Title: Stability of Structures: Kinematics, Equivalent Single-Layer Theories, and Energy-based Semi-Analytical Methods

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Event Context: Engineering Mechanics Graduate School Course, 2025

Abstract:
This part of the course details the fundamental principles of structural stability, focusing on kinematic relations and equivalent single-layer (ESL) theories for composite plates and shells. It derives three-dimensional strain-displacement equations for planar, cylindrical, conical, and spherical geometries. The formulations are systematically reduced to two-dimensional ESL models, specifically the Classical Laminated Plate Theory (CLPT), First-order Shear Deformation Theory (FSDT), and Third-order Shear Deformation Theory (TSDT). The limitations of ESL theories regarding interlaminar stress continuity are analysed, introducing Zig-Zag and Layerwise theories as advanced alternatives. Furthermore, the material covers energy-based formulation frameworks, deriving the principle of minimum potential energy, the semi-analytical Ritz method utilizing Legendre polynomials, and the neutral equilibrium criterion for extracting geometric stiffness matrices and linear buckling equations.

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Kinematics, Equivalent Single-Layer Theories, and Energy-based Semi-Analytical Methods

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Kinematics, Equivalent Single-Layer Theories, and Energy-based Semi-Analytical Methods

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Today's topics

- General kinematic relations
- Kinematics for plates, cylindrical shells, conical shells, spherical shells
- Equivalent single-layer theories
    - CLPT, FSDT, TSDT
- Principle of minimum potential energy
- Neutral equilibrium criterion
- Zig-Zag theories
- Layerwise theories

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Requirements for the upcoming examples

Python >= 3.8
jupyter notebook
sympy
numpy
scipy
composites >= 0.8.0
structsolve

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Strain-displacement (kinematic) equations for plates, cylindrical and spherical shells

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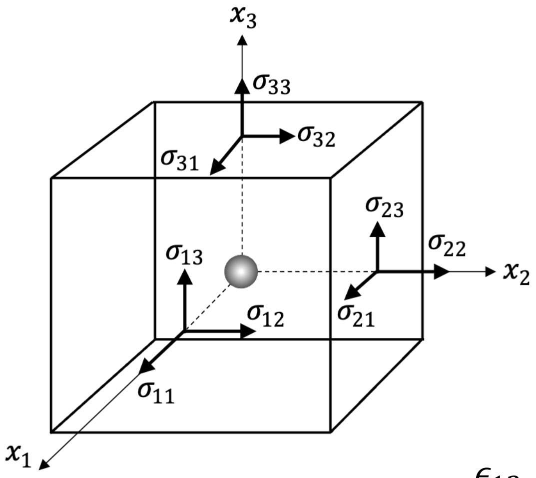
General strain-displacement relations

According to the three-dimensional (3D) elasticity theory, the strain components referred to an arbitrary orthogonal coordinate system x_1, x_2, x_3 can be written as [1]:



$$\epsilon_{11} = \frac{1}{2}\left(\left(\frac{e_{13}}{2} - \omega_2\right)^2 + \left(\frac{e_{12}}{2} + \omega_3\right)^2 + e_{11}^2\right) + e_{11}$$

$$\epsilon_{22} = \frac{1}{2}\left(\left(\frac{e_{23}}{2} + \omega_1\right)^2 + \left(\frac{e_{12}}{2} - \omega_3\right)^2 + e_{22}^2\right) + e_{22}$$

$$\epsilon_{33} = \frac{1}{2}\left(\left(\frac{e_{23}}{2} - \omega_1\right)^2 + \left(\frac{e_{13}}{2} + \omega_2\right)^2 + e_{33}^2\right) + e_{33}$$

$$\epsilon_{12} = \left(\frac{e_{23}}{2} + \omega_1\right)\left(\frac{e_{13}}{2} - \omega_2\right) + e_{11}\left(\frac{e_{12}}{2} - \omega_3\right) + e_{22}\left(\frac{e_{12}}{2} + \omega_3\right) + e_{12}$$

$$\epsilon_{13} = e_{33}\left(\frac{e_{13}}{2} - \omega_2\right) + e_{11}\left(\frac{e_{13}}{2} + \omega_2\right) + \left(\frac{e_{23}}{2} - \omega_1\right)\left(\frac{e_{12}}{2} + \omega_3\right) + e_{13}$$

$$\epsilon_{23} = e_{22}\left(\frac{e_{23}}{2} - \omega_1\right) + e_{33}\left(\frac{e_{23}}{2} + \omega_1\right) + \left(\frac{e_{13}}{2} + \omega_2\right)\left(\frac{e_{12}}{2} - \omega_3\right) + e_{23}$$

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal,, 2015. https://doi.org/10.21268/20150210-154320. View/Download Slides View/Download PhD Thesis presentation

[2] G.-Q. Zhang, Stability analysis of anisotropic conical shells, Technical University Delft - Faculty of Aerospace Engineering, 1993.

[3] V. V. Novozhilov, Foundations of the nonlinear theory of elasticity, Rochester: Graylock Press, 1953.

General strain-displacement relations

where the parameters e_{ij} and ω_i are (the conventional notation for partial derivatives $\partial/\partial x$ is used here for the sake of clarity) the following [1], with u, v, w being the displacements along directions x_1, x_2, x_3, respectively:

$$e_{11} \coloneqq \frac{\frac{\partial u}{\partial x_1}}{H_1} + \frac{v\frac{\partial H_1}{\partial x_2}}{H_1 H_2} + \frac{w\frac{\partial H_1}{\partial x_3}}{H_1 H_3}$$

$$e_{22} \coloneqq \frac{u\frac{\partial H_2}{\partial x_1}}{H_1 H_2} + \frac{\frac{\partial v}{\partial x_2}}{H_2} + \frac{w\frac{\partial H_2}{\partial x_3}}{H_2 H_3}$$

$$e_{33} \coloneqq \frac{u\frac{\partial H_3}{\partial x_1}}{H_1 H_3} + \frac{v\frac{\partial H_3}{\partial x_2}}{H_2 H_3} + \frac{\frac{\partial w}{\partial x_3}}{H_3}$$

$$e_{12} \coloneqq \frac{H_1\frac{\partial}{\partial x_2}\left(\frac{u}{H_1}\right)}{H_2} + \frac{H_2\frac{\partial}{\partial x_1}\left(\frac{v}{H_2}\right)}{H_1}$$

$$e_{13} \coloneqq \frac{H_1\frac{\partial}{\partial x_3}\left(\frac{u}{H_1}\right)}{H_3} + \frac{H_3\frac{\partial}{\partial x_1}\left(\frac{w}{H_3}\right)}{H_1}$$

$$e_{23} \coloneqq \frac{H_2\frac{\partial}{\partial x_3}\left(\frac{v}{H_2}\right)}{H_3} + \frac{H_3\frac{\partial}{\partial x_2}\left(\frac{w}{H_3}\right)}{H_2}$$

$$\omega_1 \coloneqq \frac{\frac{\partial(H_3 w)}{\partial x_2} - \frac{\partial(H_2 v)}{\partial x_3}}{2(H_2 H_3)}$$

$$\omega_2 \coloneqq \frac{\frac{\partial(H_1 u)}{\partial x_3} - \frac{\partial(H_3 w)}{\partial x_1}}{2(H_1 H_3)}$$

$$\omega_3 \coloneqq \frac{\frac{\partial(H_2 v)}{\partial x_1} - \frac{\partial(H_1 u)}{\partial x_2}}{2(H_1 H_2)}$$

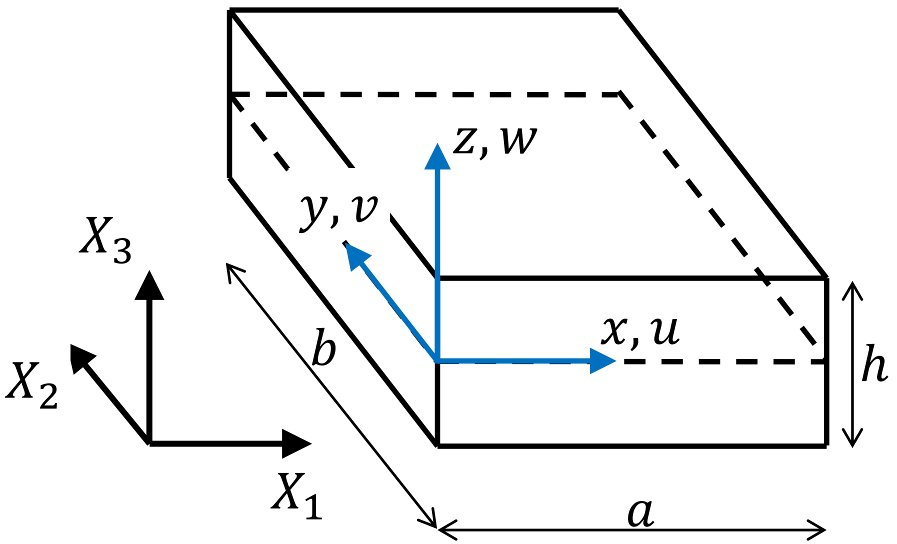
$$H_1 = \sqrt{\left(X_{1,x_1}\right)^2 + \left(X_{2,x_1}\right)^2 + \left(X_{3,x_1}\right)^2}$$

$$H_2 = \sqrt{\left(X_{1,x_2}\right)^2 + \left(X_{2,x_2}\right)^2 + \left(X_{3,x_2}\right)^2}$$

$$H_3 = \sqrt{\left(X_{1,x_3}\right)^2 + \left(X_{2,x_3}\right)^2 + \left(X_{3,x_3}\right)^2}$$

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014. https://doi.org/10.21268/20150210-154320. View/Download Slides View/Download PhD Thesis presentation

3D kinematic equations for plates



$x_1 = x \qquad X_1 = x$

$x_2 = y \qquad X_2 = y$

$x_3 = z \qquad X_3 = z$

Defining:

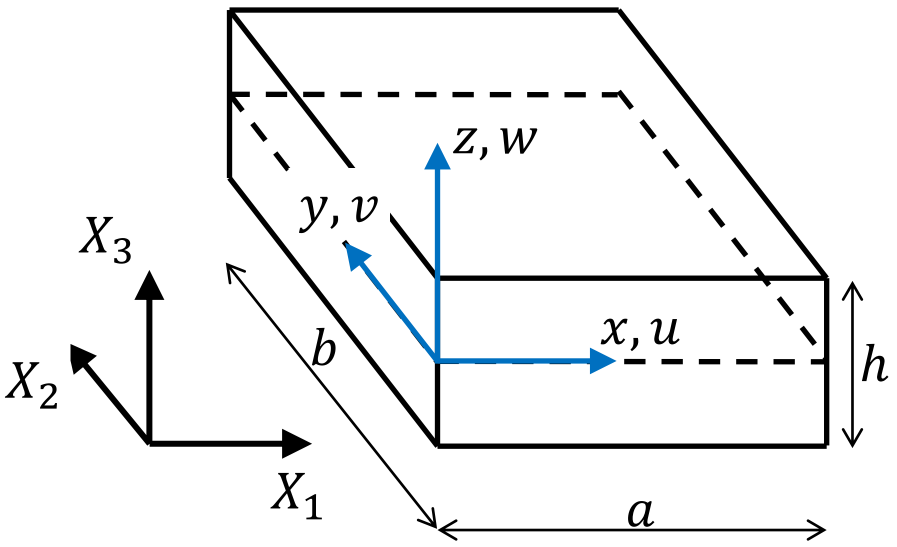
$$\varepsilon_{xx} = \epsilon_{11} \qquad \gamma_{xy} = 2\varepsilon_{xy} = \epsilon_{12}$$

$$\varepsilon_{yy} = \epsilon_{22} \qquad \gamma_{xz} = 2\varepsilon_{xz} = \epsilon_{13}$$

$$\varepsilon_{zz} = \epsilon_{33} \qquad \gamma_{yz} = 2\varepsilon_{yz} = \epsilon_{23}$$

`See: ex/kinematics_plate_3D.ipynb`

3D kinematic equations for plates



$$\varepsilon_{xx} = u_{,x} + \frac{1}{2}\left(u_{,x}^2 + v_{,x}^2 + w_{,x}^2\right)$$

$$\varepsilon_{yy} = v_{,y} + \frac{1}{2}\left(u_{,y}^2 + v_{,y}^2 + w_{,y}^2\right)$$

$$\varepsilon_{zz} = w_{,z} + \frac{1}{2}\left(u_{,z}^2 + v_{,z}^2 + w_{,z}^2\right)$$

$$\gamma_{xy} = u_{,y} + v_{,x} + \left(u_{,x}\, u_{,y} + v_{,x}\, v_{,y} + w_{,x}\, w_{,y}\right)$$

$$\gamma_{xz} = u_{,z} + w_{,x} + \left(u_{,x}\, u_{,z} + v_{,x}\, v_{,z} + w_{,x}\, w_{,z}\right)$$

$$\gamma_{yz} = v_{,z} + w_{,y} + \left(u_{,y}\, u_{,z} + v_{,y}\, v_{,z} + w_{,y}\, w_{,z}\right)$$

See: `ex/kinematics_plate_3D.ipynb`

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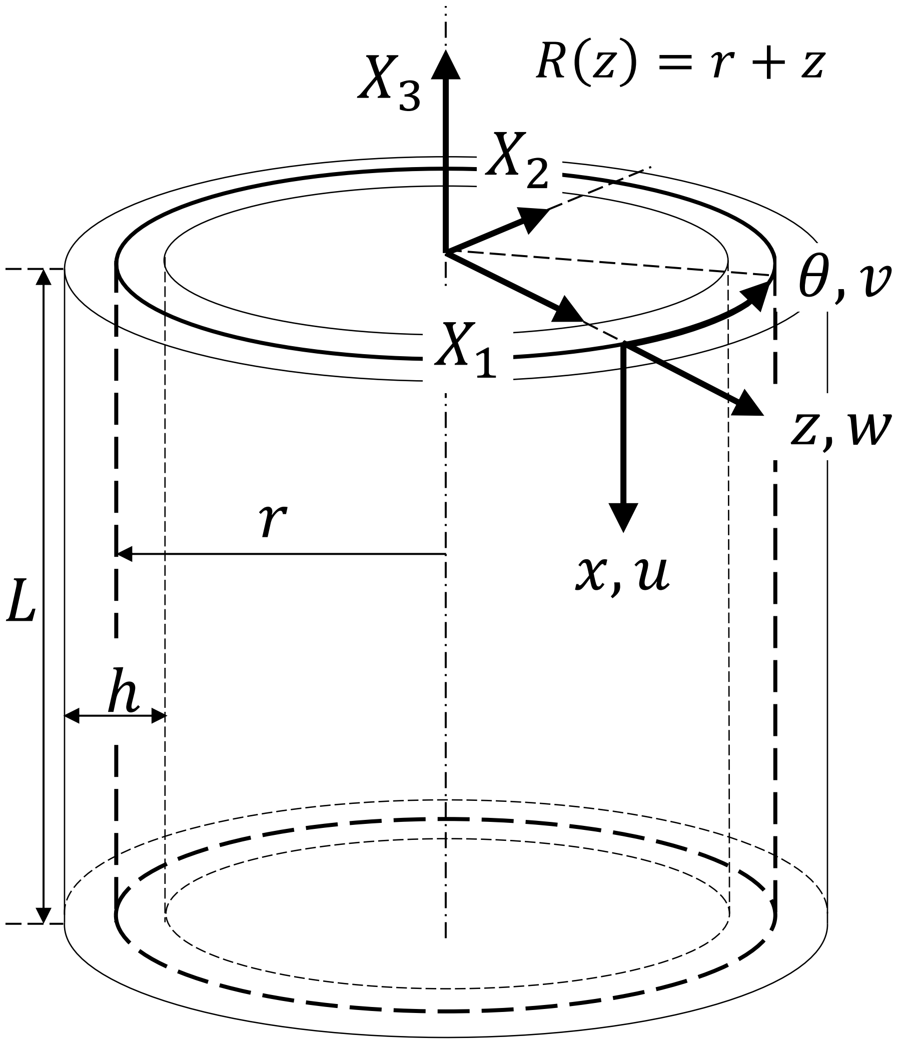
3D kinematic equations for cylindrical shells



$$x_1 = x \qquad X_1 = R(z)\cos(\theta)$$

$$x_2 = \theta \qquad X_2 = R(z)\sin(\theta)$$

$$x_3 = z \qquad X_3 = -x$$

Defining:

$$\varepsilon_{xx} = \epsilon_{11} \qquad \gamma_{x\theta} = 2\varepsilon_{x\theta} = \epsilon_{12}$$

$$\varepsilon_{\theta\theta} = \epsilon_{22} \qquad \gamma_{xz} = 2\varepsilon_{xz} = \epsilon_{13}$$

$$\varepsilon_{zz} = \epsilon_{33} \qquad \gamma_{\theta z} = 2\varepsilon_{\theta z} = \epsilon_{23}$$

See: `ex/kinematics_cylinder_3D.ipynb`

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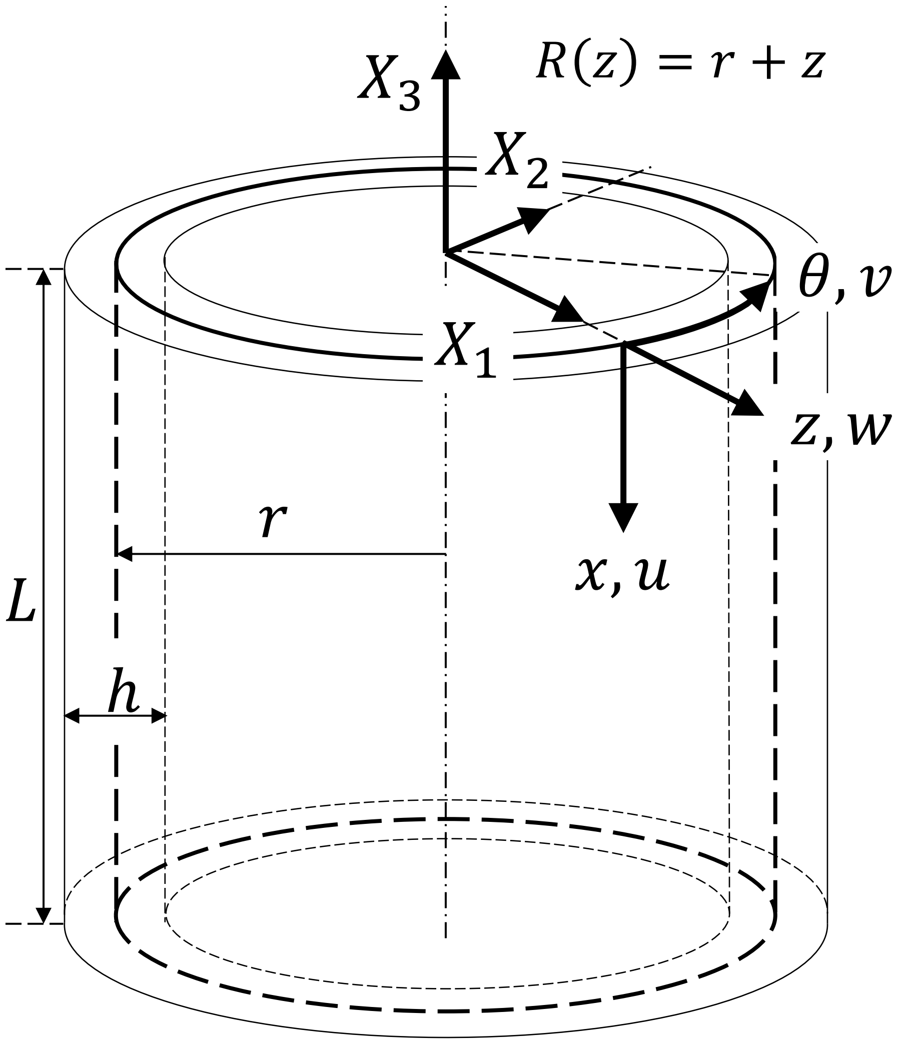
3D kinematic equations for cylindrical shells



Only linear relations shown here (NL=0):

$$\varepsilon_{xx} = u_{,x}$$

$$\varepsilon_{\theta\theta} = \frac{v_{,\theta}}{R(z)} + \frac{w}{R(z)}$$

$$\varepsilon_{zz} = w_{,z}$$

$$\gamma_{x\theta} = \frac{u_{,\theta}}{R(z)} + v_{,x}$$

$$\gamma_{xz} = u_{,z} + w_{,x}$$

$$\gamma_{\theta z} = v_{,z} + \frac{w_{,\theta}}{R(z)} - \frac{v}{R(z)}$$

See: `ex/kinematics_cylinder_3D.ipynb`

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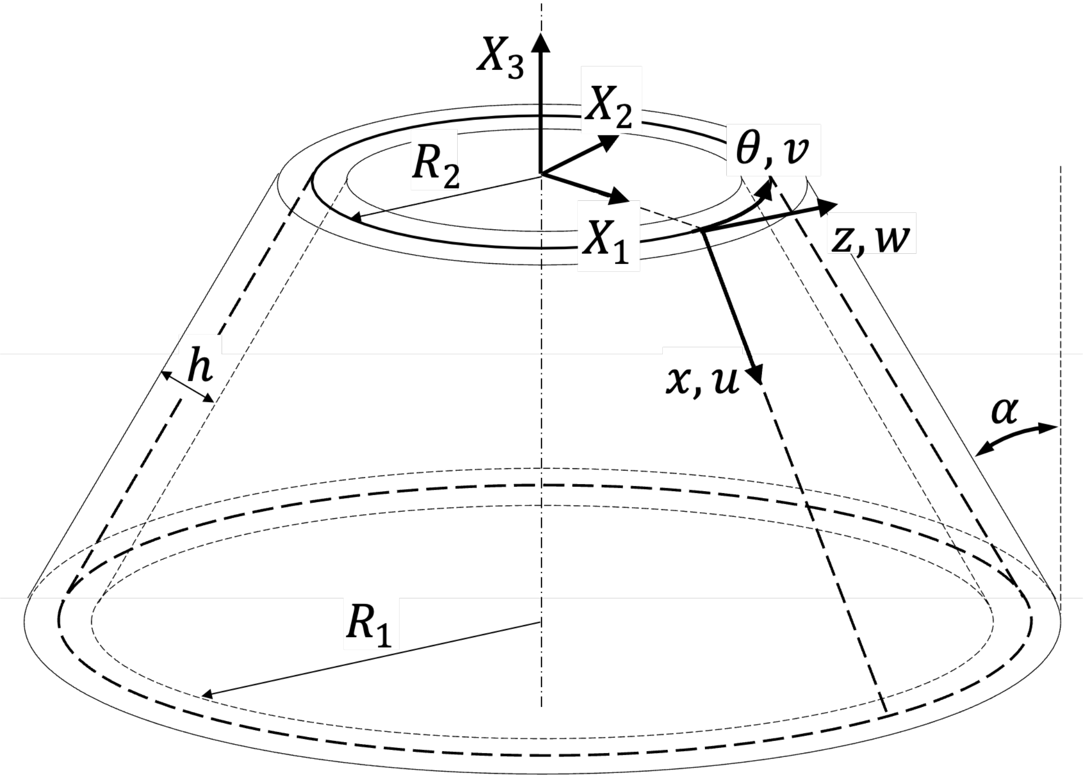
3D kinematic equations for conical shells



$$x_1 = x \qquad X_1 = R(x,z)\cos\theta$$

$$x_2 = \theta \qquad X_2 = R(x,z)\sin\theta$$

$$x_3 = z \qquad X_3 = z\sin\alpha - x\cos\alpha$$

$$R(x,z) = R_2 + x\sin\alpha + z\cos\alpha$$

Defining:

$$\varepsilon_{xx} = \epsilon_{11} \qquad \gamma_{x\theta} = 2\varepsilon_{x\theta} = \epsilon_{12}$$

$$\varepsilon_{\theta\theta} = \epsilon_{22} \qquad \gamma_{xz} = 2\varepsilon_{xz} = \epsilon_{13}$$

$$\varepsilon_{zz} = \epsilon_{33} \qquad \gamma_{\theta z} = 2\varepsilon_{\theta z} = \epsilon_{23}$$

