \mathcal{R}_{\text{mis,plug}} &\simeq V_{\text{mis}} + S_{\text{mis}} C_{\text{cal}} S_{\text{mis}}^T \\ &\quad + (b_{\text{res}} + S_{\text{mis}} m_{\text{cal}})^2. \end{aligned} \quad (36)$$

This is the MSE of the linearized bearing error. At fixed nonzero  $b_{\text{res}}$ , curvature contributes a term of the same order in  $C_{\text{cal}}$ ; Supplement Appendix E gives its form and Appendix F evaluates it for the numerical example. Joint fitting uses the full task-calibration sandwich, including each record's amplitude [12]–[14].

### C. Finite choices and deterministic design

Information-based array design includes average-CRB design with uncertain wave speed [22]. For a fixed calibration design and candidate task  $d = (f, G)$ , let  $V_0(d)$  denote the mean conditional variance over the design bearings and let  $P(d)$  denote the corresponding mean  $LC_{\text{cal}} L^T$ . In the matched, first-order plug-in model,  $R(d) = V_0(d) + P(d)$ . If  $d_0$  minimizes  $V_0$ , it also minimizes  $R$  exactly when, for every candidate  $d$ ,

$$P(d_0) - P(d) \leq V_0(d) - V_0(d_0). \quad (37)$$

Only differential calibration penalties can change this ranking; a task-independent penalty leaves it unchanged.

The design compares 42 pairs of frequencies  $\{200, 500, 800, 1100, 1400, 1700, 2000\}$  Hz and gauges  $\{0.05, 0.10, 0.25, 0.50, 0.75, 1.00\}$  m over five equally weighted bearings  $\{-60, -30, 0, 30, 60\}^\circ$ . Calibration budgets 4, 8, 18, 36 draw from 60 known-direction settings. Budget zero has no absolute rotation reference; its finite-calibration risks are undefined. Uniform selection follows one fixed random permutation; greedy selection first increases scaled information rank, then minimizes nominal joint risk. The point-crossover heuristic rounds the median point root, 1189.160079 Hz, to 1100 Hz and fixes  $G = 0.25$  m. At nominal  $\eta = 0$  and budget 18, conditional-information, plug-in-risk, and truth-informed oracle rules all choose 2000 Hz,  $G = 0.10$  m. The oracle is an evaluation reference. Computational protocol Section G specifies all source and tie rules; Sec. VI-D evaluates these choices with full-circle fits.

At budget 18 in the matched four-parameter model,  $P(d)$  varies across the 42 candidates by  $6.43 \times 10^{-9} \text{ rad}^2$  for uniform and  $6.31 \times 10^{-9} \text{ rad}^2$  for greedy sources. The conditional-variance gap from the best candidate to its nearest competitor is  $1.1924 \times 10^{-7} \text{ rad}^2$ . Thus even the largest differential calibration penalty is less than 5.4% of that gap and cannot change the minimizer in (37).

To identify the common component, partition  $C_c = C_{\text{cal}} \succ 0$  into blocks indexed by  $\gamma$  and  $\eta_- = (r, b_g, b_\varphi)$ ,

and set  $v = C_{--}^{-1}C_{-\gamma}$ . Completing the square using  $L_{\gamma} = -1$  gives

$$P(d) = c_{\text{ref}} + \langle (L_{-} - v^T)C_{--}(L_{-} - v^T)^T \rangle_d, \quad (38)$$

$$c_{\text{ref}} = C_{\gamma\gamma} - C_{\gamma-}C_{--}^{-1}C_{-\gamma},$$

Blocks refer to  $C_c$ ; brackets denote the design-bearing average. The task-invariant term is  $6.92 \times 10^{-8} \text{ rad}^2$  (uniform) or  $4.25 \times 10^{-8} \text{ rad}^2$  (greedy), over 99.8% of  $P$  at the selected task. It retains the cancelling rotation-radius-transfer cross-covariances (protocol Section L).

With efficient calibration, the zero rotation row of  $J_{\eta|\theta}$  preserves  $c_{\text{ref}}$  under joint fitting. Even removing the remaining penalty would reduce total predicted variance by less than 0.012%; (35) predicts reductions of 0.00120% and 0.000613% for uniform and greedy sources, respectively.

