Existence of Lexicographic Max-min Strategy For Pair-Valued Matrix Games

by

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Abstract

We prove the existence of a lexicographic max-min strategy for pair-valued matrix game. Such a strategy emerges from the solution of a two-level linear programming problem.

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JEL Classification Codes: C61, C72, D81

1. Introduction: The concept referred to here as a pair-valued "matrix" game originates in Radhakrishnan and Saikeerthana (2020). Associated with it in the same paper is the solution concept that is formally defined in Lahiri (2026a, b) as a lexicographic equilibrium strategy profile for pair-valued "bi-matrix" games. In Lahiri (2026a, b) examples of pair-valued bi-matrix games for which no lexicographic equilibrium strategy profile exist are discussed. The latter two papers provide several necessary and sufficient conditions for the existence of lexicographic equilibrium strategy profiles. However, the possibility of non-existence of any lexicographic equilibrium strategy profile, compels us to consider the possibility of other solution concepts for such interactive decision-making problems.

In this note we introduce the concept of a lexicographic max-min strategy and prove its existence for pair-valued "matrix" games as the solution of a two-stage (bi-level) linear programming problem. A pair-valued matrix game, is one where the decision-maker whose problem we are concerned with, chooses its strategy by considering only its own state-dependent payoffs in a bi-matrix game. The state-dependent payoff of each player depends on the actions chosen by both players. Unlike a lexicographic equilibrium strategy profile, a lexicographic max-min strategy for a decision-maker is based on a "security level" regardless of what action is chosen by the other player. Lexicographic max-min strategy for pair-valued matrix games generalizes the concept of max-min strategy for matrix games, an exhaustive coverage of the latter being chapters 7, 8, 9, 10 of Ferguson (2020).

2. The framework of analysis and lexicographic max-min strategy profile: For positive integers m, n consider an interactive decision making context between two participants- a decision maker (DM) and another agent- where the DM has 'm' actions to choose from and the other agent has 'n' actions to choose from.

A **strategy for the DM** is a point in $\Delta^{m-1} = \{x \in \mathbb{R}^m_+ | \sum_{i=1}^m x_i = 1\}$.

A strategy for the DM is a probability distribution on {1, ..., m}, with each coordinate of the strategy denoting the probability with which the DM chooses the corresponding action.

Let $\mathbb{R}^{(2,\, m\times n)}$ denote the set of all m×n matrices whose entries are pairs of real number. If $V \in \mathbb{R}^{(2,\, m\times n)}$, then for all (i, j)∈{1, …, m}×{1, ..., n}, the typical (though not invariable) notations are as follows: (i, j)th entry of V is denoted by ($v_{ij}^{(1)}$, $v_{ij}^{(2)}$), the jth column of V is denoted by V^j.

We are assuming that there are two mutually exclusive and exhaustive states of nature such that if the DM chooses action i∈{1, …, m} and the other player chooses action j∈{1, …, n}, then $v_{ij}^{(1)}$ is the pay-off received by the DM if the first state of nature is realized and $v_{ij}^{(2)}$ is the pay-off received by the DM if the second state of nature is realized.

For our purposes here, a member of $\mathbb{R}^{(2,\, m\times n)}$ will be referred to as a **pair-valued matrix game**.

Notation 1: Given $V \in \mathbb{R}^{(2,\, m\times n)}$, for r∈{1, 2} let $V^{(r)}$ be the m×n real-valued matrix whose (i, j)th term for (i, j)∈{1, …, m}×{1, ..., n} is $v_{ij}^{(r)}$. The jth column of $V^{(r)}$ for j∈{1, …, n} is denoted by $V^{(r)j}$. For $x \in \Delta^{m-1}$ and h∈{1, …, m}: $x^T V^k = (x^T V^{(1)k}, x^T V^{(2)k})$ $= \sum_{i=1}^m (x_i v_{ik}^{(1)}, x_i v_{ik}^{(2)})$. ▌

Note 1: In what follows, we will use V and ($V^{(1)}$, $V^{(2)}$) interchangeably to refer to a pair-valued matrix game. ▌

Let ≽ be a binary relation on $\mathbb{R}^2$ such that for all $(\gamma_1, \eta_1), (\gamma_2, \eta_2) \in \mathbb{R}^2$: $(\gamma_1, \eta_1) \succcurlyeq (\gamma_2, \eta_2)$ <u>if and only if</u> either $\gamma_1 > \gamma_2$ or $\gamma_1 = \gamma_2$ and $\eta_1 \geq \eta_2$.