See: `ex/kinematics_cone_3D.ipynb`

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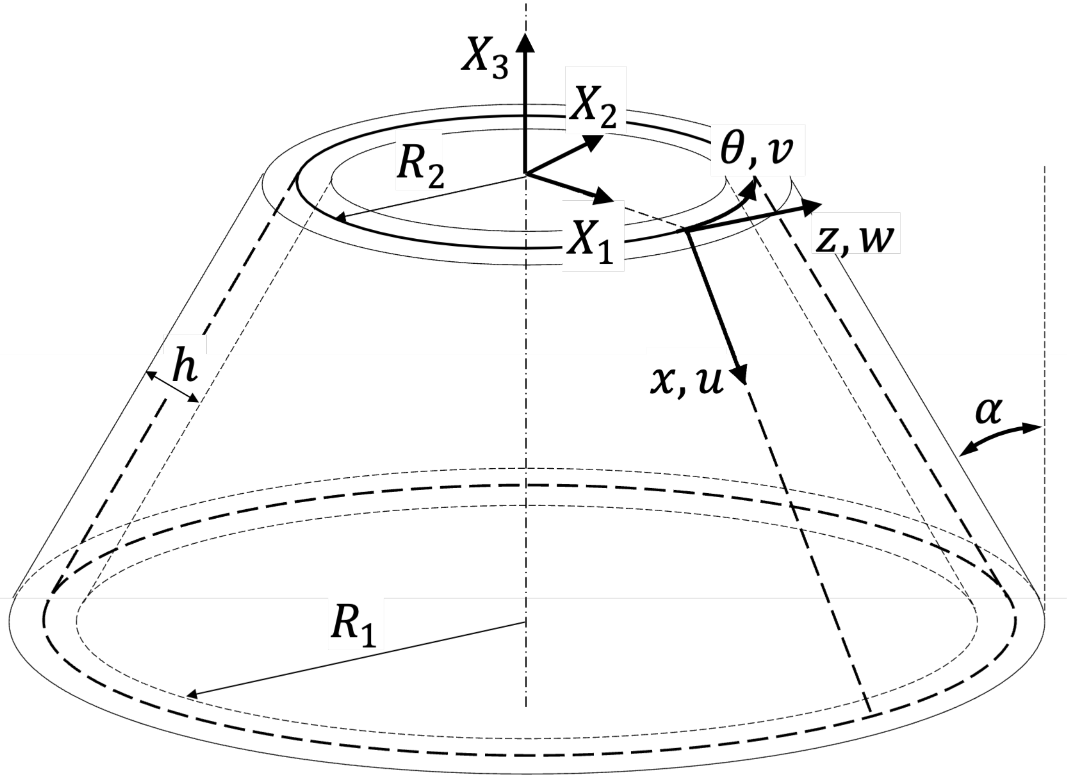
3D kinematic equations for conical shells



Only linear relations shown here (NL=0):

$$\varepsilon_{xx} = u_{,x}$$

$$\varepsilon_{\theta\theta} = \frac{v_{,\theta}}{R(x,z)} + \frac{u \sin\alpha}{R(x,z)} + \frac{w \cos\alpha}{R(x,z)}$$

$$\varepsilon_{zz} = w_{,z}$$

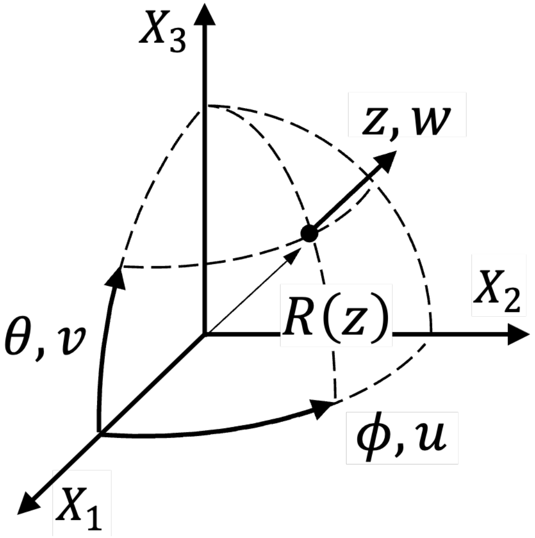
$$\gamma_{x\theta} = \frac{u_{,\theta}}{R(x,z)} + v_{,x} - \frac{v \sin\alpha}{R(x,z)}$$

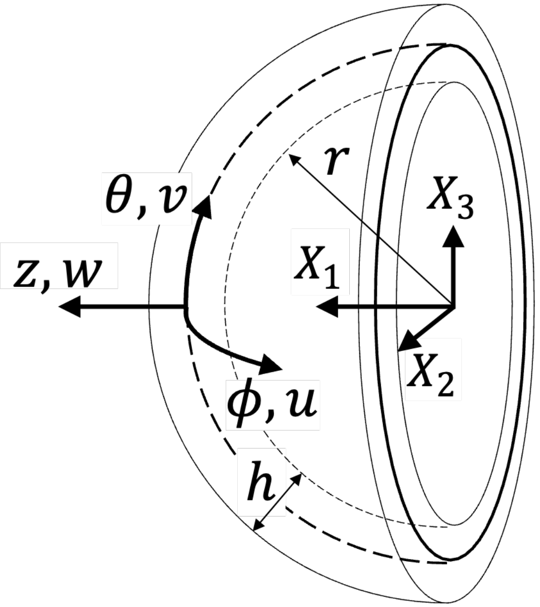
$$\gamma_{xz} = w_{,x} + u_{,z}$$

$$\gamma_{\theta z} = \frac{w_{,\theta}}{R(x,z)} + v_{,z} - \frac{v \cos\alpha}{R(x,z)}$$

See: `ex/kinematics_cone_3D.ipynb`

3D kinematic equations for spherical shells





$$R(z) = r + z$$

$$x_1 = \phi \qquad X_1 = R(z)\cos\phi\cos\theta$$

$$x_2 = \theta \qquad X_2 = R(z)\sin\phi\cos\theta$$

$$x_3 = z \qquad X_3 = R(z)\, sin\,\theta$$

Defining:

$$\varepsilon_{\phi\phi} = \epsilon_{11} \qquad \gamma_{\phi\theta} = 2\varepsilon_{\phi\theta} = \epsilon_{12}$$

$$\varepsilon_{\theta\theta} = \epsilon_{22} \qquad \gamma_{\phi z} = 2\varepsilon_{\phi z} = \epsilon_{13}$$

$$\varepsilon_{zz} = \epsilon_{33} \qquad \gamma_{\theta z} = 2\varepsilon_{\theta z} = \epsilon_{23}$$

See: `ex/kinematics_sphere_3D.ipynb`

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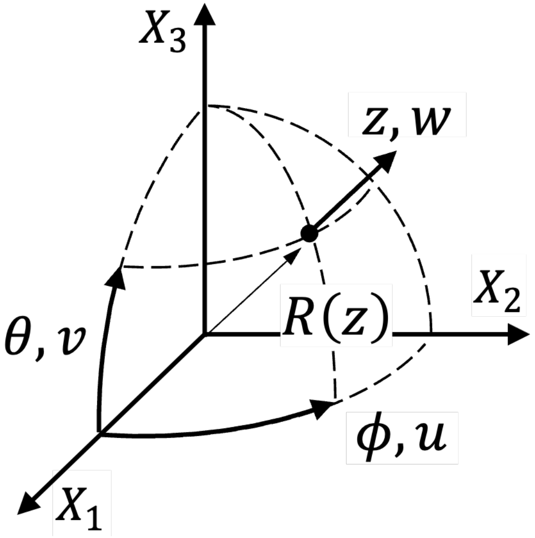
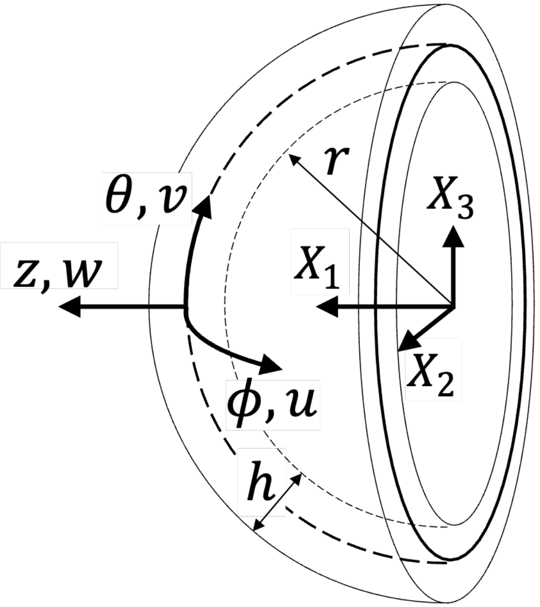
3D kinematic equations for spherical shells





$$R(z) = r + z$$

$$x_1 = \phi \qquad X_1 = R(z)\cos\phi\cos\theta$$

$$x_2 = \theta \qquad X_2 = R(z)\sin\phi\cos\theta$$

$$x_3 = z \qquad X_3 = R(z)\, sin\,\theta$$

Only linear relations shown here (NL=0):

$$\varepsilon_{\phi\phi} = \frac{1}{R(z)}\left(\frac{u_{,\phi}}{\cos\theta} + w - v\tan\theta\right)$$

$$\varepsilon_{\theta\theta} = \frac{1}{R(z)}(v_{,\theta} + w)$$

$$\varepsilon_{zz} = w_{,z}$$

$$\gamma_{\phi\theta} = \frac{1}{R(z)}\left(u_{,\theta} + \frac{v_{,\phi}}{\cos\theta} + u\tan\theta\right)$$

$$\gamma_{\phi z} = \frac{1}{R(z)}\left(\frac{w_{,\phi}}{\cos\theta} - u\right) + u_{,z}$$

$$\gamma_{\theta z} = \frac{1}{R(z)}(w_{,\theta} - v) + v_{,z}$$

```
See: ex/kinematics_sphere_3D.ipynb
```

Difficulty in using 3D kinematics

- Full discretization over the thickness
  - Mesh aspect-ratio issues (3 to 5 first-order elements needed through the thickness)
  - Poor conditioning of stiffness matrix (scales with Eh^2 for bending and Et for membrane)
  - Really high cost for laminated composite materials with multiple layers
  - Application of simply supported boundary conditions in analytical or semi-analytical models (see next examples)

- For thin-walled structures, no knowledge about the deformation kinematics is embedded into the strain-displacement relations

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Typical kinematic theories applied for composite plates

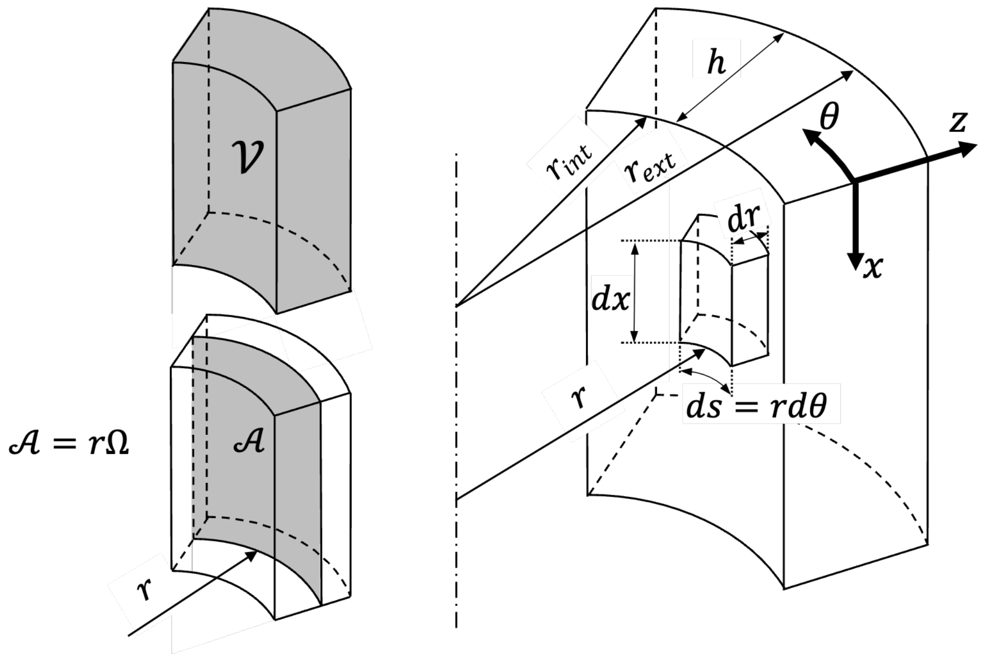
Most of the analyses performed on composite plates have been based on one of the following approaches (copied from Reddy, 2004 [4]):

- Equivalent single-layer (ESL) theories (2-D)
  - Classical laminated plate theory
  - Shear deformation laminated plate theories
- Three-dimensional elasticity theory (3-D)
  - Traditional 3-D elasticity formulations
- Layer-wise theories

Reddy states that among the ESL theories the First-order Shear Deformation Theory (FSDT) including transverse extensibility ($\varepsilon_{zz} \neq 0$) seems to provide the best compromise solution between accuracy, economy and simplicity [1, 4].

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.
[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

Equivalent single-layer for shells, illustration



Schematics to illustrate the shallow shell assumption $r \gg h$. Modified from [1]

Given a function $f(x,\theta,z)$, its integral over the 3-D domain $\mathcal{V}$ can be expressed as:

$$\int_{\mathcal{V}} f(x,\theta,z)d\mathcal{V} = \int_{r_{int}}^{r_{ext}} \int_{\Omega} f(x,\theta,z)R(x,z)d\Omega dr$$

using the following substitutions based on the figure, for cylindrical shells:

$$d\Omega = d\theta dz \qquad R(x,z) = r + z$$

$$dA = rd\Omega \qquad dr = dz$$

the previous integral becomes:

$$\int_{\mathcal{V}} f(x,\theta,z)d\mathcal{V} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{A} f(x,\theta,z)(r+z)\frac{dA}{R(x,z)}dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{A} f(x,\theta,z)\left(1+\frac{z}{r}\right)dAdz$$

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

Equivalent single-layer for shells, illustration

Repeating the previous relation:

$$\int_{\mathcal{V}} f(x,\theta,z)d\mathcal{V} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_A f(x,\theta,z)(r+z)\frac{dA}{R(x,z)}dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_A f(x,\theta,z)\left(1+\frac{z}{r}\right)dAdz$$

Applying the shallow shell theory assumption that $r \gg z$, will result in:

$$\left(1+\frac{z}{r}\right) \approx 1$$

$$(r+z) \approx r$$

leading the previous integral to:

$$\int_{\mathcal{V}} f(x,\theta,z)d\mathcal{V} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\mathcal{A}} f(x,\theta,z)d\mathcal{A}dz = \int_{z=-\frac{h}{2}}^{z=\frac{h}{2}} \int_{s=0}^{s=2\pi r} \int_{x=0}^{x=L} f(x,\theta,z)dxdsdz$$

This equation is the basis for the reduction from the 3-D domain to the 2-D domain when using the ESL theories for shells.

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

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Equivalent single-layer for shells, illustration

Thus, when the shallow shell theory assumption $r \gg z$ is valid, we have that:

$$\int_{\mathcal{V}} f(x,\theta,z) d\mathcal{V} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\mathcal{A}} f(x,\theta,z) d\mathcal{A} dz = \int_{z=-\frac{h}{2}}^{z=\frac{h}{2}} \int_{s=0}^{s=2\pi r} \int_{x=0}^{x=L} f(x,\theta,z) dx ds dz$$

Now we have paved the way to enable the integration of ESL kinematics.

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Equivalent single-layer (ESL) theories

The main ESL theories are [1]:

Classical Laminated Plate Theory (CLPT): extension of the Classical Plate Theory to composite laminates [4], also known as the Kirchhoff-Love theory.

$$u(x,y,z) = u_0(x,y) - z\, w_{,x}(x,y)$$
$$v(x,y,z) = v_0(x,y) - z\, w_{,y}(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

Transverse normals remain straight after deformation (rigid cross-section).
Transverse normals do not experience elongation ($\varepsilon_{zz} = 0$).
Transverse normals remain perpendicular to the mid-surface after deformation (no transverse shear, i.e. $\gamma_{xz} = \gamma_{yz} = 0$).

First-order Shear Deformation Theory (FSDT), also known as Reissner-Mindlin theory.

$$u(x,y,z) = u_0(x,y) + z\, \phi_x(x,y)$$
$$v(x,y,z) = v_0(x,y) + z\, \phi_y(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

Rotations disconnected from normal displacements $\phi_x(x,y) \neq -w_{,x}(x,y)$.
Transverse normals do not experience elongation ($\varepsilon_{zz} = 0$).
Transverse shear strains γ_{xz} and γ_{yz} are constant in z. Therefore, shear correction factors are needed.

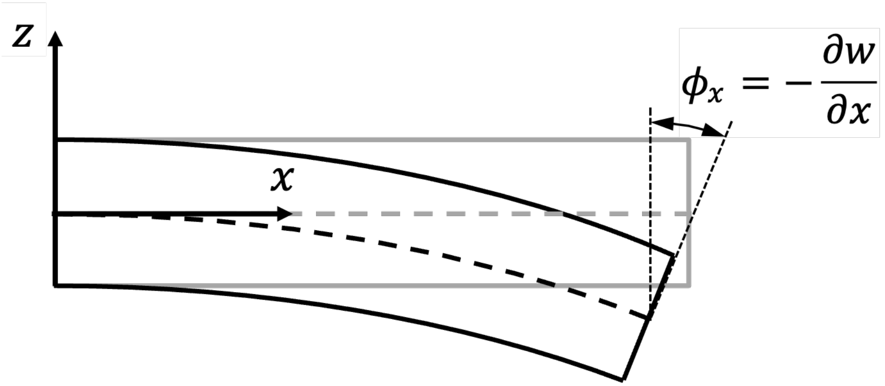
Third-order Shear Deformation Theory (TSDT).

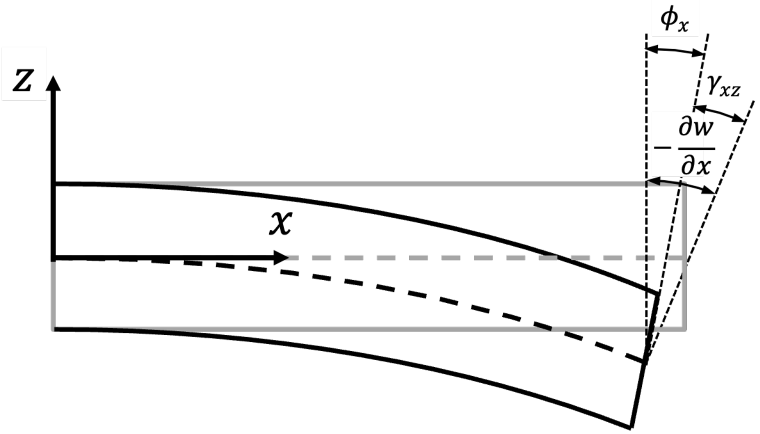
$$u(x,y,z) = u_0(x,y) + z\phi_x(x,y) + z^2\psi_x(x,y) + z^3\lambda_x(x,y)$$
$$v(x,y,z) = v_0(x,y) + z\phi_y(x,y) + z^2\psi_y(x,y) + z^3\lambda_y(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

Transverse normals do not experience elongation ($\varepsilon_{zz} = 0$).
Consistent transverse shear strains $\gamma_{xz}(x,y,z)$ and $\gamma_{yz}(x,y,z)$. Shear correction factors NOT needed.