At fixed  $G = 0.25 \text{ m}$ , raising frequency to 2000 Hz reduces predicted plug-in risk by 40.03% (uniform) and 40.94% (greedy). Both point and full-gauge models prefer that frequency. Their gauge choices differ: with the same gauge-dependent covariance, the point-response optimizer chooses 1.00 m and the full model 0.10 m. Under the full model and uniform calibration, the point-response choice has risk  $7.58694 \times 10^{-6} \text{ rad}^2$ , compared with  $0.57980 \times 10^{-6} \text{ rad}^2$  for the full-model choice. Section VI-D isolates this signal-averaging omission at fixed frequency.

A nominal energy rule maximizing  $\langle a_G^H C_G^{-1} a_G \rangle$  over the same 42 candidates selects 200 Hz,  $G = 0.10 \text{ m}$ . Its mean conditional variance is  $1.71064 \times 10^{-6} \text{ rad}^2$ , compared with  $0.510545 \times 10^{-6} \text{ rad}^2$  for the information-selected task. At fixed 2000 Hz, the energy rule also selects 0.10 m. Thus the fixed-frequency ablation tests omitted averaging, while the full-pool comparison separates signal energy from bearing information (computational protocol, Section N).

Noise location also changes the selected gauge. Hold per-channel variance at  $G_{\text{ref}} = 0.25 \text{ m}$  to  $v_{\text{ref}} = 0.0057143$  and vary  $q = q_0/(G_{\text{ref}}\sigma_r^2)$ , so  $\sigma_r^2 = v_{\text{ref}}/(1+q)$  and  $q_0 = G_{\text{ref}}q\sigma_r^2$ . Calibration stays at its original 0.1-m, 20-dB setting.

For all three assumed nuisance vectors, both information and plug-in rules choose 2000 Hz, with  $G = 0.10 \text{ m}$  at  $q \leq 1$  and  $0.25 \text{ m}$  at  $q = 10$ . Table II reports nominal risks; greedy source choice uses the same task-noise assumption.

#### D. Uncertainty-aware selection with a fixed calibration budget

The decision study uses the same 42 candidates and uniform source prefixes at budgets  $B_{\text{cal}} \in \{4, 8, 18, 36\}$ . Tasks use horizontal incidence and fixed-arc geometry. It compares raw rectangular acquisition with (10), using

TABLE II  
NOMINAL MATCHED DESIGN SENSITIVITY AT BUDGET 18.  
SELECTED FREQUENCY IS 2000 HZ THROUGHOUT; RISKS ARE IN  $10^{-6} \text{ rad}^2$ . PLUG/U AND PLUG/G DENOTE PLUG-IN RISK WITH UNIFORM AND GREEDY CALIBRATION-SOURCE SELECTION, RESPECTIVELY.

| $q$ | $G \text{ (m)}$ | Conditional | Plug/U | Plug/G |
|-----|-----------------|-------------|--------|--------|
| 0.1 | 0.10            | 0.3315      | 0.4008 | 0.3740 |
| 1   | 0.10            | 0.5105      | 0.5798 | 0.5531 |
| 10  | 0.25            | 0.6763      | 0.7456 | 0.7189 |

the same 60 raw channels and coefficient budget for both paths. The local prediction at each geometry uses  $C_{\text{cal}}(R, B_{\text{cal}}) = J_{\text{cal}}^{-1}(R, B_{\text{cal}})$  and (34). The nominal and uncertainty-aware rules minimize its maximum over the same direction set; the latter also includes  $R/R_0 \in [1 - \epsilon_R, 1 + \epsilon_R]$ . Their evaluated direction nodes are  $-60^\circ : 5^\circ : 60^\circ$ . Primary half-widths are 0, 0.5%, 1%, and 2%; 5% is a fixed stress case. An energy rule maximizes mean  $a^H C_G^{-1} a$  at nominal radius. All rules use the complete acquisition model. The continuous comparison in Theorem 3 addresses conditional information; these finite-budget scores use the stated local risk on the declared nodes. Protocol Section P specifies the inputs and off-grid verification points.

## VI. NUMERICAL VALIDATION

Section VI-A checks the analytical identities and approximations. Sections VI-B–VI-D evaluate estimator errors under model mismatch, processing effects, and finite calibration. Section VI-E compares design stability over radius uncertainty.