Clearly ≽ is a linear order on $\mathbb{R}^2$. It is generally known as the **lexicographic preference ordering (lexicographic preferences) on** $\mathbb{R}^2$ (see Varian (1978)).

We assume that the DM's preferences over state-dependent pay-offs are represented by lexicographic preferences.

A strategy x^* will be referred to as a **lexicographic max-min strategy** for the pair-valued matrix game $V = (V^{(1)}, V^{(2)})$ if $x = x^*$, $u^{(r)*}$, $r\epsilon\{1, 2\}$ solve the following problem:

Maximize (with respect to $\succcurlyeq$) $(u^{(1)}, u^{(2)})$

Subject to:

$x^TV^j \succcurlyeq (u^{(1)}, u^{(2)})$, $j\epsilon\{1, \ldots, n\}$, $x \in \Delta^{m-1}$, $u^{(1)}, u^{(2)}\epsilon\mathbb{R}$.

3. The two-stage linear programming problem and existence of lexicographic max-min strategy profile: We will now prove the existence of a lexicographic max-min strategy profile.

Notation 2: For $i\epsilon\{1, \ldots, m\}$, let $E^{(m,i)}$ be the **m-dimensional i^{th} unit coordinate column vector** (i.e., 1 in its i^{th} coordinate and 0 in all other coordinates).

Proposition 1: A lexicographic max-min strategy exists for V.

Proof: Consider the linear programming problem **(Problem 1)**:

Maximize $u^{(1)}$

Subject to:

$w^TV^{(1)j} - u^{(1)} \geq 0$, $j\epsilon\{1, \ldots, n\}$

$w\epsilon\Delta^{m-1}$, $u^{(1)}\epsilon\mathbb{R}$.

Since, $(E^{(m,1)}, \min_{j\epsilon\{1,\ldots,n\}} v_{1j}^{(1)})\epsilon\{(w, u^{(1)})\epsilon\Delta^{m-1}\times\mathbb{R}|w^TV^{(1)j} - u^{(1)} \geq 0, j\epsilon\{1, \ldots, n\}\}$ and $u^{(1)}$ is bounded above by $\max\{v_{ij}^{(1)}| (i, j)\epsilon\{1, .., m\}\times\{1, ..., n\}\}\epsilon\mathbb{R}$, Problem 1 has a solution.

Thus, there is a unique real number $u^{(1)*} = \max_{w\epsilon\Delta^{m-1}} \min_{j\epsilon\{1,\ldots,n\}} w^TV^{(1)j}$. Let $X_1 = \{w\epsilon\Delta^{m-1}| u^{(1)*} = \min_{j\epsilon\{1,\ldots,n\}} w^TV^{(1)j}\}$.

The set of solutions of Problem 1 is the non-empty and closed and bounded set $\{(x, u^{(1)*})|x\epsilon X_1\}$.

For $x\epsilon X_1$, let $J(x) = \{k| x^TV^{(1)k} = u^{(1)*}\} = \operatorname{argmin}_{k\epsilon\{1,\ldots,n\}} x^TV^{1(k)} \neq \phi$.

Since for all $x\epsilon X_1$, $J(x) \subset \{1, ..., n\}$, $\{J(x)| x\epsilon X_1\}$ is a non-empty finite collection of non-empty subsets for $\{1, \ldots, n\}$.