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.
[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

Difference between CLPT and FSDT





[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

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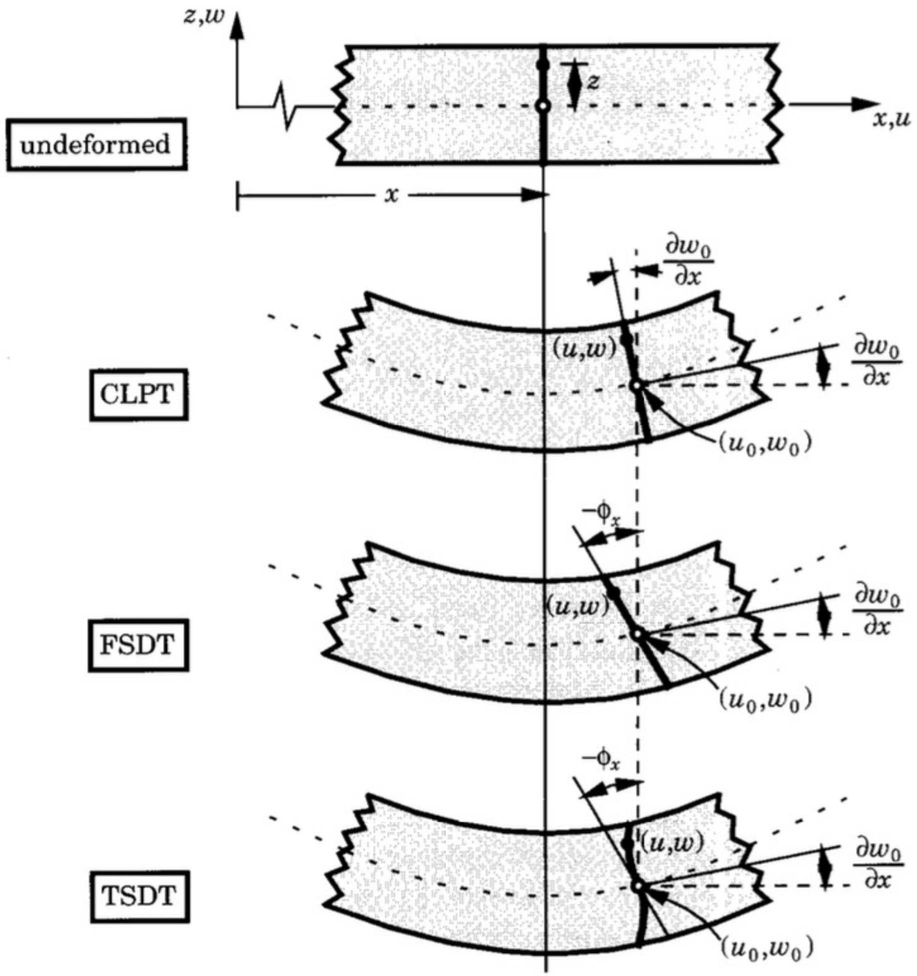
Saullo G. P. Castro

Difference between CLPT, FSDT and TSDT



[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

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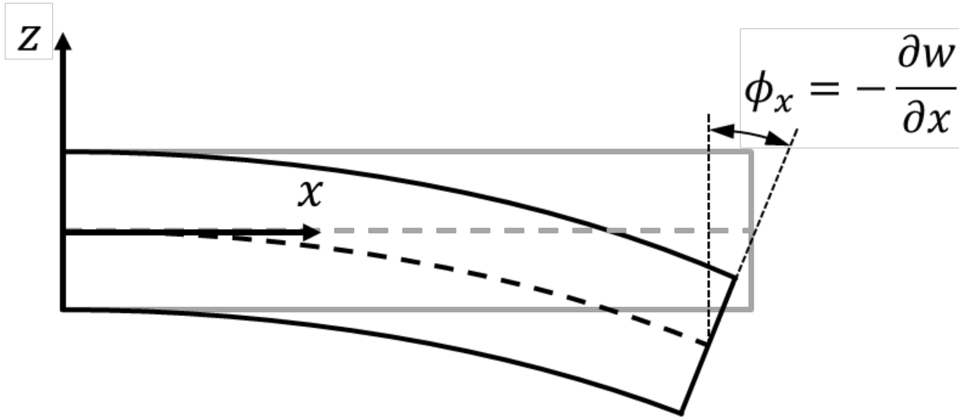
CLPT and FSDT for plates

Classical Laminated Plate Theory

The simplest of the ESL theories is the Classical Laminated Plate Theory (CLPT) which is an extension of the Classical Plate Theory to composite laminates [4], where the Kirchhoff hypotheses hold [4]:

- Transverse normals remain straight after deformation;
- Transverse normals do not experience elongation ($\varepsilon_{zz} = 0$);
- The transverse normals rotate so that they remain perpendicular to the mid-surface after deformation (no transverse shear takes place, i.e. $\gamma_{yz} = \gamma_{xz} = 0$).



Kinematics of a plate for the CLPT.
Modified from [1]

$$u(x,y,z) = u_0(x,y) - z\,w_{,x}(x,y)$$
$$v(x,y,z) = v_0(x,y) - z\,w_{,y}(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

`See: ex/kinematics_plate_CLPT.ipynb`

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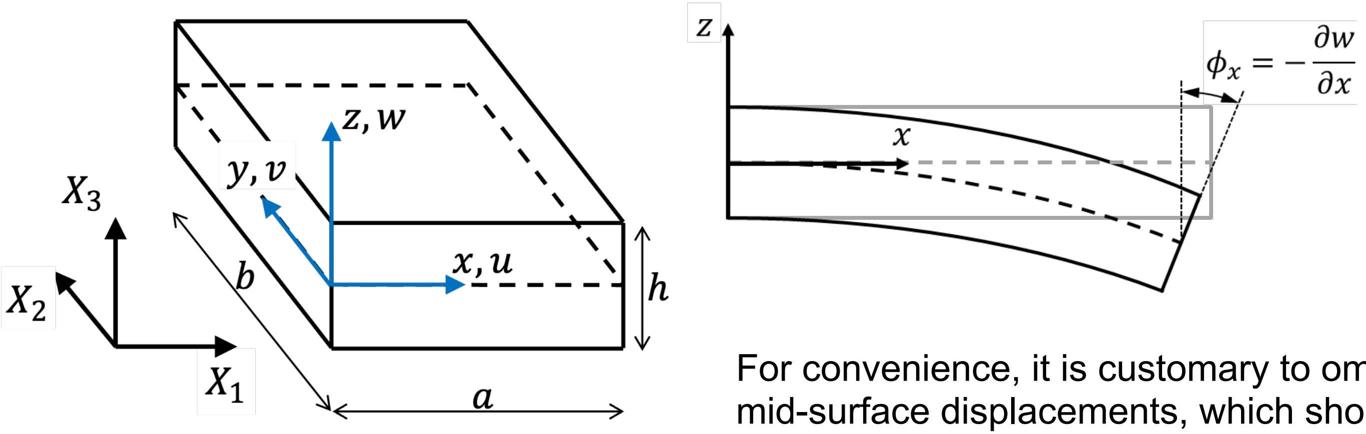
[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.
[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

CLPT general kinematic equations for plates



Kinematics of a plate for the CLPT. Modified from [1]

$$u(x,y,z) = u_0(x,y) - z\, w_{,x}(x,y)$$
$$v(x,y,z) = v_0(x,y) - z\, w_{,y}(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

For convenience, it is customary to omit the subscript "0" from the mid-surface displacements, which should be clear from the context.

$$\varepsilon_{xx} = u_{,x} - z\, w_{,xx} + \frac{1}{2}\left((z\, w_{,xx} - u_{,x})^2 + (z\, w_{,xy} - v_{,x})^2 + w_{,x}^2 \right)$$
$$\varepsilon_{yy} = v_{,y} - z\, w_{,yy} + \frac{1}{2}\left((z\, w_{,xy} - u_{,y})^2 + (z\, w_{,yy} - v_{,y})^2 + w_{,y}^2 \right)$$

$\varepsilon_{zz} = 0$ (thickness remains constant during bending)

$$\gamma_{xy} = u_{,y} + v_{,x} - 2z\, w_{,xy} + (z\, w_{,xx} - u_{,x})(z\, w_{,xy} - u_{,y}) + (z\, w_{,xy} - v_{,x})(z\, w_{,yy} - v_{,y}) + w_{,x}\, w_{,y}$$
$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 0$$

See: `ex/kinematics_plate_CLPT.ipynb`

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

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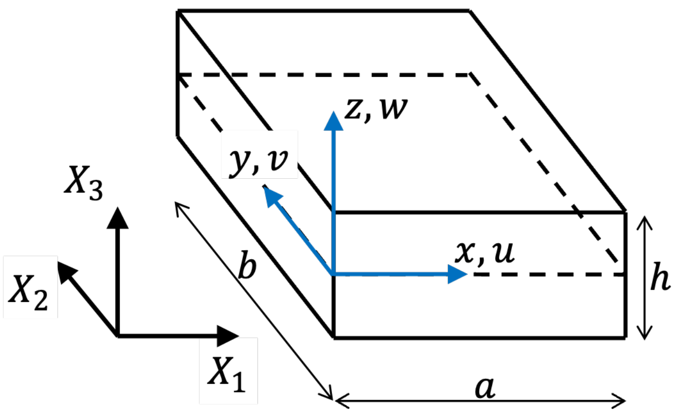
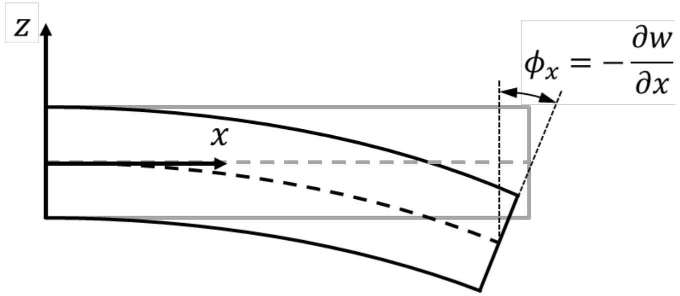
CLPT simplified kinematic equations for plates van Kármán kinematics





Kinematics of a plate for the CLPT. Modified from [1]

$$u(x, y, z) = u_0(x, y) - z\, w_{,x}(x, y)$$
$$v(x, y, z) = v_0(x, y) - z\, w_{,y}(x, y)$$
$$w(x, y, z) = w_0(x, y)$$

For convenience, it is customary to omit the subscript “0” from the mid-surface displacements, which should be clear from the context.

$$\varepsilon_{xx} = u_{,x} - z\, w_{,xx} + \frac{1}{2} w_{,x}^2$$
$$\varepsilon_{yy} = v_{,y} - z\, w_{,yy} + \frac{1}{2} w_{,y}^2$$

$\varepsilon_{zz} = 0$ (thickness remains constant during bending)

$$\gamma_{xy} = u_{,y} + v_{,x} - 2z\, w_{,xy} + w_{,x}\, w_{,y}$$
$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 0$$

`See: ex/kinematics_plate_CLPT.ipynb`

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

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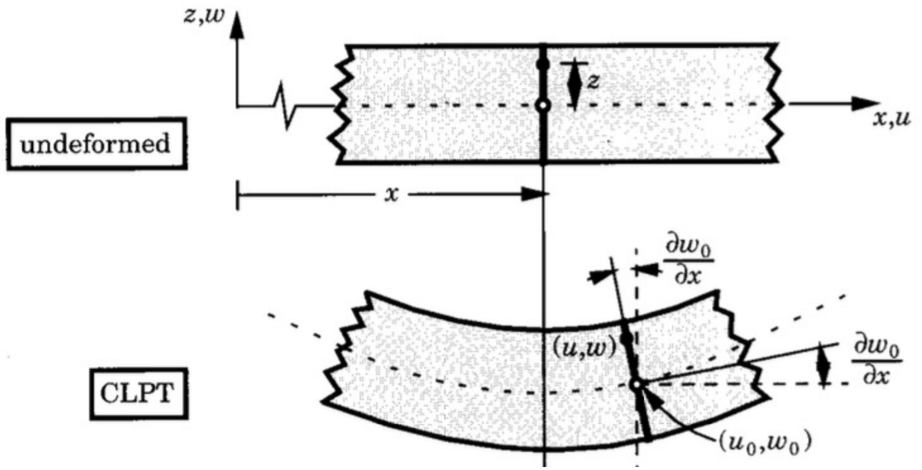
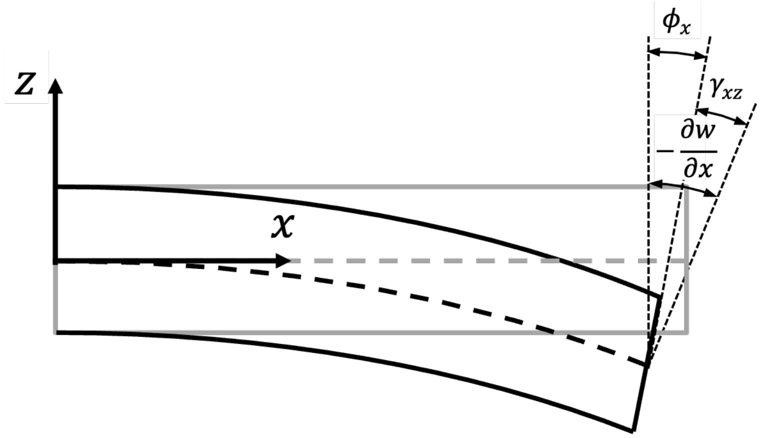
First-order Shear Deformation Theory (FSDT)

The vastly most used ESL theory within finite element codes.

- Rotations disconnected from normal displacements $\phi_x(x,y) \neq -w_{,x}(x,y)$.
- Transverse normals do not experience elongation ($\varepsilon_{zz} = 0$).
- Transverse shear strains γ_{xz} and γ_{yz} are constant in z. Therefore, shear correction factors are needed.





Kinematics of a plate for the FSDT. Modified from [1]

$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y)$$
$$w(x,y,z) = w(x,y)$$

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.
[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

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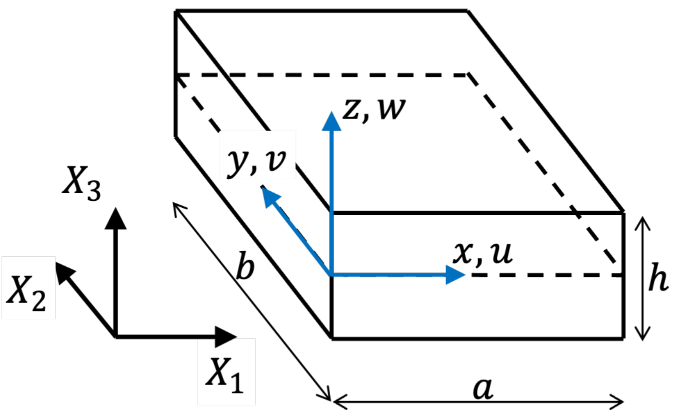
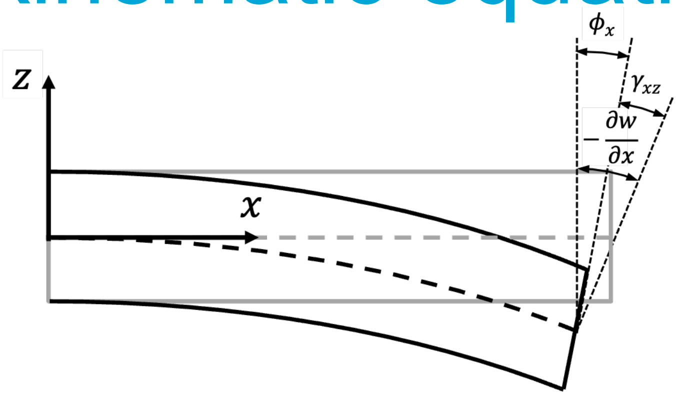
FSDT general kinematic equations for plates





Kinematics of a plate for the FSDT. Modified from [1]

$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y)$$
$$w(x,y,z) = w(x,y)$$

For convenience, it is customary to omit the subscript "0" from the mid-surface displacements, which should be clear from the context.

$$\varepsilon_{xx} = u_{,x} + z\,\phi_{x,x} + \frac{1}{2}\left(\left(z\phi_{x,x} + u_{,x}\right)^2 + \left(z\phi_{y,x} + v_{,x}\right)^2 + w_{,x}^2\right)$$
$$\varepsilon_{yy} = v_{,y} + z\,\phi_{y,y} + \frac{1}{2}\left(\left(z\phi_{x,y} + u_{,y}\right)^2 + \left(z\phi_{y,y} + v_{,y}\right)^2 + w_{,y}^2\right)$$

$\varepsilon_{zz} = 0$ (thickness remains constant during bending)

$$\gamma_{xy} = u_{,y} + v_{,x} + z\phi_{x,y} + z\phi_{y,x} + \left(z\phi_{x,x} + u_{,x}\right)\left(z\phi_{x,y} + u_{,y}\right) + \left(z\phi_{y,x} + v_{,x}\right)\left(z\phi_{y,y} + v_{,y}\right) + w_{,x}w_{,y}$$

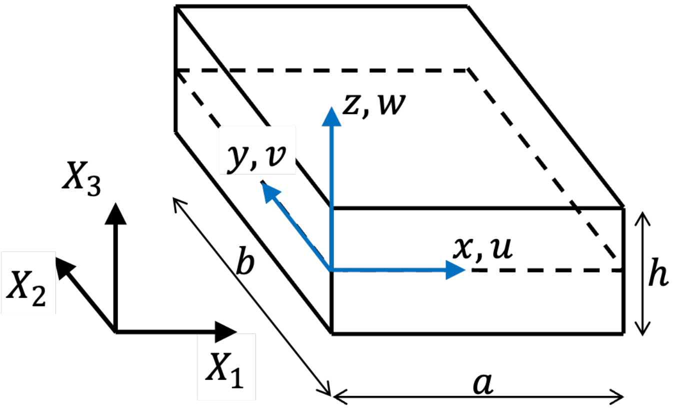
$$\gamma_{xz} = \phi_x + w_{,x} + \left(z\phi_{x,x} + u_{,x}\right)\phi_x + \left(z\phi_{y,x} + v_{,x}\right)\phi_y$$

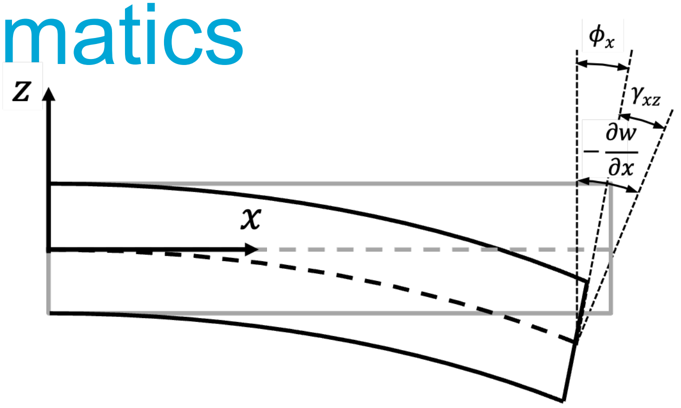
$$\gamma_{yz} = \phi_y + w_{,y} + \left(z\phi_{x,y} + u_{,y}\right)\phi_x + \left(z\phi_{y,y} + v_{,y}\right)\phi_y$$

`See: ex/kinematics_plate_FSDT.ipynb`

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

FSDT simplified kinematic equations for plates van Kármán kinematics





Kinematics of a plate for the FSDT. Modified from [1]

$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y)$$
$$w(x,y,z) = w(x,y)$$

For convenience, it is customary to omit the subscript “0” from the mid-surface displacements, which should be clear from the context.

$$\varepsilon_{xx} = u_{,x} + z\,\phi_{x,x} + \frac{1}{2}w_{,x}^2$$
$$\varepsilon_{yy} = v_{,y} + z\,\phi_{y,y} + \frac{1}{2}w_{,y}^2$$

$\varepsilon_{zz} = 0$ (thickness remains constant during bending)

$$\gamma_{xy} = u_{,y} + v_{,x} + z\phi_{x,y} + z\phi_{y,x} + w_{,x}\,w_{,y}$$

$$\gamma_{xz} = \phi_x + w_{,x}$$

$$\gamma_{yz} = \phi_y + w_{,y}$$

`See: ex/kinematics_plate_FSDT.ipynb`

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

FSDT kinematics (linear terms)

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \end{Bmatrix} + z \begin{Bmatrix} \phi_{x,x} \\ \phi_{y,y} \\ \phi_{x,y} + \phi_{y,x} \end{Bmatrix}$$

$$\boldsymbol{\gamma} = \begin{Bmatrix} 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} w_{,y} + \phi_y \\ w_{,x} + \phi_x \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + z\boldsymbol{\varepsilon}^{(1)}$$
$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(0)}$$

Usually, subscript “0” for the mid-surface expressions is omitted!