### A. Deterministic mechanism and rank checks

The 42 candidate tasks in Sec. V use gauges within the fiber interval  $[-0.5, 6.4] \text{ m}$ . Input strain-coefficient energy is fixed across task choices. A displacement-field check scales displacement amplitude as  $1/k$  to preserve that strain budget.

The deterministic checks include 200 nondegenerate derivative comparisons (E1; maximum relative difference  $4.54 \times 10^{-11}$ ), 3240 gauge/mode cases (E2), 32 crossover traces (E3), 2952 geometry cases (E4), and 18 calibration-reference/rank cases (E5). An independent MATLAB implementation covers 30 conditions and 1800 channel responses, with all rank classifications agreeing.

Among regular E2 cases, the largest relative deviation  $|F_N^{(20)}/F_\infty^{(20)} - 1|$  between finite-array and continuous information is about 0.111. The 90 horizontal one-turn calculations have maximum numerical  $F_\theta = 1.43 \times 10^{-27}$ ; the theorem establishes the exact zero. The true

TABLE III  
NEAR-TURN COEFFICIENT AND RATIO OF QUADRATURE  
INFORMATION TO ITS QUADRATIC APPROXIMATION,  
 $F_\theta/(2EK_{\text{turn}}\delta^2)$ , FOR ONE-TURN GAUGES.

| Case                  | $K_{\text{turn}}$ | $\delta = -0.01$ | $\delta = +0.01$ |
|-----------------------|-------------------|------------------|------------------|
| A: nominal horizontal | 42.6098           | 1.01852          | 0.97852          |
| B: illustration       | 37.1666           | 1.01832          | 0.97837          |
| C: oblique, colored   | 77.2132           | 0.78117          | 0.75976          |

infinite-tail Bessel bound was evaluated at four E2 cells. At  $kR = 40$ , it is too loose to certify useful information, and roundoff-limited cases remain unresolved. The independent MATLAB implementation agrees with Python to  $5.68 \times 10^{-14}$  in information.

Theorem 2 is checked at four detunings  $\delta = \pm 0.01, \pm 0.001$  in three configurations. Case A uses the nominal horizontal 800-Hz array at  $30^\circ$ . Case B uses the  $20^\circ$ -winding,  $kR = 1$  illustration geometry. Both have  $C = I, H_i = 1$ . Case C has  $p = \pi/2$  m,  $k = 8$  m $^{-1}$ ,  $\theta = \phi = 30^\circ$ , and  $C_{ij} = 0.35^{|i-j|}$ , with fixed nonuniform gains specified in protocol Section O. All use  $N = 60, R = 0.25$  m and 0.1-m center spacing. The covariance is held fixed as  $G$  varies. The entries of  $C$  are normalized by the squared strain-coefficient unit;  $K_{\text{turn}}$  is reported in the corresponding inverse-variance units per radian squared.

Including a two-turn version of A, all 16 quadrature values fall inside (20) with positive lower bounds. The discrepancy between the endpoint derivative and its central-difference estimate is below  $3.94 \times 10^{-9}$ . Case C has  $\|b_\star\| = 0.05289$ . Reducing the detuning magnitude to 0.1% gives ratios 0.99896 and 0.99487, compared with the 1% values in Table III. With same-winding-phase centers, information instead vanishes for all gauges, as predicted. For  $T_0 = T(R_0, p)$  and a  $\pm 2\%$  radius interval, the nominal one-turn exclusion band is  $G/T_0 \in [0.9800033, 1.0199968]$ . Fixed gauges  $0.99T_0$  and  $1.01T_0$  have nonzero-response information nulls at radii 0.2474996 and 0.2525004 m, respectively.

For the nominal geometry at  $\theta = 30^\circ$  and  $\phi = 0$ , refinement places the crossover at 1252.580138 Hz for point channels and 1157.102804 Hz for a 0.25-m gauge. At the tested  $G/\ell \in [10^{-3}, 0.1]$  and elevations  $0^\circ, 15^\circ, 30^\circ$ , root-shift errors are below 0.163%. At  $G = 0.25$  m,  $G/\ell \simeq 1$ : the horizontal quadratic prediction differs from the full-integral root by 10.96 Hz, or 11.5% of its 95.48-Hz shift. No crossover root was detected between 50 and 6000 Hz in four E3 scans. Independent quadrature refinement shows that two nearly coincident candidates correspond to one simple root.