For each $J\epsilon\{J(x)| x\epsilon X_1\}$, consider the linear programming problem **(Problem 2J)**

Maximize $u^{(2)}(J)$

Subject to:

$x(J)^T V^{(2)j}$ - $u^{(2)}(J) \geq 0$, $j \epsilon J$,

$x^T(J)V^{(1)j}$ - $u^{(1)*} \geq 0$, $j = 1, \ldots, n$,

$x^T(J)V^{(1)j}$ - $u^{(1)*} = 0$ for all $x \epsilon J$,

$x(J) \epsilon \Delta^{m-1}$, $u^{(2)}(J) \epsilon \mathbb{R}$

Since $u^{(2)}(J)$ is bounded above by $\max\{v_{ij}^{(2)} | (i, j) \epsilon \{1, .., m\} \times \{1, ..., n\}\} \epsilon \mathbb{R}$, for all $J \epsilon \{J(x) | x \epsilon X_1\}$, the set of solutions of Problem 2J is non-empty. For each $J \epsilon \{J(x) | x \epsilon X_1\}$, let $u^{(2)*}(J)$ denote the optimal value of Problem 2J.

Let $J^* \epsilon \underset{J \epsilon \{J(x) | x \epsilon X_1\}}{\operatorname{argmax}} u^{(2)*}(J)$.

Let $(x^*(J^*), u^{(2)*})$ be a solution of Problem $2J^*$ and let $x^* = x^*(J^*)$.

x^* is a lexicographic max-min strategy for $(V^{(1)}, V^{(2)})$. Q.E.D.

Note 2: In Lahiri (2026c) it is shown how the problem of solving for the maximum of the maximum values of several linear programming problems can be reduced to a single linear programming problem. Thus the problem of proving the existence of a lexicographic max-min strategy reduces to solving a two-stage (bi-level) linear programming problem in the following manner.

Having obtained $u^{(1)*}$, X_1, $\{J(x) | x \epsilon X_1\}$ by solving the linear programming problem Problem 1, let $u^{(2)*}$, $<(x^*(J), u^{(2)*}(J)) | J \epsilon \{J(x) | x \epsilon X_1\}>$ solve:

Maximize $2\sum_{J \epsilon \{J(x) | x \epsilon X_1\}} u^{(2)}(J)$ - $u^{(2)}$,

Subject to $u^{(2)}(J) - u^{(2)} \leq 0$, $J \epsilon \{J(x) | x \epsilon X_1\}$,

$x(J)^T V^{(2)j}$ - $u^{(2)}(J) \geq 0$, $j \epsilon J$, $J \epsilon \{J(x) | x \epsilon X_1\}$,

$x^T(J)V^{(1)j}$ - $u^{(1)*} \geq 0$, $j = 1, \ldots, n$, $J \epsilon \{J(x) | x \epsilon X_1\}$,

$x^T(J)V^{(1)j}$ - $u^{(1)*} = 0$ for all $x \epsilon J$, $J \epsilon \{J(x) | x \epsilon X_1\}$,

$x(J) \epsilon \Delta^{m-1}$, $u^{(2)}(J) \epsilon \mathbb{R}$ for all $J \epsilon \{J(x) | x \epsilon X_1\}$, $u^{(2)} \epsilon \mathbb{R}$.

From theorem 1 of Lahiri (2026c), we know that for all $J \epsilon \{J(x) | x \epsilon X_1\}$, $(x^*(J), u^{(2)*}(J))$ solves Problem 2J and that there exists $J^* \epsilon \{J(x) | x \epsilon X_1\}$ such that $u^{(2)*}(J^*) = u^{(2)*}$.

Let $x^* = x^*(J^*)$. x^* is a lexicographic max-min strategy for V. ▌

References

1. Ferguson, T. (2020): A Course in Game Theory. World Scientific Publishing Company Pte. Ltd.

2. Lahiri, S. (2026a): Lexicographic and Strictly Weighted Equilibrium Strategy Profiles with Applications to Interval-Valued Bi-matrix Games. DOI: https://doi.org/10.31224/7351

3. Lahiri, S. (2026b): Pair-valued bi-matrix games and lexicographic equilibrium strategy profile. DOI: https://doi.org/10.31224/7699

4. Lahiri, S. (2026c): The Max-max Linear Programming Problem. DOI: https://doi.org/10.13140/RG.2.2.30654.96328/1

5. Radhakrishnan, S. and Saikeerthana, D. (2020): Game Theory Problems Using Interval Parameters. Kongunadu Research Journal, Volume 7, Number 2, Pages 123-129.

6. Varian, H. (1978): Microeconomic Analysis. W.W. Norton and Company, INC.