All relations presented in this part apply to the CLPT by doing:

$$\phi_x = -w_{,x}$$
$$\phi_y = -w_{,y}$$
$$\gamma_{xz} = 0$$
$$\gamma_{yz} = 0$$

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FSDT strain energy (linear terms)

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

$$\boldsymbol{\tau} = \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$

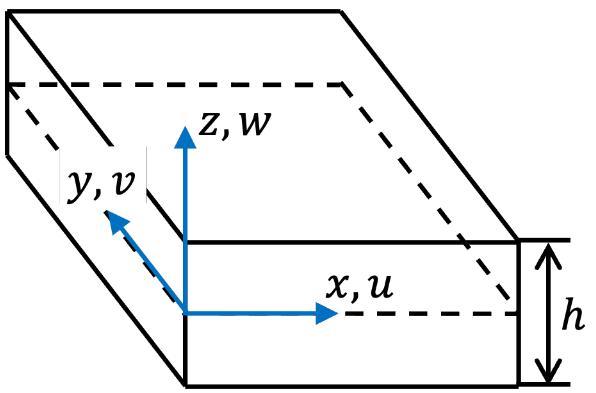
Variation of Strain Energy

$$\delta U = \int_\Omega \boldsymbol{\sigma}^\top \delta\boldsymbol{\varepsilon}\, d\Omega = \int_\Omega \boldsymbol{\sigma}^\top\left(\delta\boldsymbol{\varepsilon}^{(0)} + z\delta\boldsymbol{\varepsilon}^{(1)}\right) + \boldsymbol{\tau}^\top\left(\delta\boldsymbol{\gamma}^{(0)}\right) d\Omega$$

Separate integration through the thickness:

$$\delta U = \iint_{xy} \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma}^\top \delta\boldsymbol{\varepsilon}^{(0)} dz + \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z\boldsymbol{\sigma}^\top \delta\boldsymbol{\varepsilon}^{(1)} dz + \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau}^\top \delta\boldsymbol{\gamma}^{(0)} dz \right) dxdy$$

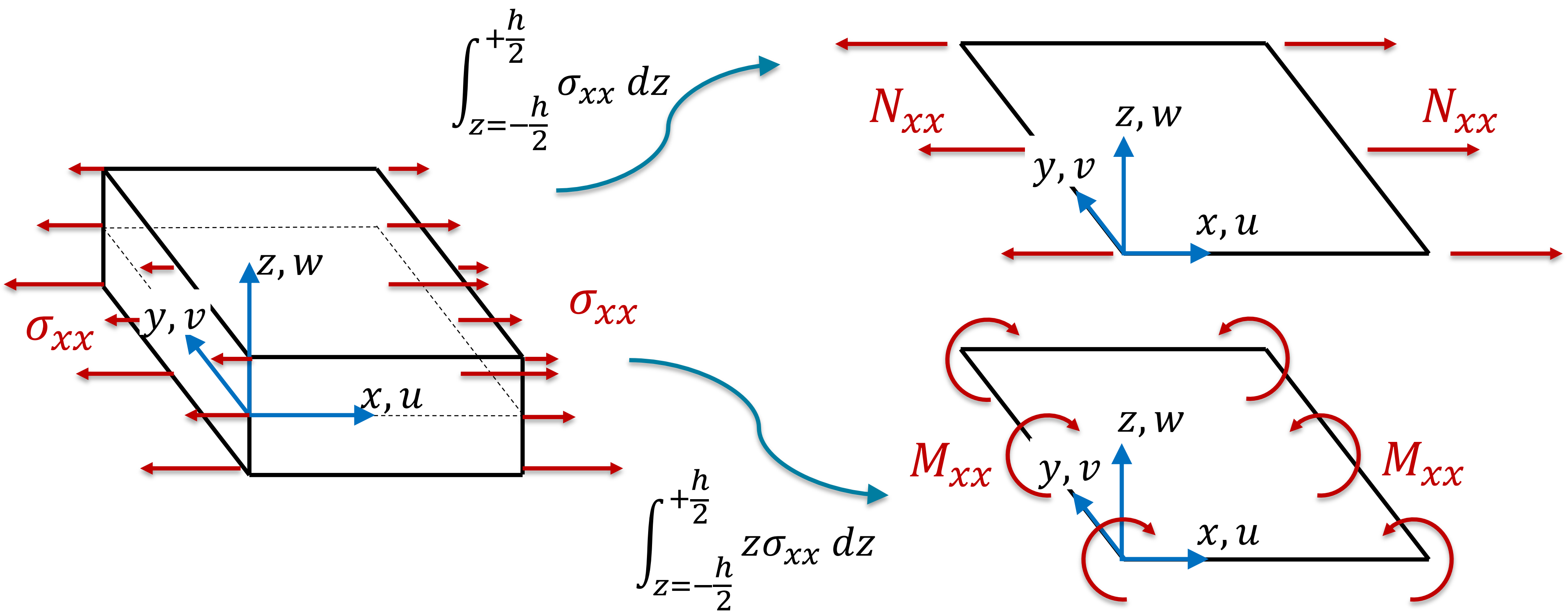
$$\delta U = \iint_{xy} \left(\left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma}^\top dz \right) \delta\boldsymbol{\varepsilon}^{(0)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z\boldsymbol{\sigma}^\top dz \right) \delta\boldsymbol{\varepsilon}^{(1)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau}^\top dz \right) \delta\boldsymbol{\gamma}^{(0)} \right) dxdy$$



What if there is an offset about the mid-plane?

FSDT integrating stresses over thickness

$$\delta U = \iint_{xy}\left(\left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma}^{\top} dz\right)\delta\boldsymbol{\varepsilon}^{(0)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z\boldsymbol{\sigma}^{\top} dz\right)\delta\boldsymbol{\varepsilon}^{(1)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau}^{\top} dz\right)\delta\boldsymbol{\gamma}^{(0)}\right) dxdy$$



FSDT strain energy

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \boldsymbol{N} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \boldsymbol{M} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} Q_{y} \\ Q_{x} \end{Bmatrix} = \boldsymbol{Q} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz$$

$$\delta U = \iint_{xy} \left(\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}^{\top} \delta \boldsymbol{\varepsilon}^{(0)} + \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}^{\top} \delta \boldsymbol{\varepsilon}^{(1)} + \begin{Bmatrix} Q_{y} \\ Q_{x} \end{Bmatrix}^{\top} \delta \boldsymbol{\gamma}^{(0)} \right) dxdy$$

$$\delta U = \iint_{xy} \left(\boldsymbol{N}^{\top} \delta \boldsymbol{\varepsilon}^{(0)} + \boldsymbol{M}^{\top} \delta \boldsymbol{\varepsilon}^{(1)} + \boldsymbol{Q}^{\top} \delta \boldsymbol{\gamma}^{(0)} \right) dxdy$$

FSDT strain energy

$$\delta U = \iint_{xy} \left(\boldsymbol{N}^{\top} \delta \boldsymbol{\varepsilon}^{(0)} + \boldsymbol{M}^{\top} \delta \boldsymbol{\varepsilon}^{(1)} + \boldsymbol{Q}^{\top} \delta \boldsymbol{\gamma}^{(0)} \right) dxdy$$

Main assumptions of this formulation?

Planar stress

FSDT (or CLPT)

Does this relation work for laminated composite materials?

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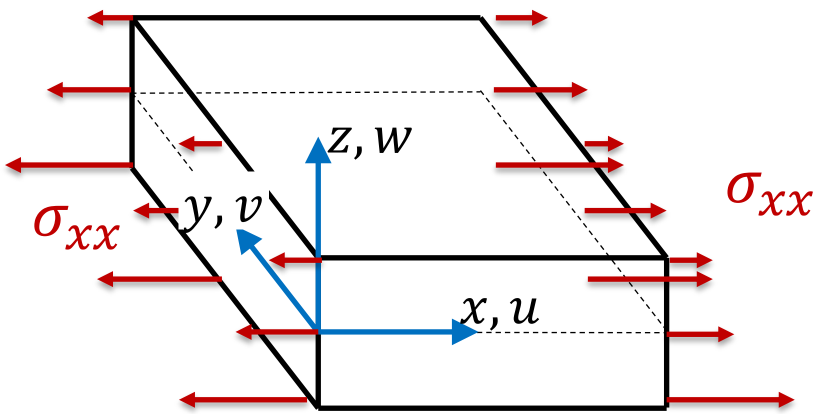
FSDT constitutive equations (isotropic)

- Find $\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{Q}$ for isotropic materials (easily extended to composites)

$$\boldsymbol{N} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\boldsymbol{M} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\boldsymbol{Q} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz$$



Note that the stresses vary along the thickness (e.g. σ_{xx})

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FSDT constitutive equations (isotropic)

Review on planar stress:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} \qquad \tau_{xy} = \frac{\gamma_{xy}}{G}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{xx}}{E} \qquad \tau_{yz} = \frac{\gamma_{yz}}{G}$$

$$G = \frac{E}{2(1+\nu)} \qquad \tau_{xz} = \frac{\gamma_{xz}}{G}$$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0 & 0 & 0\\ -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0\\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}$$

Separating in-plane from transverse shear terms:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\tau_{yz}\\ \tau_{xz}\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}\frac{1-\nu}{2} & 0\\ 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + z\boldsymbol{\varepsilon}^{(1)}$$
$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(0)}$$

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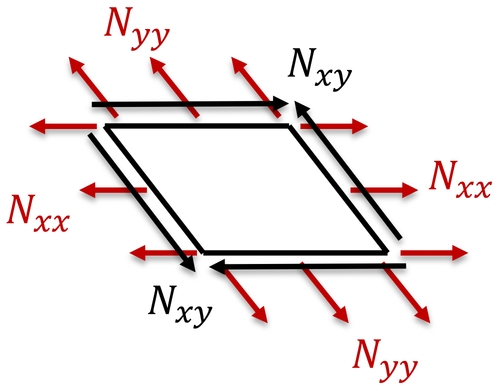
FSDT membrane stresses (isotropic)

Calculating $\boldsymbol{N}$ (membrane stresses):



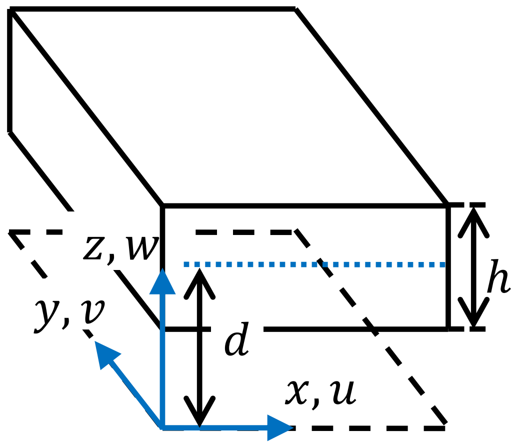
$$\boldsymbol{N} = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}$$

$$\boldsymbol{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\boldsymbol{N} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{C}\left(\boldsymbol{\varepsilon}^{(0)} + z\boldsymbol{\varepsilon}^{(1)}\right) dz = h\boldsymbol{C}\boldsymbol{\varepsilon}^{(0)}$$

With an offset d:

$$\boldsymbol{N} = \int_{-\frac{h}{2}+d}^{+\frac{h}{2}+d} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz = h\boldsymbol{C}\boldsymbol{\varepsilon}^{(0)} + dh\boldsymbol{C}\boldsymbol{\varepsilon}^{(1)}$$



Analogy with composites…

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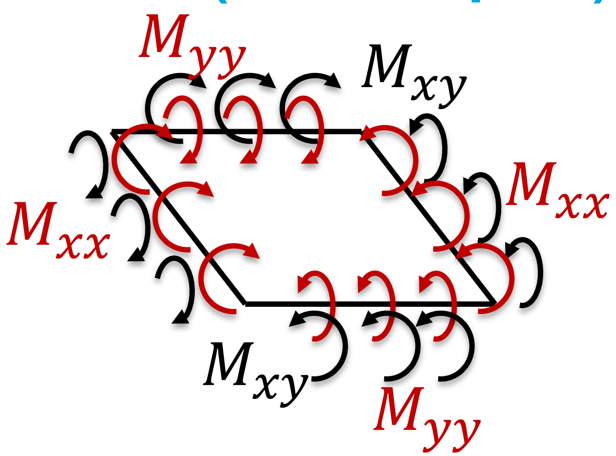
FSDT bending stresses (isotropic)

Calculating $\boldsymbol{M}$ (bending stresses):



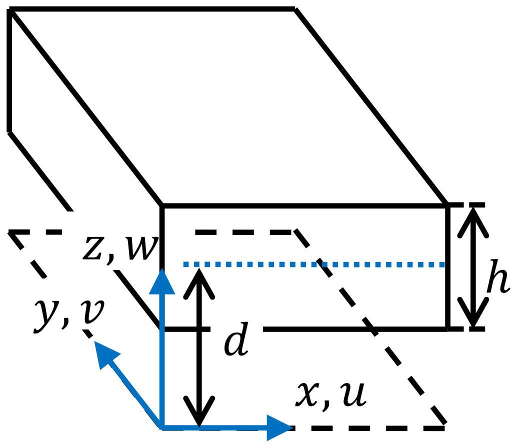
$$\boldsymbol{M} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}$$

$$\boldsymbol{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\boldsymbol{M} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{C}\left(z\boldsymbol{\varepsilon}^{(0)} + z^2\boldsymbol{\varepsilon}^{(1)}\right) dz = \frac{h^3}{12}\boldsymbol{C}\boldsymbol{\varepsilon}^{(1)}$$

Analogy with composites…

With an offset d:



$$\boldsymbol{M} = \int_{-\frac{h}{2}+d}^{+\frac{h}{2}+d} z \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz = dh\boldsymbol{C}\boldsymbol{\varepsilon}^{(0)} + h\left(d^2 + \frac{h^2}{12}\right)\boldsymbol{C}\boldsymbol{\varepsilon}^{(1)}$$

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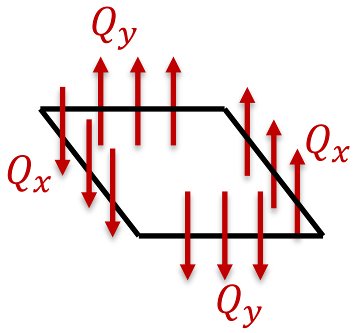
FSDT transverse stresses (isotropic)

Calculating $\boldsymbol{Q}$ (transverse stresses):



$$\boldsymbol{Q} = \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix}$$

$$\boldsymbol{S} = \frac{E}{1-\nu^2} \begin{bmatrix} \frac{1-\nu}{2} & 0 \\ 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\boldsymbol{Q} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{S}\boldsymbol{\gamma}^{(0)} \, dz = h\boldsymbol{S}\boldsymbol{\gamma}^{(0)}$$

With an offset:

$$\boldsymbol{Q} = h\boldsymbol{S}\boldsymbol{\gamma}^{(0)}$$

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FSDT stress resultants, overview

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^{(0)} \\ \boldsymbol{\varepsilon}^{(1)} \end{Bmatrix}$$

$$\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{\gamma}^{(0)}$$

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}[(z_{k+1}) - (z_k)]$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}[(z_{k+1})^2 - (z_k)^2]$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}[(z_{k+1})^3 - (z_k)^3]$$

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}(1, z, z^2)dz$$

$$\left(A_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}dz$$

[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

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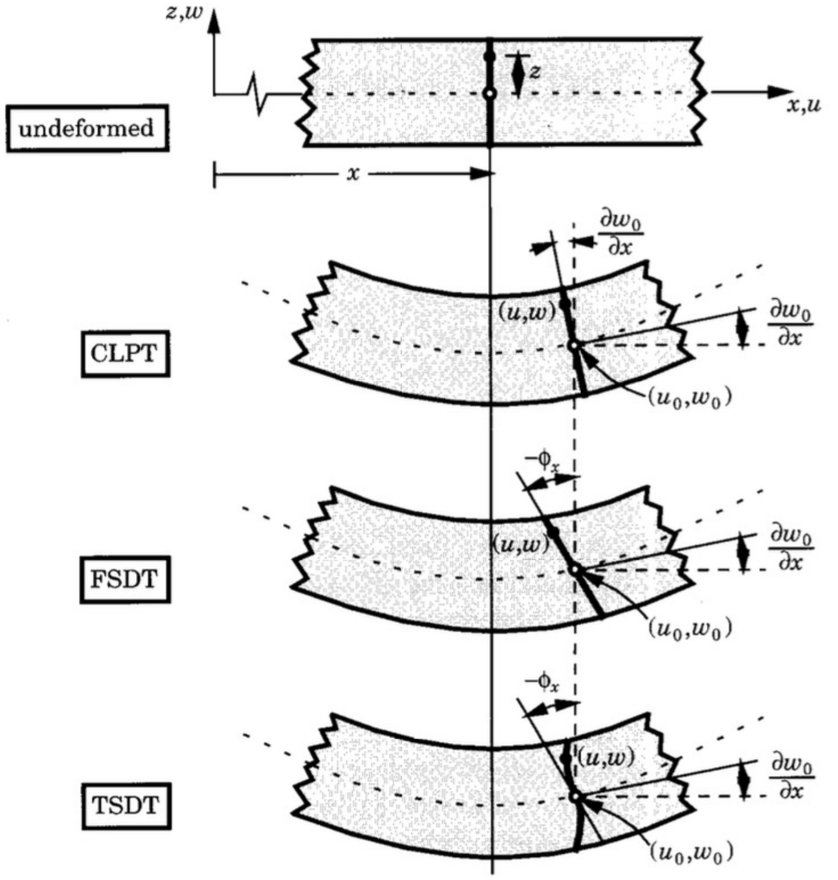
TSDT

Third-order Shear Deformation Theory (TSDT)

Assumptions:

- Rotations disconnected from normal displacements $\phi_x(x,y) \neq -w_{,x}(x,y)$.
- Transverse normals do not experience elongation ($\varepsilon_{zz} = 0$).
- Consistent transverse shear strains $\gamma_{xz}(x,y,z)$ and $\gamma_{yz}(x,y,z)$. Shear correction factors NOT needed.



Kinematics of a plate for the TSDT from [4, Chap. 11]

$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y) + z^2\theta_x(x,y) + z^3\lambda_x(x,y)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y) + z^2\theta_y(x,y) + z^3\lambda_y(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

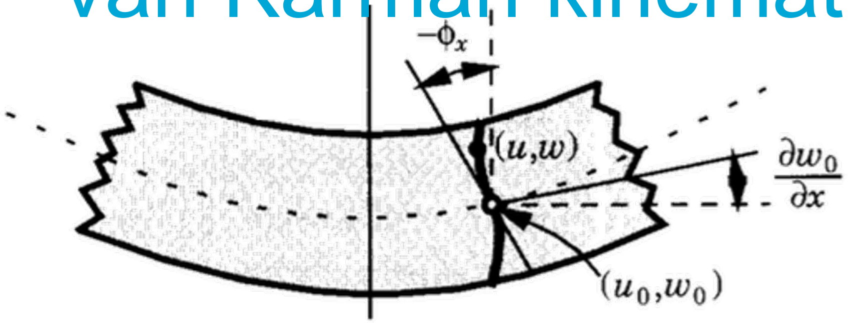
with 9 unknown field variables.

Reddy proposes in 1984 to impose 4 traction-free boundary conditions, on the bottom and top faces of the laminate:

$$\tau_{xz}\left(x,y,\pm\frac{h}{2}\right) = 0 \qquad \tau_{yz}\left(x,y,\pm\frac{h}{2}\right) = 0$$

[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

TSDT simplified kinematic equations for plates van Kármán kinematics



Kinematics of a plate for the TSDT from [4, Chap. 11]

$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y) - \frac{4}{3h^2} z^3\left(\phi_x(x,y) + w_{,x}(x,y)\right)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y) - \frac{4}{3h^2} z^3\left(\phi_y(x,y) + w_{,y}(x,y)\right)$$
$$w(x,y,z) = w(x,y)$$

For convenience, it is customary to omit the subscript “0” from the mid-surface displacements, which should be clear from the context.

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2} w_{,x}^2 + z\phi_{x,x} + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_{x,x} + w_{,xx}\right)$$
$$\varepsilon_{yy} = v_{,y} + \frac{1}{2} w_{,y}^2 + z\phi_{y,y} + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_{y,y} + w_{,yy}\right)$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x} w_{,y} + z\phi_{x,y} + z\phi_{y,x} + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_{x,y} + \phi_{y,x} + 2w_{,xy}\right)$$

$$\gamma_{xz} = \phi_x + w_{,x} + z^2\left(-\frac{4}{h^2}\right)\left(\phi_x + w_{,x}\right)$$

$$\gamma_{yz} = \phi_y + w_{,y} + z^2\left(-\frac{4}{h^2}\right)\left(\phi_y + w_{,y}\right)$$

See: `ex/kinematics_plate_TSDT.ipynb`

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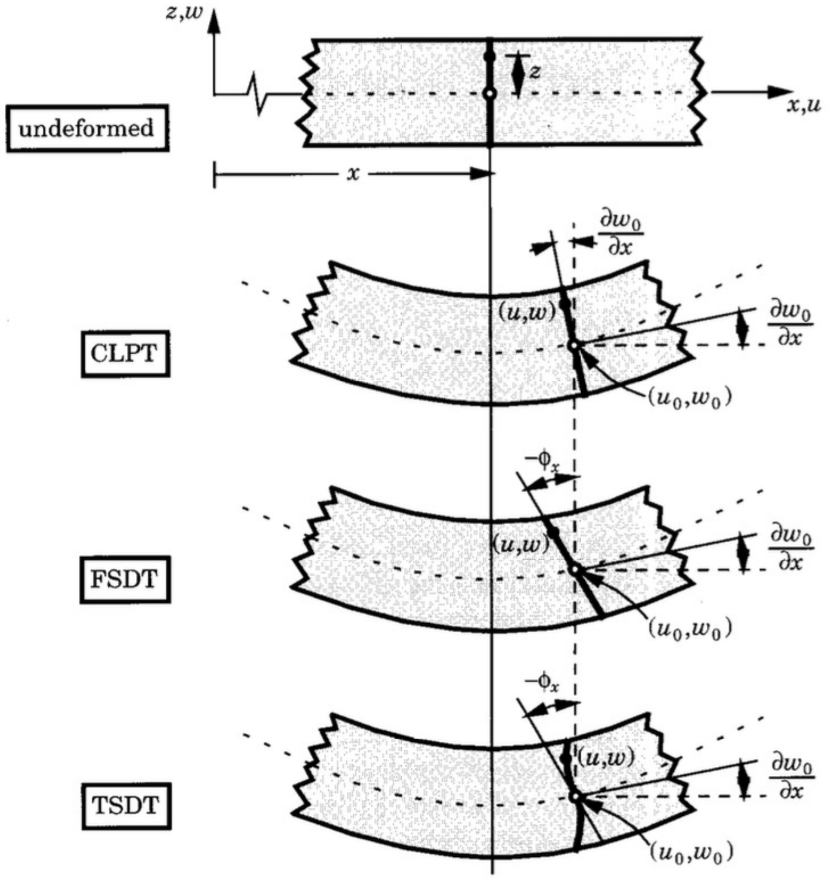
[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

Third-order Shear Deformation Theory (TSDT)

which results in:

$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y) - \frac{4}{3h^2} z^3\big(\phi_x(x,y) + w_{,x}(x,y)\big)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y) - \frac{4}{3h^2} z^3\big(\phi_y(x,y) + w_{,y}(x,y)\big)$$
$$w(x,y,z) = w(x,y)$$

with 5 unknown field variables.



[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

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TSDT for plates

TSDT kinematics (linear terms)

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \end{Bmatrix} + z \begin{Bmatrix} \phi_{x,x} \\ \phi_{y,y} \\ \phi_{x,y} + \phi_{y,x} \end{Bmatrix} + z^3 \left(-\frac{4}{3h^2}\right) \begin{Bmatrix} \phi_{x,x} + w_{,xx} \\ \phi_{y,y} + w_{,yy} \\ \phi_{x,y} + \phi_{y,x} + 2w_{,xy} \end{Bmatrix}$$

$$\boldsymbol{\gamma} = \begin{Bmatrix} 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} w_{,y} + \phi_y \\ w_{,x} + \phi_x \end{Bmatrix} + z^2 \left(-\frac{4}{h^2}\right) \begin{Bmatrix} w_{,y} + \phi_y \\ w_{,x} + \phi_x \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + z\boldsymbol{\varepsilon}^{(1)} + z^3\boldsymbol{\varepsilon}^{(3)}$$
$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^{(0)} + z^2\boldsymbol{\gamma}^{(2)}$$

Usually, subscript “0” for the mid-surface expressions is omitted!