E5 uses  $\theta = 20^\circ, f = 800$  Hz,  $K = 100$ , and either  $G = 0.25$  m or one complete turn. Four

known-direction references use  $G_{\text{cal}} = 0.1$  m at  $(\theta, \phi, f)$  of  $(-60^\circ, -30^\circ, 200 \text{ Hz})$ ,  $(-30^\circ, 0, 800 \text{ Hz})$ ,  $(30^\circ, 30^\circ, 1400 \text{ Hz})$ , and  $(60^\circ, -30^\circ, 2000 \text{ Hz})$ . Without these references, unknown rotation produces the predicted null direction. With them, the ordinary-gauge oblique case at  $\phi = 20^\circ$  has bearing variances  $2.23855 \times 10^{-6}$  and  $2.24561 \times 10^{-6}$  rad $^2$  for known and unknown elevation. An oblique complete-turn case has a second full-circle peak with normalized gap  $4.68 \times 10^{-6}$ . In four fixed- $\eta$  two-dimensional searches, the horizontal complete-turn case has an azimuth ridge, while an oblique case has a near-equivalent alternative.

### B. Residual transfer mismatch

We evaluate local misspecified risk under a fixed higher-harmonic transfer at  $15^\circ, 800$  Hz,  $G = 0.25$  m, and  $K = 100$ . Across 30 calibration batches per budget and five paired task draws, increasing the number of calibration records from four to 18 lowers plug-in MSE from 0.005332 to 0.004297 deg $^2$ ; joint-fitting MSEs are 0.005354 and 0.004299. The larger budget also increases calibration energy. Supplement Appendix F reports full-covariance predictions, batch intervals, and a separate evaluation of 50 transfers, including three undefined unconditional-risk entries.

### C. Local approximation and processing checks

E6 compares six scenarios: point and 0.25-m matched responses, two  $+2\%$  radius-registration paths, residual transfer, and a near-turn  $G = 0.95(2\pi\ell)$  gauge. Using the shared-amplitude Gaussian mean statistic (protocol Section J), we evaluate 72 conditions combining the six scenarios,  $K \in \{20, 100, 500\}$ , and four input SNR levels from  $-10$  to  $20$  dB. Each condition starts with 2000 realizations, with three near-turn cells extended to 5000, for a total of 153000 realizations and 306000 fits (Supplement Appendix G). Fixed winding uses its deformed material-gauge length. Each observation is fitted over  $[-90^\circ, 90^\circ]$  and over the  $\pm 5^\circ$  window around its pseudo-true bearing, using a  $0.25^\circ$  grid and three-point refinement. Table IV compares the resulting distinct estimators.

At 0 dB the near-turn case has 66.96% of sector estimates more than  $5^\circ$  from the pseudo-true bearing and MSE 420.0 deg $^2$  (95% interval  $[371.7, 471.4]$ ). Its noiseless normalized peak gap is 0.00561, versus 0.75691 at 0.25 m. This gap is independent of noise level and cannot replace error evaluation.

For the fixed-arc  $+2\%$  radius condition with  $G = 0.25$  m and  $(K, \text{SNR}) = (20, -10 \text{ dB})$ , the 2000 sector-wide estimates have an MSE of 202.151 deg $^2$ . This comprises 33.759 deg $^2$  of mean squared error about

TABLE IV

LOCAL PREDICTIONS DESCRIBE HIGH-SNR ERRORS BUT A CONSTRAINED SEARCH CAN CONCEAL NEAR-TURN FAILURES. E6 MATCHED CASES AT  $K = 100, 800$  HZ, AND  $\theta_0 = 30^\circ$ ; VARIANCES AND MSE ARE IN  $\text{deg}^2$ , EXCEEDANCES IN PERCENT.  $T = 2\pi\ell$ . LOCAL MEANS A  $\pm 5^\circ$  WINDOW, AND SECTOR MEANS  $[-90^\circ, 90^\circ]$ . EACH ROW HAS 2000 REPEATS EXCEPT  $0.95T$  AT 0 dB (5000).