TSDT strain energy (linear terms)

Variation of Strain Energy

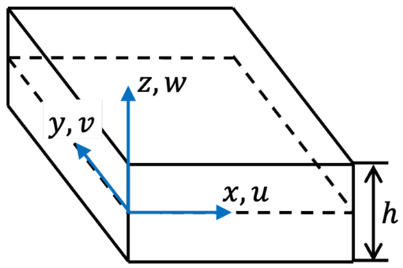
$$\delta U = \int_\Omega \boldsymbol{\sigma}^\top \delta\boldsymbol{\varepsilon}\, d\Omega = \int_\Omega \boldsymbol{\sigma}^\top\left(\delta\boldsymbol{\varepsilon}^{(0)} + z\delta\boldsymbol{\varepsilon}^{(1)} + z^3\delta\boldsymbol{\varepsilon}^{(3)}\right) + \boldsymbol{\tau}^\top\left(\delta\boldsymbol{\gamma}^{(0)} + z^2\delta\boldsymbol{\gamma}^{(2)}\right) d\Omega$$

$$\boldsymbol{\sigma} = \begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}$$

$$\boldsymbol{\tau} = \begin{Bmatrix}\tau_{yz}\\ \tau_{xz}\end{Bmatrix}$$

Separate integration through the thickness:

$$\delta U = \iint_{xy}\left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma}^\top\delta\boldsymbol{\varepsilon}^{(0)} dz + \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z\boldsymbol{\sigma}^\top\delta\boldsymbol{\varepsilon}^{(1)} dz + \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^{\mathbf{3}}\, \boldsymbol{\sigma}^\top\delta\boldsymbol{\varepsilon}^{(3)} dz\right.$$
$$\left. + \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau}^\top\delta\boldsymbol{\gamma}^{(0)} dz + \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^2\boldsymbol{\tau}^\top\delta\boldsymbol{\gamma}^{(2)} dz\right) dxdy$$



$$\delta U = \iint_{xy}\left(\left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma}^\top dz\right)\delta\boldsymbol{\varepsilon}^{(0)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z\boldsymbol{\sigma}^\top dz\right)\delta\boldsymbol{\varepsilon}^{(1)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^{\mathbf{3}}\,\boldsymbol{\sigma}^\top dz\right)\delta\boldsymbol{\varepsilon}^{(3)}\right.$$
$$\left. + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau}^\top dz\right)\delta\boldsymbol{\gamma}^{(0)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^2\boldsymbol{\tau}^\top dz\right)\delta\boldsymbol{\gamma}^{(2)}\right) dxdy$$

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TSDT integrating stresses over thickness

$$\delta U = \iint_{xy} \left(\left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma}^{\top} dz \right) \delta \boldsymbol{\varepsilon}^{(0)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z \boldsymbol{\sigma}^{\top} dz \right) \delta \boldsymbol{\varepsilon}^{(1)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^{\mathbf{3}} \, \boldsymbol{\sigma}^{\top} dz \right) \delta \boldsymbol{\varepsilon}^{(3)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau}^{\top} dz \right) \delta \boldsymbol{\gamma}^{(0)} + \left(\int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^2 \boldsymbol{\tau}^{\top} dz \right) \delta \boldsymbol{\gamma}^{(2)} \right) dxdy$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \boldsymbol{N} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} Q_{y} \\ Q_{x} \end{Bmatrix} = \boldsymbol{Q} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \boldsymbol{M} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} R_{y} \\ R_{x} \end{Bmatrix} = \boldsymbol{R} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^2 \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz$$

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \boldsymbol{P} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^3 \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

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TSDT strain energy

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \boldsymbol{N} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \boldsymbol{M} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \boldsymbol{P} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^3 \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} Q_{y} \\ Q_{x} \end{Bmatrix} = \boldsymbol{Q} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz$$

$$\begin{Bmatrix} R_{y} \\ R_{x} \end{Bmatrix} = \boldsymbol{R} = \int_{z=-\frac{h}{2}}^{+\frac{h}{2}} z^2 \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz$$

$$\delta U = \iint_{xy} \Big(\boldsymbol{N}^\top \delta \boldsymbol{\varepsilon}^{(0)} + \boldsymbol{M}^\top \delta \boldsymbol{\varepsilon}^{(1)} + \boldsymbol{P}^\top \delta \boldsymbol{\varepsilon}^{(3)} + \boldsymbol{Q}^\top \delta \boldsymbol{\gamma}^{(0)} + \boldsymbol{R}^\top \delta \boldsymbol{\gamma}^{(2)} \Big) dx dy$$

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TSDT stress resultants, overview

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \\ \boldsymbol{P} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{E} \\ \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{F} \\ \boldsymbol{E} & \boldsymbol{F} & \boldsymbol{H} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^{(0)} \\ \boldsymbol{\varepsilon}^{(1)} \\ \boldsymbol{\varepsilon}^{(3)} \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{Q} \\ \boldsymbol{R} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{D} \\ \boldsymbol{D} & \boldsymbol{F} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\gamma}^{(0)} \\ \boldsymbol{\gamma}^{(2)} \end{Bmatrix}$$

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} (1, z, z^2, z^3, z^4, z^6) dz$$

$$\left(A_{ij}, D_{ij}, F_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} (1, z^2, z^4) dz$$

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} [(z_{k+1}) - (z_k)]$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} [(z_{k+1})^2 - (z_k)^2]$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} [(z_{k+1})^3 - (z_k)^3]$$

$$E_{ij} = \frac{1}{4} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} [(z_{k+1})^4 - (z_k)^4]$$

$$F_{ij} = \frac{1}{5} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} [(z_{k+1})^5 - (z_k)^5]$$

$$H_{ij} = \frac{1}{7} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)} [(z_{k+1})^7 - (z_k)^7]$$

[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

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Next steps…

- We will start solving some practical problems
- Principle of minimum potential energy
- Legendre polynomials
- Neutral equilibrium criterion

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Principle of Minimum Potential Energy

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Principle of Minimum Potential Energy

The total potential energy V can be decomposed into elastic energy U, and the work due to external forces W_{ext}:

$$V = U - W_{ext}$$

This potential becomes stationary when:

$$\delta V = \delta U - \delta W_{ext} = 0$$

Strain Energy (linear)

$$U = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}^{\top} \boldsymbol{\varepsilon}\, d\Omega$$

Work due to external forces

$$W_{ext} = \int_{\Omega} \boldsymbol{b}^{\top} \boldsymbol{u}\, d\Omega + \int_{\delta\Omega} \left(\overline{\boldsymbol{\sigma}}^{\top} \boldsymbol{u}\right) d(\delta\Omega)$$

Variation of Strain Energy

$$\delta U = \int_{\Omega} \boldsymbol{\sigma}^{\top} \delta\boldsymbol{\varepsilon}\, d\Omega$$

Variation of work due to external forces

$$\delta W_{ext} = \int_{\Omega} \boldsymbol{b}^{\top} \delta\boldsymbol{u}\, d\Omega + \int_{\delta\Omega} \left(\overline{\boldsymbol{\sigma}}^{\top} \delta\boldsymbol{u}\right) d(\delta\Omega)$$

Note that, here, $\boldsymbol{u}$ represents the 3D displacement field (with no approximations made).

$$\boldsymbol{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

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Strain energy

Strain Energy

$$\delta V = \boxed{\int_{\Omega} \boldsymbol{\sigma}^{\top} \delta \boldsymbol{\varepsilon} \, d\Omega} - \int_{\Omega} (\boldsymbol{b}^{\top} \delta \boldsymbol{u}) d\Omega - \int_{\delta\Omega} (\boldsymbol{\sigma}^{\top} \delta \boldsymbol{u}) d(\delta\Omega) = 0$$

To calculate the strain energy:

- Kinematic equations: how strains relate to displacements

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{u}$$

- Constitutive relations: how stresses relate to strains

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$$

$\boldsymbol{B} \equiv$ differentiation operator matrix

$\boldsymbol{u}(x, y, z) \equiv$ continuous displacement field

$\boldsymbol{C} \equiv$ constitutive matrix

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Strain energy

Kinematic equations: how strains relate to displacements

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{u}$$

Constitutive relations: how stresses relate to strains

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$$

Voigt notation

For plane stress, CLPT:

$$\boxed{\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \sigma_{xx}, \sigma_{yy}, \tau_{xy}}$$

$$\boxed{\boldsymbol{\varepsilon}^{\mathrm{T}} = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}\} \qquad \boldsymbol{\sigma}^{\mathrm{T}} = \{\sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy}\}}$$

2nd-order tensors represented as vectors

For plane stress, FSDT:

$$\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$$

$$\sigma_{xx}, \sigma_{yy}, \tau_{xy}, \tau_{yz}, \tau_{xz}$$

$$\boxed{\boldsymbol{\varepsilon}^{\mathrm{T}} = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{xz}\} \qquad \boldsymbol{\sigma}^{\mathrm{T}} = \{\sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{xz}\}}$$

Strain energy
Finite elements

- Interpolation functions: how displacement field is approximated

$$\boldsymbol{u} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \boldsymbol{S}(x,y,z)\,\bar{\boldsymbol{u}}$$

$\bar{\boldsymbol{u}} \equiv$ nodal displacements

$\boldsymbol{S}(x,y,z) \equiv$ interpolation (shape) functions

Example for quadrilateral elements:

$$\boldsymbol{u} = \boldsymbol{S}_1\,\bar{\boldsymbol{u}}_1 + \boldsymbol{S}_2\,\bar{\boldsymbol{u}}_2 + \boldsymbol{S}_3\,\bar{\boldsymbol{u}}_3 + \boldsymbol{S}_4\,\bar{\boldsymbol{u}}_4$$

- From the kinematic equations: $\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{u}$, the differentiation operator $\boldsymbol{B}$ will contain the proper derivatives of the interpolation functions corresponding to a given finite element

- The stress strain relation $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$ will highly depend on each case. For trusses and beam (uniaxial stress), it can be simply $\sigma_{xx} = E\varepsilon_{xx}$, for plates this becomes more complicated, as covered later

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Strain energy
Finite elements

$$\delta U = \int_\Omega \boldsymbol{\sigma}^\top \delta\boldsymbol{\varepsilon}\, d\Omega$$

Replacing $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$, for $\boldsymbol{C} = \boldsymbol{C}^\top$:

$$\delta U = \int_\Omega \boldsymbol{\varepsilon}^\top \boldsymbol{C}\, \delta\boldsymbol{\varepsilon}\, d\Omega$$

Replacing and $\boldsymbol{\varepsilon} = \boldsymbol{B}\bar{\boldsymbol{u}}$:

$$\delta U = \bar{\boldsymbol{u}}^T \int_\Omega \boldsymbol{B}^\top \boldsymbol{C}\, \boldsymbol{B}\, d\Omega\;\, \delta\bar{\boldsymbol{u}}$$

$$\delta U = \bar{\boldsymbol{u}}^\top \boldsymbol{K}\, \delta\bar{\boldsymbol{u}}$$

$$\boxed{\boldsymbol{K} = \int_\Omega \boldsymbol{B}^\top \boldsymbol{C}\, \boldsymbol{B}\, d\Omega}$$

Constitutive stiffness matrix

The rows and columns of $\boldsymbol{K}$ correspond to the degrees-of-freedom built by the assembly of all finite elements.

The integration over the 3-dimensional domain Ω is performed in a piece-wise manner within the domain of each finite element Ω_e.

$$\boldsymbol{K} = \sum_{e=1}^{n_e} \boldsymbol{K_e}$$

$$\boldsymbol{K_e} = \int_{\Omega_e} \boldsymbol{B}^\top \boldsymbol{C_e}\, \boldsymbol{B}\, d\Omega_\mathrm{e}$$

The integration of $\boldsymbol{K_e}$ can be efficiently done numerically due to the local support of the integration points (only affect the stiffness of the corresponding element).

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Strain energy
Energy-based methods, Ritz method

- Shape functions: how displacement field is approximated

$$\boldsymbol{u} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \boldsymbol{S}(x,y,z)\,\bar{\boldsymbol{c}}$$

$\bar{\boldsymbol{c}} \equiv$ amplitude of each term of the shape functions

$\boldsymbol{S}(x,y,z) \equiv$ shape functions

Example for deflection of simply supported plate $\xi = x/a,\ \ \eta = y/b$:

$$w(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} \sin i\pi\xi \sin j\pi\eta = \begin{bmatrix} \sin i\pi\xi \sin \pi\eta & \sin i\pi\xi \sin 2\pi\eta & \cdots \end{bmatrix} \begin{Bmatrix} c_{11} \\ c_{12} \\ \vdots \end{Bmatrix}$$

$$\bar{\boldsymbol{c}} = \begin{Bmatrix} c_{11} \\ c_{12} \\ \vdots \end{Bmatrix}$$

- From the kinematic equations: $\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{u}$, the differentiation operator $\boldsymbol{B}$ will contain the proper derivatives of the shape functions

Strain energy
Energy-based methods, Ritz method

$$\delta U = \int_\Omega \boldsymbol{\sigma}^\top \delta\boldsymbol{\varepsilon}\, d\Omega$$

Replacing $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$, for $\boldsymbol{C} = \boldsymbol{C}^\top$:

$$\delta U = \int_\Omega \boldsymbol{\varepsilon}^\top \boldsymbol{C}\, \delta\boldsymbol{\varepsilon}\, d\Omega$$

Replacing and $\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{c}$:

$$\delta U = \boldsymbol{c}^\top \int_\Omega \boldsymbol{B}^\top \boldsymbol{C}\, \boldsymbol{B}\, d\Omega\ \delta\boldsymbol{c}$$

$$\delta U = \boldsymbol{c}^\top \boldsymbol{K}\, \delta\boldsymbol{c}$$

$$\boxed{\boldsymbol{K} = \int_\Omega \boldsymbol{B}^\top \boldsymbol{C}\, \boldsymbol{B}\, d\Omega}$$

Constitutive stiffness matrix

The rows and columns of $\boldsymbol{K}$ correspond to the degrees-of-freedom that depend on the number of function terms used in the displacement approximation.

In the Ritz method, there is only one integration domain Ω. And the integration is usually performed analytically.

When numerical integration is needed, for instance due to variable stiffness or nonlinear analyses, the non-local support of the integration can create a large disadvantage of the Ritz method when compared to the finite element. The non-local support comes from the fact that the approximation functions represent the whole domain.

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Work of external forces

Work from external forces

$$\delta V = \int_{\Omega} \boldsymbol{\sigma}^{\top} \delta \boldsymbol{\varepsilon} \, d\Omega - \int_{\Omega} (\boldsymbol{b}^{\top} \delta \boldsymbol{u}) d\Omega - \int_{\delta\Omega} (\overline{\boldsymbol{\sigma}}^{\top} \delta \boldsymbol{u}) d(\delta\Omega) = 0$$

$$\delta W_{ext} = \int_{\Omega} (\boldsymbol{b}^{\top} \delta \boldsymbol{u}) d\Omega + \int_{\delta\Omega} (\overline{\boldsymbol{\sigma}}^{\top} \delta \boldsymbol{u}) d(\delta\Omega) = \boldsymbol{F}^{\top} \delta \boldsymbol{u}$$

$\boldsymbol{F} \equiv$ external force vector, including body ($\boldsymbol{b}$) and boundary forces ($\overline{\boldsymbol{\sigma}}$)

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Principle of Minimum Potential Energy

Back to the stationary total potential energy functional:

$$\delta V = \delta U - \delta W_{ext} = 0$$

the state of V depends on a nodal displacement vector $\bar{\boldsymbol{u}}$ in the case of displacement-based finite elements. In the Ritz method, we can make $\bar{\boldsymbol{c}} = \bar{\boldsymbol{u}}$ such that $\bar{\boldsymbol{u}}$ represents the Ritz coefficients $\bar{\boldsymbol{c}}$ that contain the amplitude of each term of the shape functions.

We can represent δV using Fréchet's derivatives. In the case of a linear analyses:

$$\delta V = {V'}^{\top} \delta \bar{\boldsymbol{u}} = 0$$

Thus, $V' = \boldsymbol{R}$, or a residual force vector, with V' being the first Fréchet's derivative of V. Using the definition for the total potential energy, the first variation becomes:

$$ {V'}^{\top} \delta \bar{\boldsymbol{u}} = \int_{\Omega} \boldsymbol{\sigma}^{\top} \delta \boldsymbol{\varepsilon} \, d\Omega - \int_{\Omega} \left(\hat{\boldsymbol{b}}^{\top} \delta \boldsymbol{u} \right) d\Omega - \int_{\delta\Omega} \left(\hat{\boldsymbol{\sigma}}^{\top} \delta \boldsymbol{u} \right) d(\delta\Omega) = 0$$

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Legendre Polynomials

Legendre Polynomials

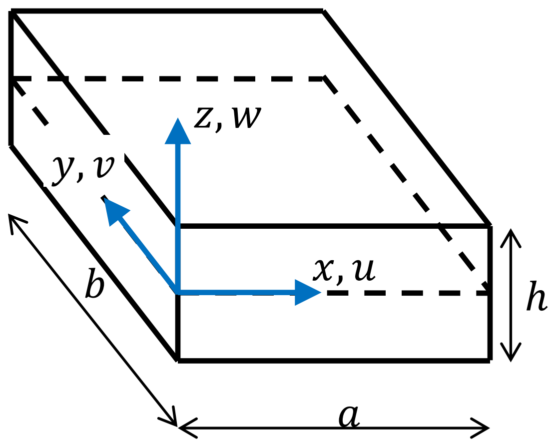
Shape functions: how displacement field is approximated

$$\boldsymbol{u} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \boldsymbol{S}(x,y,z)\,\bar{\boldsymbol{c}} = \begin{Bmatrix} \boldsymbol{S}^{\boldsymbol{u}}(x,y,z) \\ \boldsymbol{S}^{\boldsymbol{v}}(x,y,z) \\ \boldsymbol{S}^{\boldsymbol{w}}(x,y,z) \end{Bmatrix} \bar{\boldsymbol{c}}$$

using Legendre polynomials (summation convention for repeated indices):

$$\begin{aligned} u(x,y,z) &= c_{ijk}^{u} P_i(\xi) P_j(\eta) P_k(\zeta) \\ v(x,y,z) &= c_{ijk}^{v} P_i(\xi) P_j(\eta) P_k(\zeta) \\ w(x,y,z) &= c_{ijk}^{z} P_i(\xi) P_j(\eta) P_k(\zeta) \end{aligned}$$

with, for a rectangular plate:



$$\xi = \frac{2x}{a} - 1$$
$$\eta = \frac{2y}{b} - 1$$
$$\zeta = \frac{2z}{h} - 1$$

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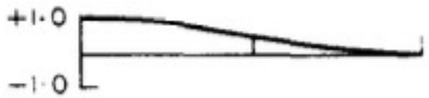
Legendre Polynomials, Rodrigues' form

The Rodrigues' form of the Legendre polynomials is [1c, 1d]:

$$P_1(\chi) = \left(\frac{1}{2} - \frac{3}{4}\chi + \frac{1}{4}\chi^3\right)\delta_{t1}$$



$$P_2(\chi) = \left(\frac{1}{8} - \frac{1}{8}\chi - \frac{1}{8}\chi^2 + \frac{1}{8}\chi^3\right)\delta_{r1}$$

$$P_3(\chi) = \left(\frac{1}{2} + \frac{3}{4}\chi - \frac{1}{4}\chi^3\right)\delta_{t2}$$

$$P_4(\chi) = \left(-\frac{1}{8} - \frac{1}{8}\chi + \frac{1}{8}\chi^2 + \frac{1}{8}\chi^3\right)\delta_{r2}$$

with $\chi \in \{\xi, \eta, \zeta\}$, and for any $i > 4$:

$$P_i(\chi) = \sum_{p=0}^{i/2} \frac{(-1)^p (2i-2p-7)!!}{2^p p!\,(i-2p-1)!} \chi^{i-2p-1}$$

$P_5(\chi) =$

$P_6(\chi) =$

$P_7(\chi) =$

$P_8(\chi) =$

where $q!! = q(q-2)\ldots(2 \text{ or } 1)$ such that $0!! = 1$, and $(i/2)$ in the summation is an integer division.

[1c] Castro S.G.P., Donadon, M.V., "Assembly of Semi-Analytical models to Address Linear Buckling and Vibration of Stiffened Composite Panels with Debonding Defect". Composite Structures, 2017. 10.1016/j.compstruct.2016.10.026
[1d] Bardell N.S., "Free vibration analysis of a flat plate using the hierarchical finite element method". Journal of Sound and Vibration, 1991. https://doi.org/10.1016/0022-460X(91)90855-E

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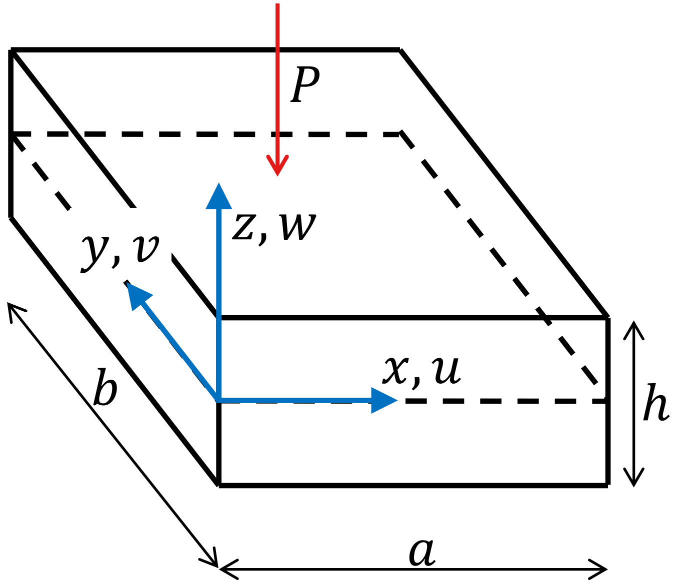
Examples CLPT, FSDT, TSDT

Deflection of a plate, 3D elasticity, van Kármán nonlinear terms



$$\varepsilon_{xx} = u_{,x} + \frac{1}{2} w_{,x}^2$$

$$\varepsilon_{yy} = v_{,y} + \frac{1}{2} w_{,y}^2$$

$$\varepsilon_{zz} = w_{,z} + \frac{1}{2} w_{,z}^2$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}\, w_{,y}$$
$$\gamma_{xz} = u_{,z} + w_{,x} + w_{,x}\, w_{,z}$$
$$\gamma_{yz} = v_{,z} + w_{,y} + w_{,y}\, w_{,z}$$

```
See:
ex/deflection_plate_3D_elasticity.ipynb
```

$u(x,y,z)$
$v(x,y,z)$
$w(x,y,z)$

Using the Ritz method:

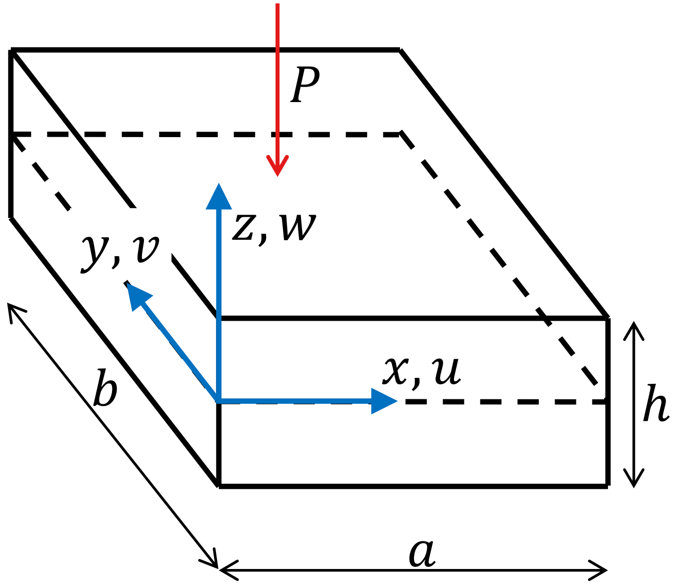
$u(x,y,z) = \boldsymbol{S}^u\,\overline{\boldsymbol{u}}$
$v(x,y,z) = \boldsymbol{S}^v\,\overline{\boldsymbol{u}}$
$w(x,y,z) = \boldsymbol{S}^w\,\overline{\boldsymbol{u}}$

The linear strains then become:

$\varepsilon_{xx} = \boldsymbol{S}^u_{,x}\,\overline{\boldsymbol{u}}$
$\varepsilon_{yy} = \boldsymbol{S}^v_{,y}\,\overline{\boldsymbol{u}}$
$\gamma_{xy} = \boldsymbol{S}^u_{,y} + \boldsymbol{S}^v_{,x}\,\overline{\boldsymbol{u}}$

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\,\overline{\boldsymbol{u}}$$

Deflection of a plate, 3D elasticity, van Kármán nonlinear terms



See: `ex/deflection_plate_3D_elasticity.ipynb`

The stiffness matrix becomes, using the 3D constitutive matrix $\boldsymbol{C}$.

$$\boldsymbol{K} = \iiint_{x,y,z} \boldsymbol{B}^\top \boldsymbol{C}\, \boldsymbol{B}\, dx\, dy\, dz$$

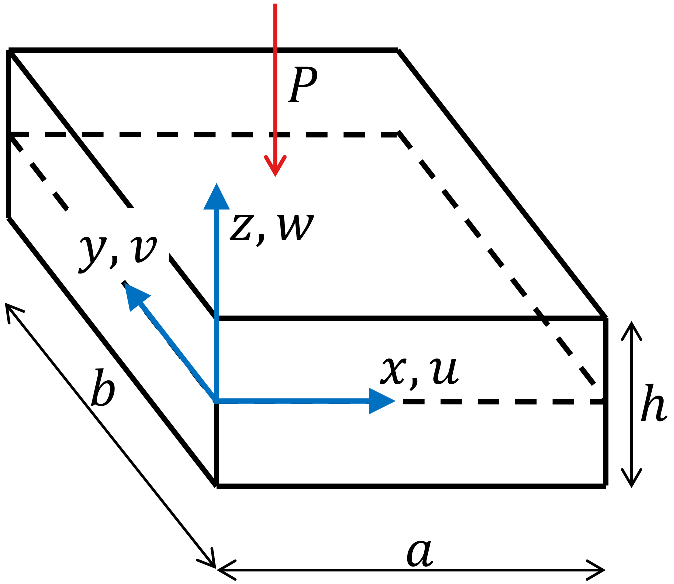
$$\boldsymbol{F_{ext}} = P\boldsymbol{S}^w \Big|_{\substack{x=a/2\\ y=b/2\\ z=+h/2}}$$

$$\overline{\boldsymbol{u}} = \boldsymbol{K}^{-1}\boldsymbol{F_{ext}}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\,\overline{\boldsymbol{u}}$$

Deflection of a plate, CLPT

See: ex/deflection_plate_CLPT.ipynb



$$\varepsilon_{xx} = u_{,x} - z\, w_{,xx} + \frac{1}{2} w_{,x}^2$$
$$\varepsilon_{yy} = v_{,y} - z\, w_{,yy} + \frac{1}{2} w_{,y}^2$$
$$\gamma_{xy} = u_{,y} + v_{,x} - 2z\, w_{,xy} + w_{,x}\, w_{,y}$$

$$u(x,y,z) = u_0(x,y) - z\, w_{,x}(x,y)$$
$$v(x,y,z) = v_0(x,y) - z\, w_{,y}(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

Using the Ritz method:

$$u(x,y,z) = \left(\boldsymbol{S^u} - z\boldsymbol{S^w_{,x}}\right)\bar{\boldsymbol{u}}$$
$$v(x,y,z) = \left(\boldsymbol{S^v} - z\boldsymbol{S^w_{,y}}\right)\bar{\boldsymbol{u}}$$
$$w(x,y) = \boldsymbol{S^w}\,\bar{\boldsymbol{u}}$$

The linear strains then become:

$$\varepsilon_{xx} = \left(\boldsymbol{S^u_{,x}} - z\boldsymbol{S^w_{,xx}}\right)\bar{\boldsymbol{u}}$$
$$\varepsilon_{yy} = \left(\boldsymbol{S^v_{,y}} - z\boldsymbol{S^w_{,yy}}\right)\bar{\boldsymbol{u}}$$
$$\gamma_{xy} = \left(\boldsymbol{S^u_{,y}} + \boldsymbol{S^v_{,x}} - 2z\boldsymbol{S^w_{,xy}}\right)\bar{\boldsymbol{u}}$$

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \left(\begin{bmatrix} \boldsymbol{S^u_{,x}} \\ \boldsymbol{S^v_{,y}} \\ \boldsymbol{S^u_{,y}} + \boldsymbol{S^v_{,x}} \end{bmatrix} + z \begin{bmatrix} -\boldsymbol{S^w_{,xx}} \\ -\boldsymbol{S^w_{,yy}} \\ -2\boldsymbol{S^w_{,xy}} \end{bmatrix} \right) \bar{\boldsymbol{u}}$$

$$\boldsymbol{\varepsilon} = \left(\boldsymbol{B_m} + z\boldsymbol{B_b}\right)\bar{\boldsymbol{u}}$$

Engineering Mechanics graduate school course

Stability of Structures

2025

Kinematics, Equivalent Single-Layer Theories, and Energy-based Semi-Analytical Methods

Saullo G. P. Castro

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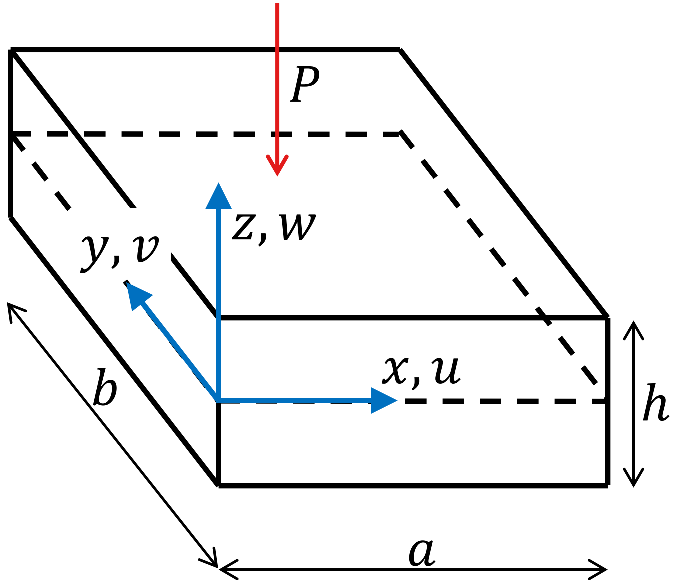
Deflection of a plate, CLPT

See: ex/deflection_plate_CLPT.ipynb



The stiffness matrix becomes, using the laminate constitutive matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$:

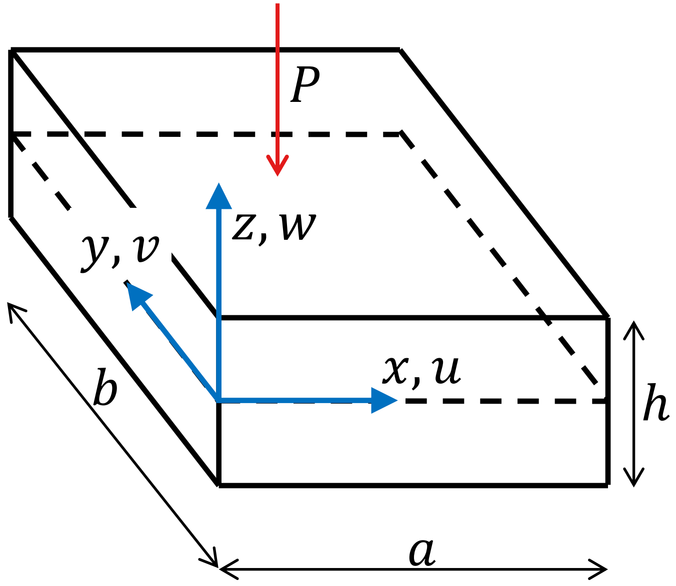
$$\boldsymbol{K} = \iint_{x,y} (\boldsymbol{B}_{\boldsymbol{m}}^{\top} \boldsymbol{A}\, \boldsymbol{B}_{\boldsymbol{m}}) + (\boldsymbol{B}_{\boldsymbol{b}}^{\top} \boldsymbol{D}\, \boldsymbol{B}_{\boldsymbol{b}}) + (\boldsymbol{B}_{\boldsymbol{m}}^{\top} \boldsymbol{B}\, \boldsymbol{B}_{\boldsymbol{b}}) + (\boldsymbol{B}_{\boldsymbol{b}}^{\top} \boldsymbol{B}\, \boldsymbol{B}_{\boldsymbol{m}}) dxdy$$

$$\boldsymbol{F}_{\boldsymbol{ext}} = P \boldsymbol{S}^{w} \Big|_{\substack{x=a/2 \\ y=b/2}}$$

$$\overline{\boldsymbol{u}} = \boldsymbol{K}^{-1} \boldsymbol{F}_{\boldsymbol{ext}}$$

Deflection of a plate, FSDT

See: ex/deflection_plate_FSDT.ipynb



$$\varepsilon_{xx} = u_{,x} + z\,\phi_{x,x} + \frac{1}{2} w_{,x}^2$$
$$\varepsilon_{yy} = v_{,y} + z\,\phi_{y,y} + \frac{1}{2} w_{,y}^2$$

$$\gamma_{xy} = u_{,y} + v_{,x} + z\phi_{x,y} + z\phi_{y,x} + w_{,x}\, w_{,y}$$

$$\gamma_{xz} = \phi_x + w_{,x}$$

$$\gamma_{yz} = \phi_y + w_{,y}$$

$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y)$$
$$w(x,y,z) = w(x,y)$$

Using the Ritz method:

$$u(x,y,z) = \left(\boldsymbol{S}^{u} + z\boldsymbol{S}^{\phi_x}\right)\bar{\boldsymbol{u}}$$
$$v(x,y,z) = \left(\boldsymbol{S}^{v} + z\boldsymbol{S}^{\phi_y}\right)\bar{\boldsymbol{u}}$$
$$w(x,y) = \boldsymbol{S}^{w}\,\bar{\boldsymbol{u}}$$

The linear strains then become:

$$\varepsilon_{xx} = \left(\boldsymbol{S}^{u}_{,x} + z\boldsymbol{S}^{\phi_x}_{,x}\right)\bar{\boldsymbol{u}}$$
$$\varepsilon_{yy} = \left(\boldsymbol{S}^{v}_{,y} + z\boldsymbol{S}^{\phi_y}_{,y}\right)\bar{\boldsymbol{u}}$$
$$\gamma_{xy} = \left(\boldsymbol{S}^{u}_{,y} + \boldsymbol{S}^{v}_{,x} + z\boldsymbol{S}^{\phi_x}_{,y} + z\boldsymbol{S}^{\phi_y}_{,x}\right)\bar{\boldsymbol{u}}$$
$$\gamma_{yz} = \left(\boldsymbol{S}^{\phi_y} + \boldsymbol{S}^{w}_{,y}\right)\bar{\boldsymbol{u}}$$
$$\gamma_{xz} = \left(\boldsymbol{S}^{\phi_x} + \boldsymbol{S}^{w}_{,x}\right)\bar{\boldsymbol{u}}$$

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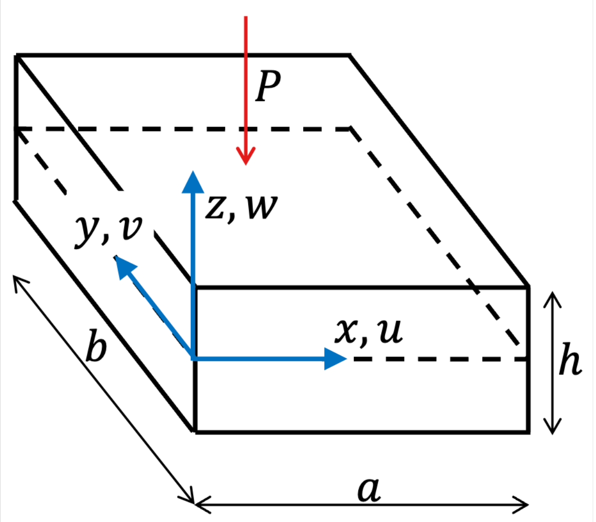
Deflection of a plate, FSDT

See: ex/deflection_plate_FSDT.ipynb



For the FSDT the variation of the strain energy is:

$$\delta U = \iint_{xy} \left(\boldsymbol{N}^\top \begin{Bmatrix} \delta\varepsilon_{xx}^{(0)} \\ \delta\varepsilon_{yy}^{(0)} \\ \delta\gamma_{xy}^{(0)} \end{Bmatrix} + \boldsymbol{M}^\top \begin{Bmatrix} \delta\varepsilon_{xx}^{(1)} \\ \delta\varepsilon_{yy}^{(1)} \\ \delta\gamma_{xy}^{(1)} \end{Bmatrix} + \boldsymbol{Q}^\top \begin{Bmatrix} \delta\gamma_{yz}^{(0)} \\ \delta\gamma_{xz}^{(0)} \end{Bmatrix} \right) dxdy$$

Such that the stiffness matrix becomes, using the laminate constitutive matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$:

$$\boldsymbol{B}^N = \boldsymbol{A} \begin{Bmatrix} \boldsymbol{B}^{\varepsilon_{xx}^{(0)}} \\ \boldsymbol{B}^{\varepsilon_{yy}^{(0)}} \\ \boldsymbol{B}^{\gamma_{xy}^{(0)}} \end{Bmatrix} + \boldsymbol{B} \begin{Bmatrix} \boldsymbol{B}^{\varepsilon_{xx}^{(1)}} \\ \boldsymbol{B}^{\varepsilon_{yy}^{(1)}} \\ \boldsymbol{B}^{\gamma_{xy}^{(1)}} \end{Bmatrix}$$

$$K = \iint_{xy} \left(\boldsymbol{B^N}^\top \begin{Bmatrix} \boldsymbol{B}^{\varepsilon_{xx}^{(0)}} \\ \boldsymbol{B}^{\varepsilon_{yy}^{(0)}} \\ \boldsymbol{B}^{\gamma_{xy}^{(0)}} \end{Bmatrix} + \boldsymbol{B^M}^\top \begin{Bmatrix} \boldsymbol{B}^{\varepsilon_{xx}^{(1)}} \\ \boldsymbol{B}^{\varepsilon_{yy}^{(1)}} \\ \boldsymbol{B}^{\gamma_{xy}^{(1)}} \end{Bmatrix} + \boldsymbol{B^Q}^\top \begin{Bmatrix} \boldsymbol{B}^{\gamma_{yz}^{(0)}} \\ \boldsymbol{B}^{\gamma_{xz}^{(0)}} \end{Bmatrix} \right) dxdy$$

$$\boldsymbol{B}^M = \boldsymbol{B} \begin{Bmatrix} \boldsymbol{B}^{\varepsilon_{xx}^{(0)}} \\ \boldsymbol{B}^{\varepsilon_{yy}^{(0)}} \\ \boldsymbol{B}^{\gamma_{xy}^{(0)}} \end{Bmatrix} + \boldsymbol{D} \begin{Bmatrix} \boldsymbol{B}^{\varepsilon_{xx}^{(1)}} \\ \boldsymbol{B}^{\varepsilon_{yy}^{(1)}} \\ \boldsymbol{B}^{\gamma_{xy}^{(1)}} \end{Bmatrix}$$

$$\boldsymbol{F_{ext}} = P\boldsymbol{S}^w \Big|_{\substack{x=a/2 \\ y=b/2}}$$

$$\boldsymbol{B}^Q = \boldsymbol{A} \begin{Bmatrix} \boldsymbol{B}^{\gamma_{yz}^{(0)}} \\ \boldsymbol{B}^{\gamma_{xz}^{(0)}} \end{Bmatrix}$$

$$\bar{\boldsymbol{u}} = \boldsymbol{K}^{-1}\boldsymbol{F_{ext}}$$

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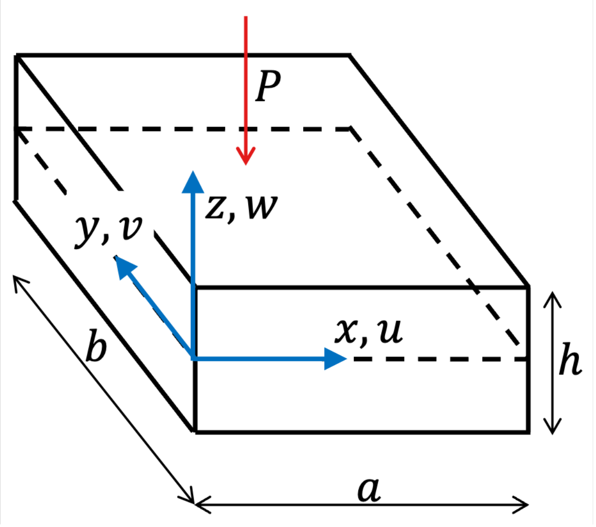
Deflection of a plate, TSDT

See: ex/deflection_plate_TSDT.ipynb



$$u(x,y,z) = u_0(x,y) + z\,\phi_x(x,y) - \frac{4}{3h^2} z^3\left(\phi_x(x,y) + w_{,x}(x,y)\right)$$
$$v(x,y,z) = v_0(x,y) + z\,\phi_y(x,y) - \frac{4}{3h^2} z^3\left(\phi_y(x,y) + w_{,y}(x,y)\right)$$
$$w(x,y,z) = w(x,y)$$

Using the Ritz method:

$$u(x,y,z) = \left(\boldsymbol{S^u} + z\boldsymbol{S^{\phi_x}} + z^3\left(-\frac{4}{3h^2}\right)\left(\boldsymbol{S^{\phi_x}} + \boldsymbol{S^w_{,x}}\right)\right)\bar{\boldsymbol{u}}$$
$$v(x,y,z) = \left(\boldsymbol{S^v} + z\boldsymbol{S^{\phi_y}} + z^3\left(-\frac{4}{3h^2}\right)\left(\boldsymbol{S^{\phi_y}} + \boldsymbol{S^w_{,y}}\right)\right)\bar{\boldsymbol{u}}$$
$$w(x,y) = \boldsymbol{S^w}\,\bar{\boldsymbol{u}}$$

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2}w_{,x}^2 + z\phi_{x,x} + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_{x,x} + w_{,xx}\right)$$
$$\varepsilon_{yy} = v_{,y} + \frac{1}{2}w_{,y}^2 + z\phi_{y,y} + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_{y,y} + w_{,yy}\right)$$

$$\tau_{xy} = u_{,y} + v_{,x} + w_{,x}\,w_{,y} + z\phi_{x,y} + z\phi_{y,x} + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_{x,y} + \phi_{y,x} + 2w_{,xy}\right)$$

$$\tau_{xz} = \phi_x + w_{,x} + z^2\left(-\frac{4}{h^2}\right)\left(\phi_x + w_{,x}\right)$$

$$\tau_{yz} = \phi_y + w_{,y} + z^2\left(-\frac{4}{h^2}\right)\left(\phi_y + w_{,y}\right)$$

The linear strains then become:

$$\varepsilon_{xx} = \left(\boldsymbol{S^u_{,x}} + z\boldsymbol{S^{\phi_x}_{,x}} + z^3\left(-\frac{4}{3h^2}\right)\left(\boldsymbol{S^{\phi_x}_{,x}} + \boldsymbol{S^w_{,xx}}\right)\right)\bar{\boldsymbol{u}}$$
$$\varepsilon_{yy} = \left(\boldsymbol{S^v_{,y}} + z\boldsymbol{S^{\phi_y}_{,y}} + z^3\left(-\frac{4}{3h^2}\right)\left(\boldsymbol{S^{\phi_y}_{,y}} + \boldsymbol{S^w_{,yy}}\right)\right)\bar{\boldsymbol{u}}$$
$$\gamma_{xy} = \left(\boldsymbol{S^u_{,y}} + \boldsymbol{S^v_{,x}} + z\boldsymbol{S^{\phi_x}_{,y}} + z\boldsymbol{S^{\phi_y}_{,x}} + z^3\left(-\frac{4}{3h^2}\right)\left(\boldsymbol{S^{\phi_x}_{,y}} + \boldsymbol{S^{\phi_y}_{,x}} + 2\boldsymbol{S^w_{,xy}}\right)\right)\bar{\boldsymbol{u}}$$
$$\gamma_{yz} = \left(\boldsymbol{S^{\phi_y}} + \boldsymbol{S^w_{,y}} + z^2\left(-\frac{4}{h^2}\right)\left(\boldsymbol{S^{\phi_y}} + \boldsymbol{S^w_{,y}}\right)\right)\bar{\boldsymbol{u}}$$
$$\gamma_{xz} = \left(\boldsymbol{S^{\phi_x}} + \boldsymbol{S^w_{,x}} + z^2\left(-\frac{4}{h^2}\right)\left(\boldsymbol{S^{\phi_x}} + \boldsymbol{S^w_{,x}}\right)\right)\bar{\boldsymbol{u}}$$