| $G$     | SNR (dB) | Pred.    | Local var. | Sector MSE | $> 5^\circ$ |
|---------|----------|----------|------------|------------|-------------|
| 0.25 m  | 0        | 0.6058   | 0.6104     | 0.6103     | 0           |
| $0.95T$ | 0        | 145.4    | 18.56      | 420.0      | 66.96       |
| 0.25 m  | 10       | 0.06058  | 0.05821    | 0.05828    | 0           |
| $0.95T$ | 10       | 14.54    | 10.14      | 15.30      | 19.75       |
| 0.25 m  | 20       | 0.006058 | 0.006160   | 0.006157   | 0           |
| $0.95T$ | 20       | 1.454    | 1.432      | 1.432      | 0           |

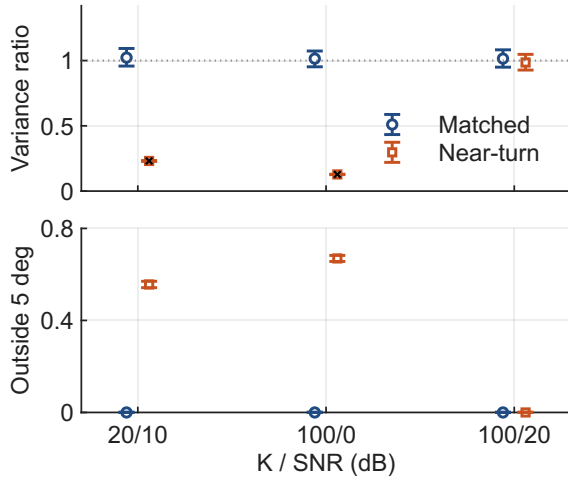


Fig. 4. Near complete-turn cancellation, low constrained variance can coexist with frequent errors outside the local window. Top: variance of the window-constrained estimator divided by its sandwich prediction. Bottom: the sector-wide estimator's probability of lying more than  $5^\circ$  from the pseudo-true bearing. The search sector is  $[-90^\circ, 90^\circ]$ . Crosses flag the two conditions whose sector-MSE relative interval half-width exceeds 10%; these MSE intervals are assessed separately from the plotted variance ratios and exceedances. Matched denotes point channels; near-turn uses  $G = 0.95(2\pi\ell)$ . Error bars show nominal 95% intervals; Supplement Appendix G gives the interval methods and sample-size rule.

the array pseudo-true bearing, a squared pseudo-true offset of  $176.546 \text{ deg}^2$ , and a finite-sample cross term of  $-8.154 \text{ deg}^2$ .

The processing checks assess whether array-level bias persists after phase-to-bearing estimation. Hann-DFT-MUSIC separates physical-bearing error from processing offset (E8); an independently differentiated displacement field checks the finite-gauge response (E9). Supplement Appendix H gives the estimators and refinement results.

A separate displacement-field processing study evaluates combinations of gauges  $\{0.25, 0.75\}$  m, bearings  $\{-30, 30\}^\circ$ , three output SNRs, and eight frozen matched/mismatched conditions, with 200 realizations

per cell (19200 total; protocol Section K). Its full-circle MUSIC grid is  $0.25^\circ$ . Noise is normalized to received phase power, whereas Sec. VI-D fixes incident coefficient energy  $E$ . At 0 dB, matched RMSE ranges are  $0.166\text{--}0.175^\circ$  and  $0.624\text{--}0.727^\circ$  for the two gauges; at 20 dB they are  $0.0170\text{--}0.0173^\circ$  and  $0.0680\text{--}0.0683^\circ$ . A fixed-arc 2% radius change retains about  $13^\circ$  physical bias. No estimate lies more than  $5^\circ$  from its array target. The lower AR(1)-noise RMSE reflects reduced noise power near 800 Hz at the same total variance.

#### D. Calibration-source and frequency-gauge decisions

E7 uses nominal  $\eta = 0$  and four evaluation bearings  $\{-45, -15, 15, 45\}^\circ$ , distinct from the design bearings. Each setting has 200 independent blocks with independently generated calibration at each bearing and 20 paired task draws per bearing. Calibration and tasks use  $K = 100$ . The source pool combines the five design azimuths with  $\{-30, 0, 30\}^\circ$  elevations and  $\{200, 800, 1400, 2000\}$  Hz; calibration uses  $G = 0.1$  m and 20-dB coefficient SNR. Task comparisons fix  $E$  and retain each gauge's covariance. Pointwise 95% Student- $t$  intervals use the 200 block means. Source policies share record-indexed noise; task rules share calibration and paired task noise. Five of the 6400 calibration batches used for the budget curves initially failed. A separate recovery procedure obtained accepted fits using the original observations and seeds while preserving the initial failure records (Supplement Appendix E). None of 1600 task-selection batches failed, and the study includes 384000 task fits.

At 800 Hz and  $G = 0.25$  m, four-record greedy selection lowers plug-in MSE from  $0.005498$  to  $0.004942 \text{ deg}^2$  against the fixed uniform sequence; at 36 records, the values are  $0.004640$  and  $0.004607 \text{ deg}^2$ . Increasing budget also increases total calibration energy.

At budget 18, the three gauge-aware rules coincide at 2000 Hz, 0.10 m. Relative to the 1100-Hz, 0.25-m heuristic, their uniform/greedy plug-in MSEs fall from  $0.00396084/0.00383488$  to  $0.00192012/0.00180349 \text{ deg}^2$ . Paired differences have 95% intervals  $[-0.002106, -0.001976]$  and  $[-0.002095, -0.001968] \text{ deg}^2$ . Joint-minus-plug-in intervals,  $[-3.898, 0.139] \times 10^{-7}$  and  $[-1.828, 0.497] \times 10^{-7} \text{ deg}^2$ , do not resolve a difference. All recorded fits converge, with no absolute circular error of  $90^\circ$  or more. For each E7 setting, zero such events across 200 independent blocks give a one-sided 95% upper bound of 1.49% on the probability that a block contains at least one such error. The nuisance sensitivity cases also select the same candidate (protocol Section G).

TABLE V

THE 1-M POINT-RESPONSE CHOICE HAS 13.26 TIMES THE EMPIRICAL MSE OF THE 0.1-M FULL-GAUGE CHOICE AT 2000 HZ, WITH  $q = 1$ ,  $K = 100$ , AND 18 CALIBRATION RECORDS. VALUES ARE IN  $10^{-6} \text{ rad}^2$ , AVERAGED OVER THE FOUR EVALUATION BEARINGS. INTERVALS USE 200 INDEPENDENT SHARED-CALIBRATION BATCH MEANS.

| Gauge | Local plug-in | MSE [95% interval]      |
|-------|---------------|-------------------------|
| 1.0 m | 7.5908        | 7.6053 [7.4268, 7.7837] |
| 0.1 m | 0.5797        | 0.5735 [0.5564, 0.5906] |

*Signal-averaging control:* At fixed 2000 Hz, the point-response optimizer chooses 1 m, whereas the full-gauge optimizer chooses 0.1 m. Both are evaluated with the correct finite-gauge plug-in estimator, common  $E$ , and their respective  $C_G$ . Each of 200 independent batches shares one 18-record uniform calibration fit across four evaluation bearings and 20 paired task draws per bearing. All records use  $K = 100$ ; calibration uses a 0.1-m gauge and 20-dB reference noise. Nominal and  $2^\circ$ -rotation starts capture a local calibration branch. The protocol and seed were fixed before generation; the five design bearings determined task choices.

The local prediction gives an MSE ratio of 13.09. The paired difference is  $7.0317 \times 10^{-6} \text{ rad}^2$  (95% interval  $[6.8564, 7.2071] \times 10^{-6}$ ). All 32000 full-circle searches converge without a  $5^\circ$  error. Eight prescribed grid refinements change the optimum by at most  $3.09 \times 10^{-7}$  degree (protocol Section L).

#### E. Decision stability over radius uncertainty

Both processing paths select 2000 Hz and  $G = 0.10 \text{ m}$  at every tested calibration budget and radius half-width. The energy rule selects 200 Hz and the same gauge. At  $\epsilon_R = 2\%$ , the sampled gap between the best and second local-risk scores is 14.7–20.9%, depending on budget. The original conditional-information choice also remains unchanged. Fig. 5 shows the exclusion band and score margins.