Deflection of a plate, TSDT

See: `ex/deflection_plate_TSDT.ipynb`

$$\boldsymbol{B^N} = \boldsymbol{A}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(0)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(0)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(0)}}\end{Bmatrix} + \boldsymbol{B}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(1)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(1)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(1)}}\end{Bmatrix} + \boldsymbol{E}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(3)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(3)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(3)}}\end{Bmatrix}$$

$$\boldsymbol{B^M} = \boldsymbol{B}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(0)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(0)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(0)}}\end{Bmatrix} + \boldsymbol{D}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(1)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(1)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(1)}}\end{Bmatrix} + \boldsymbol{F}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(3)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(3)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(3)}}\end{Bmatrix}$$

$$\boldsymbol{B^P} = \boldsymbol{E}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(0)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(0)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(0)}}\end{Bmatrix} + \boldsymbol{F}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(1)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(1)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(1)}}\end{Bmatrix} + \boldsymbol{H}\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(3)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(3)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(3)}}\end{Bmatrix}$$

$$\boldsymbol{B^Q} = \boldsymbol{A}\begin{Bmatrix}\boldsymbol{B}^{\gamma_{yz}^{(0)}}\\ \boldsymbol{B}^{\gamma_{xz}^{(0)}}\end{Bmatrix} + \boldsymbol{D}\begin{Bmatrix}\boldsymbol{B}^{\gamma_{yz}^{(2)}}\\ \boldsymbol{B}^{\gamma_{xz}^{(2)}}\end{Bmatrix}$$

$$\boldsymbol{B^R} = \boldsymbol{D}\begin{Bmatrix}\boldsymbol{B}^{\gamma_{yz}^{(0)}}\\ \boldsymbol{B}^{\gamma_{xz}^{(0)}}\end{Bmatrix} + \boldsymbol{F}\begin{Bmatrix}\boldsymbol{B}^{\gamma_{yz}^{(2)}}\\ \boldsymbol{B}^{\gamma_{xz}^{(2)}}\end{Bmatrix}$$

For the TSDT the variation of the strain energy is:

$$\delta U = \iint_{xy}\left(\boldsymbol{N}^\top\begin{Bmatrix}\delta\varepsilon_{xx}^{(0)}\\ \delta\varepsilon_{yy}^{(0)}\\ \delta\gamma_{xy}^{(0)}\end{Bmatrix} + \boldsymbol{M}^\top\begin{Bmatrix}\delta\varepsilon_{xx}^{(1)}\\ \delta\varepsilon_{yy}^{(1)}\\ \delta\gamma_{xy}^{(1)}\end{Bmatrix} + \boldsymbol{P}^\top\begin{Bmatrix}\delta\varepsilon_{xx}^{(3)}\\ \delta\varepsilon_{yy}^{(3)}\\ \delta\gamma_{xy}^{(3)}\end{Bmatrix}\right.$$
$$\left. + \boldsymbol{Q}^\top\begin{Bmatrix}\delta\gamma_{yz}^{(0)}\\ \delta\gamma_{xz}^{(0)}\end{Bmatrix} + \boldsymbol{R}^\top\begin{Bmatrix}\delta\gamma_{yz}^{(2)}\\ \delta\gamma_{xz}^{(2)}\end{Bmatrix}\right) dxdy$$

Such that the stiffness matrix becomes, using the laminate constitutive matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}$:

$$K = \iint_{xy}\left(\boldsymbol{B^N}^\top\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(0)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(0)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(0)}}\end{Bmatrix} + \boldsymbol{B^M}^\top\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(1)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(1)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(1)}}\end{Bmatrix} + \boldsymbol{B^P}^\top\begin{Bmatrix}\boldsymbol{B}^{\varepsilon_{xx}^{(3)}}\\ \boldsymbol{B}^{\varepsilon_{yy}^{(3)}}\\ \boldsymbol{B}^{\gamma_{xy}^{(3)}}\end{Bmatrix}\right.$$
$$\left. + \boldsymbol{B^Q}^\top\begin{Bmatrix}\boldsymbol{B}^{\gamma_{yz}^{(0)}}\\ \boldsymbol{B}^{\gamma_{xz}^{(0)}}\end{Bmatrix} + \boldsymbol{B^R}^\top\begin{Bmatrix}\boldsymbol{B}^{\gamma_{yz}^{(2)}}\\ \boldsymbol{B}^{\gamma_{xz}^{(2)}}\end{Bmatrix}\right) dxdy$$

$$\boldsymbol{F_{ext}} = P\boldsymbol{S}^w\Big|_{\substack{x=a/2\\ y=b/2}}$$

$$\bar{\boldsymbol{u}} = \boldsymbol{K}^{-1}\boldsymbol{F_{ext}}$$

Saullo G. P. Castro

Neutral Equilibrium Criterion

Engineering Mechanics graduate school course

Stability of Structures

2025

Kinematics, Equivalent Single-Layer Theories, and Energy-based Semi-Analytical Methods

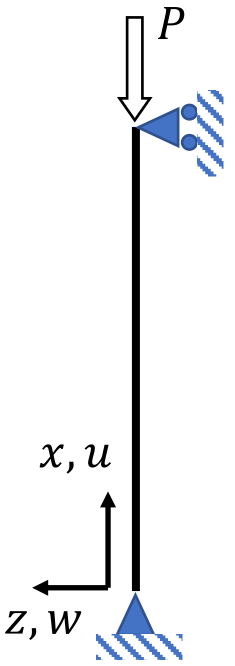
Saullo G. P. Castro

TU Delft

Neutral Equilibrium Criterion

Equilibrium
$\delta V = 0$

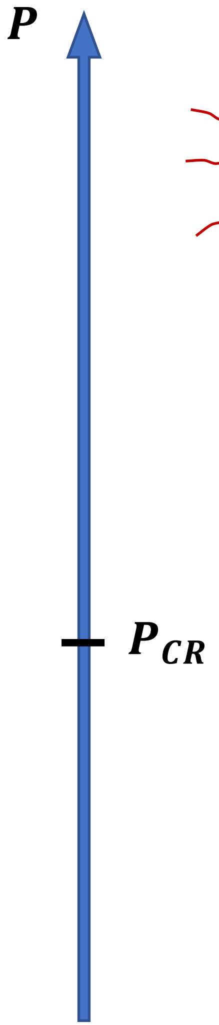
$$\varepsilon_{xx} = u_{,x} + \frac{1}{2} w_{,x}^2$$

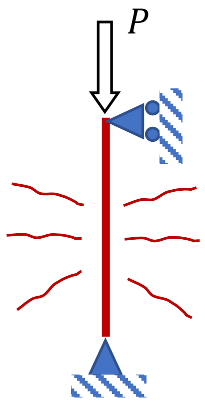


Geometric nonlinearity:
axial strains ⇔ beam rotation

Negative ε_{xx} ⇒ Reduced bending stiffness

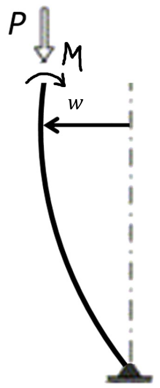
Positive ε_{xx} ⇒ Increased bending stiffness





Unstable

$\delta^2 V < 0$



Bifurcation

Neutral Equilibrium Criterion

$\delta^2 V = 0$

Stable

$\delta^2 V > 0$

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Neutral Equilibrium Criterion

Given the expression for V':

$$V'^{\top}\delta\bar{\boldsymbol{u}} = \int_{\Omega} \boldsymbol{\sigma}^{\top}\delta\boldsymbol{\varepsilon}\, d\Omega - \int_{\Omega} (\hat{\boldsymbol{b}}^{\top}\delta\boldsymbol{u}) d\Omega - \int_{\delta\Omega} (\hat{\boldsymbol{\sigma}}^{\top}\delta\boldsymbol{u}) d(\delta\Omega)$$

The second variation of V becomes:

$$\delta^2 V = V''\delta\bar{\boldsymbol{u}}\delta\bar{\boldsymbol{u}} =$$
$$= \int_{\Omega} \delta\boldsymbol{\sigma}^{\top}\delta\boldsymbol{\varepsilon}\, d\Omega + \int_{\Omega} \boldsymbol{\sigma}^{\top}\delta^2\boldsymbol{\varepsilon}\, d\Omega - \int_{\Omega} (\delta\hat{\boldsymbol{b}}^{\top}\delta\boldsymbol{u}) d\Omega - \int_{\Omega} (\hat{\boldsymbol{b}}^{\top}\delta^2\boldsymbol{u}) d\Omega - \int_{\delta\Omega} (\delta\hat{\boldsymbol{\sigma}}^{\top}\delta\boldsymbol{u}) d(\delta\Omega) - \int_{\delta\Omega} (\hat{\boldsymbol{\sigma}}^{\top}\delta^2\boldsymbol{u}) d(\delta\Omega)$$

In a system without follower forces $\delta\hat{\boldsymbol{\sigma}} = \delta\hat{\boldsymbol{b}} = \boldsymbol{0}$, and we can consider $\delta^2\boldsymbol{u} \ll \delta\boldsymbol{u}$:

$$V''\delta\bar{\boldsymbol{u}}\delta\bar{\boldsymbol{u}} = \int_{\Omega} \delta\boldsymbol{\sigma}^{\top}\delta\boldsymbol{\varepsilon}\, d\Omega + \int_{\Omega} \boldsymbol{\sigma}^{\top}\delta^2\boldsymbol{\varepsilon}\, d\Omega$$

The first integral represents the nonlinear constitutive stiffness, whereas the second integral represents the geometric stiffness matrix. If the strains can be represented as:

$$\boldsymbol{\varepsilon} = \left(\boldsymbol{B_L} + \frac{1}{2}\boldsymbol{B_{NL}}\right)\bar{\boldsymbol{u}} \qquad \text{and} \qquad \delta\boldsymbol{\varepsilon} = (\boldsymbol{B_L} + \boldsymbol{B_{NL}})\delta\bar{\boldsymbol{u}}$$

And the stresses as:

$$\boldsymbol{\sigma} = \boldsymbol{C}\,\boldsymbol{\varepsilon}$$

Then:

$$V''\delta\bar{\boldsymbol{u}}\delta\bar{\boldsymbol{u}} = \delta\bar{\boldsymbol{u}}^{\top} \int_{\Omega} (\boldsymbol{B_L} + \boldsymbol{B_{NL}})^{\top}\boldsymbol{C}(\boldsymbol{B_L} + \boldsymbol{B_{NL}}) d\Omega\, \delta\bar{\boldsymbol{u}} + \int_{\Omega} \boldsymbol{\sigma}^{\top}\delta^2\boldsymbol{\varepsilon}\, d\Omega$$

Geometric Stiffness Matrix for beams

Using the full nonlinear Green-Lagrange strain relation:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

The first variation becomes, using $\partial(\cdot)/\partial x = (.)_{,x}$

$$\delta\varepsilon_{xx} = \delta u_{,x} + u_{,x}\,\delta u_{,x} + v_{,x}\,\delta v_{,x} + w_{,x}\,\delta w_{,x}$$

In terms of nodal displacements (or Ritz coefficients) $\bar{\boldsymbol{u}}$:

$$\delta\varepsilon_{xx} = \bar{\boldsymbol{B}}\delta\bar{\boldsymbol{u}} = (\boldsymbol{B_L} + \boldsymbol{B_{NL}})\delta\bar{\boldsymbol{u}}$$

with:

$$\boldsymbol{B_L} = \frac{\partial \boldsymbol{S}^u}{\partial x} \qquad \boldsymbol{B_{NL}} = \frac{\partial u}{\partial x}\frac{\partial \boldsymbol{S}^u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial \boldsymbol{S}^v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \boldsymbol{S}^w}{\partial x}$$

The second variation becomes:

$$\delta^2\varepsilon_{xx} = \delta^2 u_{,x} + \delta u_{,x}\,\delta u_{,x} + u_{,x}\,\delta^2 u_{,x} + \delta v_{,x}\,\delta v_{,x} + v_{,x}\,\delta^2 v_{,x} + \delta w_{,x}\,\delta w_{,x} + w_{,x}\,\delta^2 w_{,x}$$

Note that $\delta^2(\cdot)_{,x} \ll \delta(\cdot)_{,x}$, leading to:

$$\delta^2\varepsilon_{xx} = \delta u_{,x}\,\delta u_{,x} + \delta v_{,x}\,\delta v_{,x} + \delta w_{,x}\,\delta w_{,x}$$

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Geometric Stiffness Matrix for beams

With $\delta^2\varepsilon_{xx}$ defined as:

$$\delta^2\varepsilon_{xx} = \delta u_{,x}\,\delta u_{,x} + \delta v_{,x}\,\delta v_{,x} + \delta w_{,x}\,\delta w_{,x}$$

using the finite element (or Ritz method) shape functions:

$$\delta^2\varepsilon_{xx} = \delta\bar{\boldsymbol{u}}^\top\left[\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right)^\top\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right) + \left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right)^\top\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right) + \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^\top\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)\right]\delta\bar{\boldsymbol{u}}$$

Then:

$$\delta\bar{\boldsymbol{u}}^\top\boldsymbol{K}_{\boldsymbol{G}}\delta\bar{\boldsymbol{u}} = \int_\Omega \hat{\sigma}_{xx}\delta^2\varepsilon_{xx}\,d\Omega = \delta\bar{\boldsymbol{u}}^\top \underbrace{\int_\Omega \hat{\sigma}_{xx}\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right)^\top\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right) + \hat{\sigma}_{xx}\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right)^\top\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right) + \hat{\sigma}_{xx}\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^\top\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) d\Omega}\,\delta\bar{\boldsymbol{u}}$$

$\boldsymbol{K}_{\boldsymbol{G}} \equiv$ geometric stiffness matrix
$\hat{\sigma}_{xx} \equiv$ known pre-stress level

$\hat{\sigma}_{xx}$ can be given or calculated from an initial nodal displacement state $\hat{\boldsymbol{u}}$ (pre-buckling of fundamental state):

$$\hat{\sigma}_{xx} = E\left(\boldsymbol{B}_{\boldsymbol{L}} + \frac{1}{2}\boldsymbol{B}_{\boldsymbol{NL}}\right)\hat{\boldsymbol{u}}$$

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Geometric Stiffness Matrix for plates (full nonlinear kinematics)

Using the full nonlinear Green-Lagrange strain relation:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

The first variation becomes, using $\partial(\cdot)/\partial x = (.)_{,x}$

$$\delta\varepsilon_{xx} = \delta u_{,x} + u_{,x}\,\delta u_{,x} + v_{,x}\,\delta v_{,x} + w_{,x}\,\delta w_{,x}$$

$$\delta\varepsilon_{yy} = \delta v_{,y} + u_{,y}\,\delta u_{,y} + v_{,y}\,\delta v_{,y} + w_{,y}\,\delta w_{,y}$$

$$\delta\gamma_{xy} = \delta u_{,y} + \delta v_{,x} + \delta u_{,x}\,u_{,y} + u_{,x}\,\delta u_{,y} + \delta v_{,x}\,v_{,y} + v_{,x}\,\delta v_{,y} + \delta w_{,x}\,w_{,y} + w_{,x}\,\delta w_{,y}$$

The second variation becomes:

$$\delta^2\varepsilon_{xx} = \delta^2 u_{,x} + \delta u_{,x}\,\delta u_{,x} + u_{,x}\,\delta^2 u_{,x} + \delta v_{,x}\,\delta v_{,x} + v_{,x}\,\delta^2 v_{,x} + \delta w_{,x}\,\delta w_{,x} + w_{,x}\,\delta^2 w_{,x}$$

$$\delta^2\varepsilon_{yy} = \delta^2 v_{,y} + \delta u_{,y}\,\delta u_{,y} + u_{,y}\,\delta^2 u_{,y} + \delta v_{,y}\,\delta v_{,y} + v_{,y}\,\delta^2 v_{,y} + \delta w_{,y}\,\delta w_{,y} + w_{,y}\,\delta^2 w_{,y}$$

$$\delta^2\gamma_{xy} = \delta^2 u_{,y} + \delta^2 v_{,x} + \delta^2 u_{,x}\,u_{,y} + 2\delta u_{,x}\,\delta u_{,y} + u_{,x}\,\delta^2 u_{,y} + \delta^2 v_{,x}\,v_{,y} + 2\delta v_{,x}\,\delta v_{,y} + v_{,x}\,\delta^2 v_{,y} + \delta^2 w_{,x}\,w_{,y} + 2\delta w_{,x}\,\delta w_{,y} + w_{,x}\,\delta^2 w_{,y}$$

Note that $\delta^2(\cdot)_{,x} \ll \delta(\cdot)_{,x}$ and $\delta^2(\cdot)_{,y} \ll \delta(\cdot)_{,y}$, leading to:

$$\delta^2\varepsilon_{xx} = \delta u_{,x}\,\delta u_{,x} + \delta v_{,x}\,\delta v_{,x} + \delta w_{,x}\,\delta w_{,x}$$

$$\delta^2\varepsilon_{yy} = \delta u_{,y}\,\delta u_{,y} + \delta v_{,y}\,\delta v_{,y} + \delta w_{,y}\,\delta w_{,y}$$

$$\delta^2\gamma_{xy} = 2\delta u_{,x}\,\delta u_{,y} + 2\delta v_{,x}\,\delta v_{,y} + 2\delta w_{,x}\,\delta w_{,y}$$

Geometric Stiffness Matrix for plates (full nonlinear kinematics)

With $\delta^2\boldsymbol{\varepsilon}$ defined as:

$$\delta^2\varepsilon_{xx} = \delta u_{,x}\,\delta u_{,x} + \delta v_{,x}\,\delta v_{,x} + \delta w_{,x}\,\delta w_{,x}$$
$$\delta^2\varepsilon_{yy} = \delta u_{,y}\,\delta u_{,y} + \delta v_{,y}\,\delta v_{,y} + \delta w_{,y}\,\delta w_{,y}$$
$$\delta^2\gamma_{xy} = 2\delta u_{,x}\,\delta u_{,y} + 2\delta v_{,x}\,\delta v_{,y} + 2\delta w_{,x}\,\delta w_{,y}$$

using the finite element shape functions:

$$\delta^2\varepsilon_{xx} = \delta\bar{\boldsymbol{u}}^{\top}\left[\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right) + \left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right) + \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)\right]\delta\bar{\boldsymbol{u}}$$

$$\delta^2\varepsilon_{yy} = \delta\bar{\boldsymbol{u}}^{\top}\left[\left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right) + \left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right) + \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)\right]\delta\bar{\boldsymbol{u}}$$

$$\delta^2\gamma_{xy} = \delta\bar{\boldsymbol{u}}^{\top}\left[\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right) + \left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right) + \left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right) + \left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right) + \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right) + \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)\right]\delta\bar{\boldsymbol{u}}$$

Then:

$$\delta\bar{\boldsymbol{u}}^{\top}\boldsymbol{K_G}\delta\bar{\boldsymbol{u}} = \delta\bar{\boldsymbol{u}}^{\top}\int_{\Omega}\hat{\sigma}_{xx}\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right) + \hat{\sigma}_{xx}\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right) + \hat{\sigma}_{xx}\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)$$
$$+\hat{\sigma}_{yy}\left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right) + \hat{\sigma}_{yy}\left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right) + \hat{\sigma}_{yy}\left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)$$
$$\hat{\tau}_{xy}\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right) + \hat{\tau}_{xy}\left(\frac{\partial \boldsymbol{S}^u}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^u}{\partial x}\right) + \hat{\tau}_{xy}\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right) + \hat{\tau}_{xy}\left(\frac{\partial \boldsymbol{S}^v}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^v}{\partial x}\right) + \hat{\tau}_{xy}\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right) + \hat{\tau}_{xy}\left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^{\top}\left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) d\Omega \quad \delta\bar{\boldsymbol{u}}$$

$\hat{\sigma}_{xx}, \hat{\sigma}_{yy}, \hat{\tau}_{xy}$ can be calculated for plate elements from an initial nodal displacement state $\hat{\boldsymbol{u}}$ (pre-buckling of fundamental state):

$$\begin{Bmatrix}\hat{\sigma}_{xx}\\ \hat{\sigma}_{yy}\\ \hat{\tau}_{xy}\end{Bmatrix} = \frac{1}{h}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{B} & \boldsymbol{D}\end{bmatrix}\left(\boldsymbol{B_L} + \frac{1}{2}\boldsymbol{B_{NL}}\right)\hat{\boldsymbol{u}}$$

Geometric Stiffness Matrix for plates (van Kármán)

applying the van Kármán simplifications, $\delta^2 \boldsymbol{\varepsilon}$ is defined as:

$$\delta^2 \varepsilon_{xx} = \delta w_{,x} \, \delta w_{,x}$$
$$\delta^2 \varepsilon_{yy} = \delta w_{,y} \, \delta w_{,y}$$
$$\delta^2 \gamma_{xy} = 2 \delta w_{,x} \, \delta w_{,y}$$

using the finite element shape functions:

$$\delta^2 \varepsilon_{xx} = \delta \overline{\boldsymbol{u}}^\top \left[\left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right) \right] \delta \overline{\boldsymbol{u}}$$

$$\delta^2 \varepsilon_{yy} = \delta \overline{\boldsymbol{u}}^\top \left[\left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right) \right] \delta \overline{\boldsymbol{u}}$$

$$\delta^2 \gamma_{xy} = \delta \overline{\boldsymbol{u}}^\top \left[\left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right) + \left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right) \right] \delta \overline{\boldsymbol{u}}$$

Then:

$$\begin{aligned} \delta \overline{\boldsymbol{u}}^\top \boldsymbol{K_G} \delta \overline{\boldsymbol{u}} = \delta \overline{\boldsymbol{u}}^\top \int_\Omega & \hat{\sigma}_{xx} \left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right) \\ & + \hat{\sigma}_{yy} \left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right) \\ & + \hat{\tau}_{xy} \left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right) + \hat{\tau}_{xy} \left(\frac{\partial \boldsymbol{S}^w}{\partial y} \right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial x} \right) d\Omega \;\; \delta \overline{\boldsymbol{u}} \end{aligned}$$

$\hat{\sigma}_{xx}, \hat{\sigma}_{yy}, \hat{\tau}_{xy}$ can be calculated for plate elements from an initial nodal displacement state $\widehat{\boldsymbol{u}}$:

$$\begin{Bmatrix} \hat{\sigma}_{xx} \\ \hat{\sigma}_{yy} \\ \hat{\tau}_{xy} \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \left(\boldsymbol{B_L} + \frac{1}{2} \boldsymbol{B_{NL}} \right) \widehat{\boldsymbol{u}}$$

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Neutral Equilibrium Criterion, Linear Buckling Equation

The neutral equilibrium criterion states that:

$$\delta^2 V = 0$$

$$\delta^2 V = V'' \delta\bar{\boldsymbol{u}} \delta\bar{\boldsymbol{u}} = \delta\bar{\boldsymbol{u}}^\top \int_\Omega (\boldsymbol{B_L} + \boldsymbol{B_{NL}})^\top \boldsymbol{C} (\boldsymbol{B_L} + \boldsymbol{B_{NL}}) d\Omega \, \delta\bar{\boldsymbol{u}} + \delta\bar{\boldsymbol{u}}^\top \boldsymbol{K_G} \delta\bar{\boldsymbol{u}} = 0$$

This term represents the constitutive stiffness matrix:

$$\boldsymbol{K_C} = \int_\Omega (\boldsymbol{B_L} + \boldsymbol{B_{NL}})^\top \boldsymbol{C} (\boldsymbol{B_L} + \boldsymbol{B_{NL}}) d\Omega$$

From a linear fundamental (pre-buckling) state we have that $\boldsymbol{B_{NL}} = \boldsymbol{0}$:

$$\boldsymbol{K_C} = K_0 = \int_\Omega \boldsymbol{B_L}^\top \, \boldsymbol{C} \, \boldsymbol{B_L} d\Omega$$

such that:

$$\delta^2 V = \delta\bar{\boldsymbol{u}}^\top \boldsymbol{K_0} \delta\bar{\boldsymbol{u}} + \delta\bar{\boldsymbol{u}}^\top \boldsymbol{K_G} \delta\bar{\boldsymbol{u}} = 0$$

which must be true for any variation $\delta\bar{\boldsymbol{u}}$, such that $\boldsymbol{K_0} + \lambda \boldsymbol{K_G}$ must be singular for this to be true:

$$\det(\boldsymbol{K_0} + \lambda \boldsymbol{K_G}) = 0$$

where λ is a load multiplier applied to the initial stress state defining $\boldsymbol{K_G}$.