For the continuous domain  $\theta \in [-60^\circ, 60^\circ]$ ,  $R \in [0.245, 0.255] \text{ m}$ , 24,000 cells per path give  $\underline{m}_{(2000,0.1)} = 1.615859 \times 10^6 \text{ rad}^{-2}$  for raw channels and  $1.613406 \times 10^6$  for spatial averaging. The largest competing upper bounds are  $1.502407 \times 10^6$  and  $1.500341 \times 10^6$ , respectively, both from 1700 Hz,  $G = 0.10 \text{ m}$ . Thus (25) preserves the unique worst-case conditional-information choice across the entire domain. The calculation uses  $|n| \leq 16$ , the complete-series tail bound, and analytic variation over each cell. Positive response-norm bounds also verify regularity of all 42 candidates over the domain (protocol P).

An off-grid verification study evaluates the 16 combinations of  $\theta = \{-52.5, -22.5, 7.5, 37.5\}^\circ$  and relative

TABLE VI

OFF-GRID FULL-CIRCLE COMPARISON USING RAW CHANNELS. MSE AND 95% STUDENT- $t$  INTERVALS ARE IN  $10^{-6} \text{ rad}^2$ , USING 16 INDEPENDENT FOUR-RADIUS BLOCK MEANS. NOMINAL AND ROBUST RULES SHARE THE SAME SELECTED ACTION AND OUTPUTS.

| Rule         | $f$ (Hz) | $G$ (m) | MSE [95% interval]      |
|--------------|----------|---------|-------------------------|
| Nominal risk | 2000     | 0.10    | 0.5593 [0.4987, 0.6199] |
| Energy       | 200      | 0.10    | 1.7084 [1.5750, 1.8418] |
| Robust risk  | 2000     | 0.10    | 0.5593 [0.4987, 0.6199] |

radius  $\{-0.0175, -0.0075, 0.0075, 0.0175\}$ . Sixteen independent blocks each contain one 18-record calibration per radius, shared across four bearings and five paired task draws. All records use  $K = 100$ . Both actions have  $G = 0.10 \text{ m}$  and use that calibration's fitted nuisance vector. The same plug-in estimator searches the full azimuth circle and refines detected peaks. The study comprises 64 calibration fits and 2560 task fits, with no unresolved fit or error above  $5^\circ$ .

The task selected by the energy rule has 3.05 times the observed MSE of the nominal choice. Its paired excess is  $1.1491 \times 10^{-6} \text{ rad}^2$  (95% interval  $[1.0322, 1.2660] \times 10^{-6}$ ). The robust and nominal rules select the same action, so the robust rule gives zero additional error reduction. At  $\epsilon_R = 2\%$ , it evaluates nine radius nodes per candidate instead of one. The three-point path reduces the output dimension from 60 to 58 and preserves this choice in the information and local-risk calculations.

## VII. DISCUSSION AND LIMITATIONS

### A. Engineering interpretation

At horizontal incidence with fixed physical gauge length and pitch, first exclude gauges that become integer-turn averages at an allowed radius. Evaluate the remaining candidates with the projected bearing derivative and their noise covariances. Near commensurate nulls, Theorem 2 and its finite bound apply under the stated fixed-covariance condition.

Radius sensitivity must follow the relevant registration path. On a continuing branch, define  $S_R = \text{Re}(a_\theta^H S_a a_{\theta,R}) / I_\theta$ . The first-order bias allocation  $|S_R \delta R| \leq \epsilon_\theta$  gives candidate tolerances for evaluation with the full profiled objective. Compare frequency-gauge choices at a common incident coefficient budget  $E$  (Table V).

Calibration covariance then enters each task score. Equations (37) and (38) explain why the nominal choices coincide and why joint fitting yields only a small local gain. Local scores alone do not establish estimation reliability: at  $G = 0.95T$ , 0 dB, and  $K = 100$ , 66.96% of the sector estimates in Sec. VI-C lie more than  $5^\circ$  from the pseudo-true bearing. Final candidates therefore

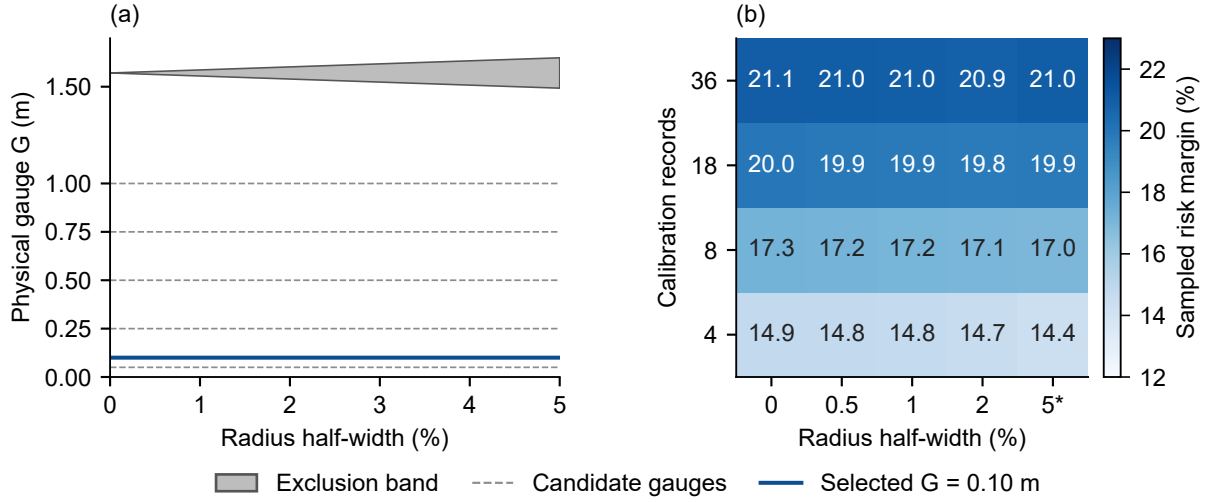


Fig. 5. The original candidate set retains the same selected task at horizontal incidence,  $R_0 = 0.25$  m and  $p = 0.02$  m. (a) Complete-turn exclusion band as the radius interval widens; the six candidate gauges lie below it. (b) Relative gap  $(S_2 - S_1)/S_1$  between the two smallest sampled local-risk scores, taking the smaller gap across raw and three-point processing. All cells select 2000 Hz,  $G = 0.10$  m. The asterisk marks the prespecified 5% stress case.

require error evaluation over the intended angular search domain and SNR range, as illustrated in Secs. VI-C–E. The continuous bounds in Sec. VI-E establish that the conditional-information choice persists over the domain; finite-budget local-risk stability is evaluated on the declared nodes.

#### B. Connection to instrument characterization

An instrument-specific calculation requires calibrated complex directional responses, gauge kernels, and noise covariances. Surface seismic measurements of helically wound and straight fibers have found sensitivity ratios that differ from wrapping-angle predictions [23]. Direction-independent channel weights reweight information; directional coupling adds response derivatives. An empirical response determines total information but does not uniquely separate amplitude and propagation mechanisms. For fixed covariance, let  $\delta b = C^{-1/2}\delta a$  and  $b = C^{-1/2}a$ . On the regular branch of (31), Cauchy–Schwarz gives

$$|\delta\theta_{\text{pt}}| \leq \frac{\|P_b^\perp \delta b\|_2}{\sqrt{I_\theta}} + O(\|\delta a\|^2). \quad (39)$$

Thus weak projected information amplifies the response error that cannot be absorbed by source amplitude.

#### C. Limitations

The prescribed host-strain model excludes cable–medium mechanics.

The analytic bounds hold in real arithmetic. Their numerical evaluation uses floating-point arithmetic without a roundoff enclosure. Separate quadrature calculations and finite frequency/angle searches have no rigorous integration-error or global peak-completeness guarantee.

The calibration intervals condition on four evaluation bearings, one fixed uniform source sequence, and initialization near the nominal nuisance vector. They describe noise variability for that design; variation across source sequences and distant initializations is untested.

### VIII. CONCLUSION

The finite-gauge angular filter identifies nonzero responses with no bearing information. The endpoint coefficient and finite remainder bound quantify recovery near these nulls; radius uncertainty gives excluded gauge bands. Registration-specific derivatives separate accumulated winding rotation from shape changes and relate radius error to bearing bias. Continuous-domain bounds preserve the nominal choice over the tested radius interval. Interval-aware selection gives the same action and error, while energy-based selection incurs higher error under equal acquisition and calibration budgets.

#### DATA AND CODE AVAILABILITY

The original code and synthetic data are publicly available on Zenodo (version 2.0.0), doi: 10.5281/zenodo.22992148. The accompanying code archive also contains the additional deterministic energy-selection, registration, near-turn, and interval-decision studies.

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