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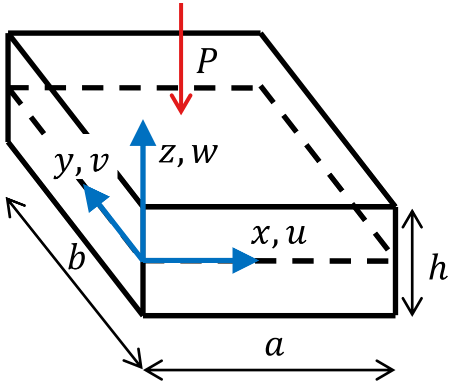
Examples CLPT, FSDT, TSDT

Buckling of a plate, full 3D elasticity



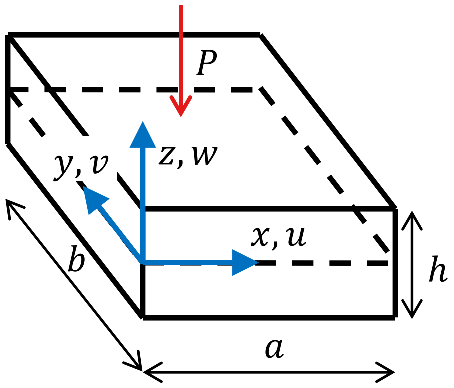
For the 3D elasticity case, the following expression can be used to calculate the geometric stiffness matrix for plates, using van Kármán kinematics:

$$\boldsymbol{K_G} = \iiint_{x,y,z} \hat{\sigma}_{xx} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) + \hat{\sigma}_{yy} \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right) + \hat{\sigma}_{xy} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right) + \hat{\sigma}_{xy} \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^\top \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) \; dxdydz$$

See:

`ex/buckling_plate_3D_elasticity.ipynb`

Buckling of a plate, CLPT, FSDT, TSDT



For all 3 ESL theories previously discussed, the following expression can be used to calculate the geometric stiffness matrix for plates, using van Kármán kinematics:

$$\boldsymbol{K_G} = \iint_{x,y} \widehat{N}_{xx} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^\mathsf{T} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) + \widehat{N}_{yy} \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^\mathsf{T} \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right) + \widehat{N}_{xy} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^\mathsf{T} \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right) + \widehat{N}_{xy} \left(\frac{\partial \boldsymbol{S}^w}{\partial y}\right)^\mathsf{T} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) \, dxdy$$

See:

`ex/buckling_plate_FSDT.ipynb`

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CLPT, FSDT, TSDT for cylindrical shells

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CLPT kinematic equations for cylinders Sanders kinematics

Kinematics of a cylinder for the CLPT.
Modified from [1, 1b]

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) - z\, w_{,x}(x,y)\\
v(x,y,z) &= v_0(x,y) - z\, w_{,y}(x,y)\\
w(x,y,z) &= w_0(x,y)
\end{aligned}
$$

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2} w_{,x}^2 - z\, w_{,xx}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r} v_{,\theta} + \frac{1}{r} w + \frac{1}{2}\frac{1}{r^2} w_{,\theta}^2 \boxed{+ \frac{1}{2}\frac{1}{r^2} v^2 - \frac{1}{r^2} v\, w_{,\theta}} - z \frac{1}{r^2} w_{,\theta\theta} \boxed{+ z \frac{1}{r^2} v_{,\theta}}$$

$$\gamma_{x\theta} = \frac{1}{r} u_{,\theta} + v_{,x} + \frac{1}{r} w_{,x}\, w_{,\theta} \boxed{- \frac{1}{r} v\, w_{,x}} - 2z \frac{1}{r} w_{,x\theta} \boxed{+ z \frac{1}{r} v_{,x}}$$

If you remove the terms highlighted, you obtain the simplest type of nonlinear kinematics for cylinders, known as Donnell-type or Kirchhoff-Love.

`See: ex/kinematics_cylinder_CLPT.ipynb`

WARNING: Sanders' linear terms related to $\boldsymbol{\varepsilon}^{(1)}$ are not purely obtained from the CLPT, which implies small rotations. There are corrections [1a] aimed at extending the applicability.

[1a] Sanders, J. L., "Nonlinear theories for thin shells," Quarterly of Applied Mathematics, Vol. 21, No. 1, 1963, pp. 21–36. URL https://www.jstor.org/stable/43634948
[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.
[1b] Castro, S. G., & Jansen, E. L. (2022). Displacement-based multi-modal formulation of Koiter's method applied to cylindrical shells. In AIAA SCITECH 2022 Forum. American Institute of Aeronautics and Astronautics. AIAA SCITECH 2022 Forum. https://doi.org/10.2514/6.2022-0256.

Engineering Mechanics graduate school course

Stability of Structures

2025

Kinematics, Equivalent Single-Layer Theories, and Energy-based Semi-Analytical Methods

Saullo G. P. Castro

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FSDT kinematic equations for cylinders Sanders kinematics

Kinematics of a cylinder for the FSDT.
Modified from [1, 1b]

$$u(x,y,z) = u_0(x,y) - z\, w_{,x}(x,y)$$
$$v(x,y,z) = v_0(x,y) - z\, w_{,y}(x,y)$$
$$w(x,y,z) = w_0(x,y)$$

$$\varepsilon_{xx} = u_{,x} + \frac{1}{2} w_{,x}^2 + z\, \phi_{x,x}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r} v_{,\theta} + \frac{1}{r} w + \frac{1}{2}\frac{1}{r^2} w_{,\theta}^2 + \boxed{\frac{1}{2}\frac{1}{r^2} v^2 - \frac{1}{r^2} v\, w_{,\theta}} + z \frac{1}{r} \phi_{\theta,\theta} + \boxed{z \frac{1}{r^2} v_{,\theta}}$$

$$\gamma_{x\theta} = \frac{1}{r} u_{,\theta} + v_{,x} + \frac{1}{r} w_{,x} w_{,\theta} \boxed{- \frac{1}{r} v\, w_{,x}} + z \frac{1}{r} \phi_{x,\theta} + z \phi_{\theta,x} + \boxed{z \frac{1}{r} v_{,x}}$$

$$\gamma_{xz} = \phi_x + w_{,x}$$

$$\gamma_{yz} = \phi_y + w_{,y}$$

If you remove the terms highlighted, you obtain the simplest type of nonlinear kinematics for cylinders, known as Donnell-type or Kirchhoff-Love.

See: `ex/kinematics_cylinder_FSDT.ipynb`

WARNING: Sanders' linear terms related to $\boldsymbol{\varepsilon}^{(1)}$ are not purely obtained from the FSDT, which implies small rotations. There are corrections [1a] aimed at extending the applicability.

[1a] Sanders, J. L., "Nonlinear theories for thin shells," Quarterly of Applied Mathematics, Vol. 21, No. 1, 1963, pp. 21–36. URL https://www.jstor.org/stable/43634948
[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.
[1b] Castro, S. G., & Jansen, E. L. (2022). Displacement-based multi-modal formulation of Koiter's method applied to cylindrical shells. In AIAA SCITECH 2022 Forum. American Institute of Aeronautics and Astronautics. AIAA SCITECH 2022 Forum. https://doi.org/10.2514/6.2022-0256.

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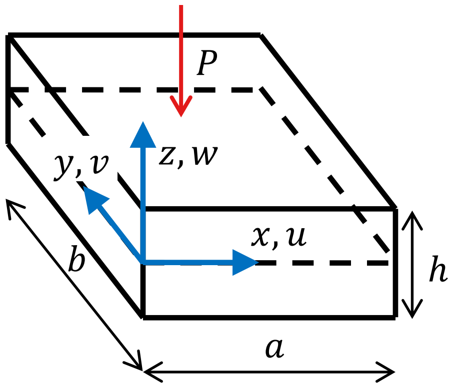
Examples

Buckling of a cylinder, CLPT, FSDT, TSDT



For all 3 ESL theories previously discussed, the following expression can be used to calculate the geometric stiffness matrix for cylinders, here using Donnell's equations [1]:

$$\boldsymbol{K_G} = \iint_{x,\theta} \hat{N}_{xx} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^{\mathsf{T}} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) + \hat{N}_{\theta\theta} \left(\frac{1}{r^2}\right) \left(\frac{\partial \boldsymbol{S}^w}{\partial \theta}\right)^{\mathsf{T}} \left(\frac{\partial \boldsymbol{S}^w}{\partial \theta}\right)$$
$$+ \hat{N}_{xy} \left(\frac{1}{r}\right) \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right)^{\mathsf{T}} \left(\frac{\partial \boldsymbol{S}^w}{\partial \theta}\right) + \hat{N}_{xy} \left(\frac{1}{r}\right) \left(\frac{\partial \boldsymbol{S}^w}{\partial \theta}\right)^{\mathsf{T}} \left(\frac{\partial \boldsymbol{S}^w}{\partial x}\right) (r)\, dx d\theta$$

```
See:

ex/buckling_cylinder_FSDT.ipynb
```

[1] Saullo G. P. Castro. "Semi-Analytical Tools for the Analysis of Laminated Composite Cylindrical and Conical Imperfect Shells under Various Loading and Boundary Conditions". Technische Universität Clausthal, 9th of December, 2014.

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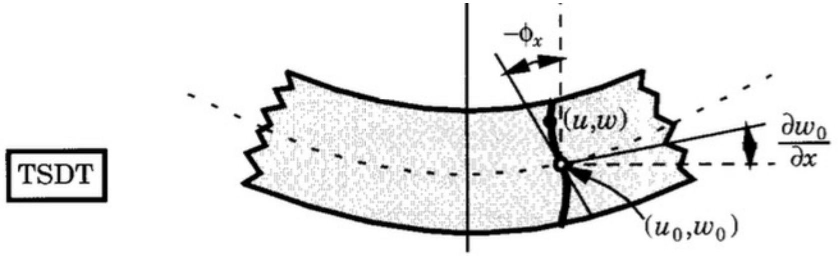
Limitations of ESL theories

Back to the Figure (TSDT) by Reddy [4]:



We can see that ESL theories are kinematically homogeneous [5], despite being able to describe transverse shear and normal strains, including transverse warping of cross section.

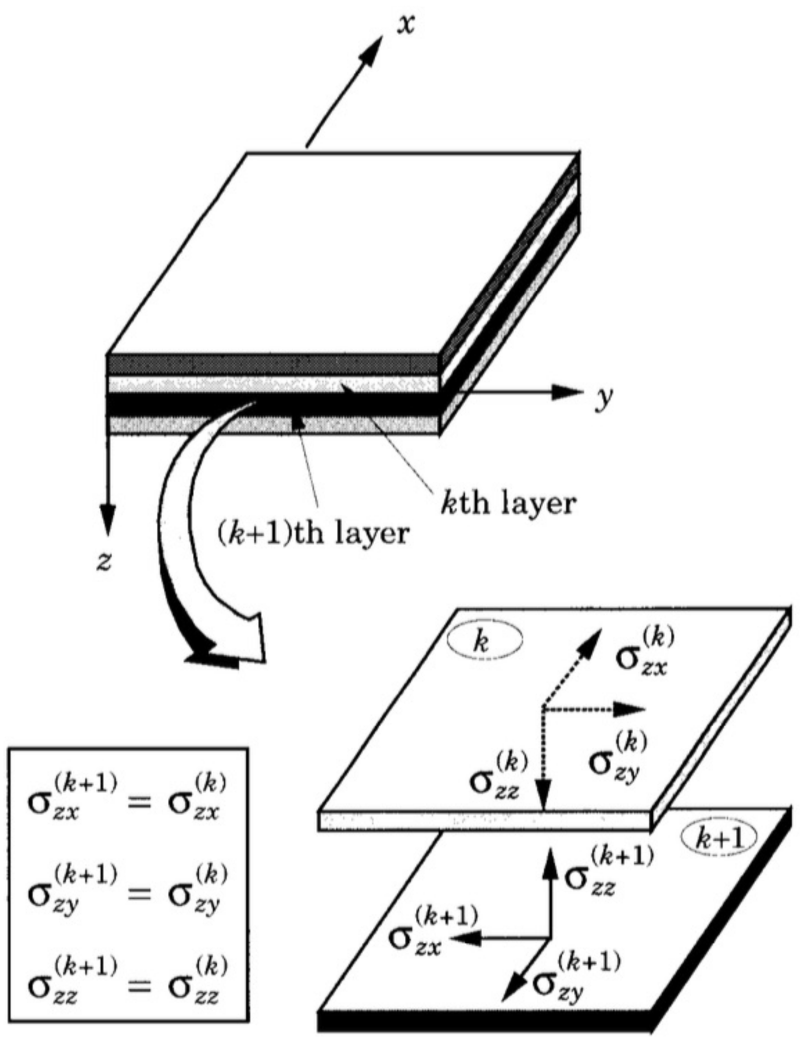
The kinematic homogeneity of **ESL theories** make them **very adequate** to predict global responses of the laminated component [4]. The most used ESL theories are the CLPT and FSDT.

- Gross deflections
- Critical buckling loads and associated buckling modes
- Fundamental vibration frequencies and associated mode shapes

For primary structural components, the goal of the analysis must be broadened to include a highly accurate assessment of more local failure modes, such as damage criteria, to understand where and how damage will initiate.

[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.
[5] Carrera, E. (2003). Historical review of Zig-Zag theories for multilayered plates and shells. Applied Mechanics Reviews, 56(3), 287–308. https://doi.org/10.1115/1.1557614.

Limitations of ESL theories



Equilibrium of interlaminar stresses [4].

Equilibrium of interlaminar stresses:

$$\begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k)} = \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\\ \sigma_{zz}\end{Bmatrix}^{(k+1)}$$

For the general case of $[\bar{Q}^{(k)}] \neq [\bar{Q}^{(k+1)}]$

$$\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{Bmatrix}^{(k)} \neq \begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\\ \varepsilon_{zz}\end{Bmatrix}^{(k+1)}$$

All ESL theories assume continuous functions over z, thus leading to a non-physical homogenized behaviour.

What happens with the interlaminar stresses in a general case?

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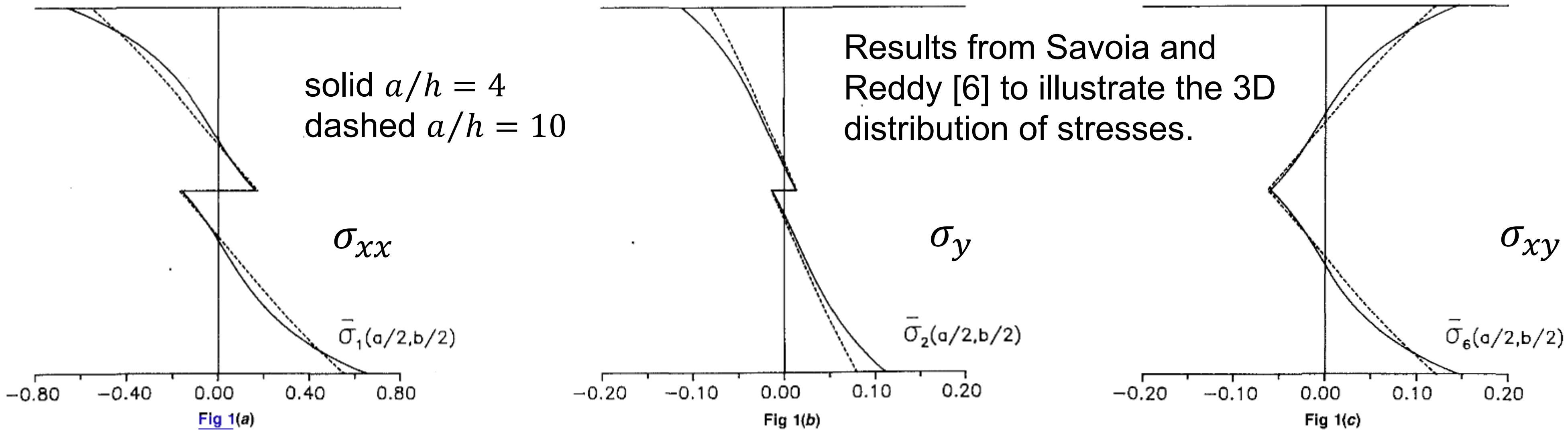
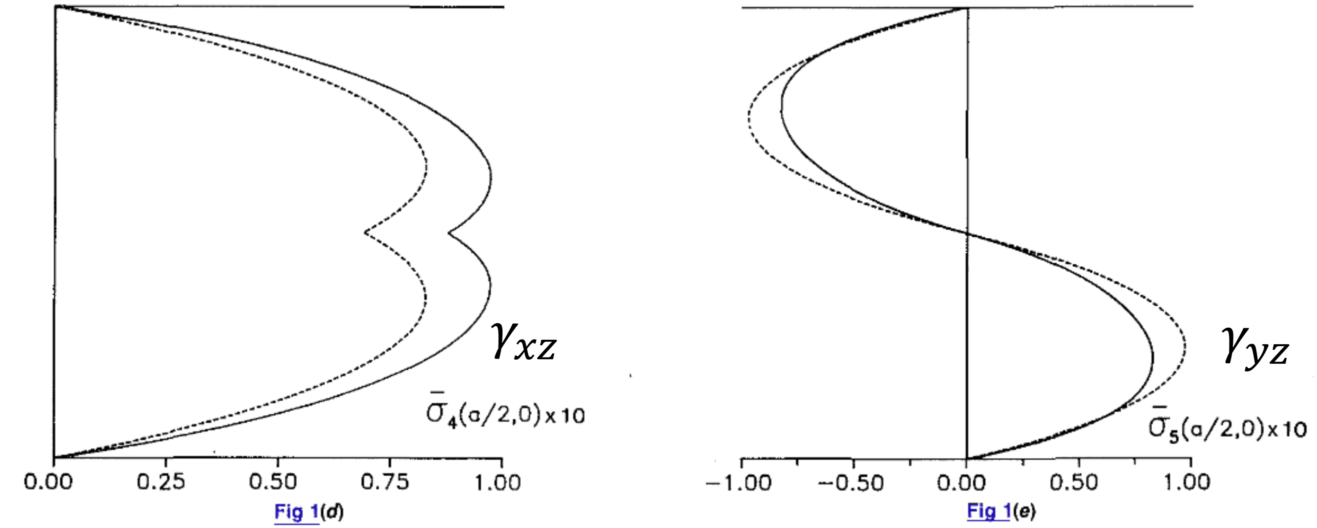
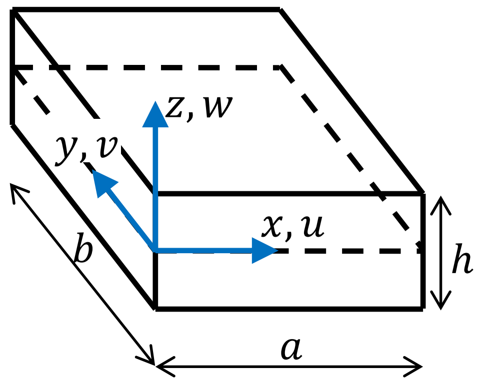
[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

Limitations of ESL theories

Results from Savoia and Reddy [6] to illustrate the 3D distribution of stresses.







[6] Savoia, M., & Reddy, J. N. (1992). A Variational Approach to Three-Dimensional Elasticity Solutions of Laminated Composite Plates. Journal of Applied Mechanics, 59(2S), S166–S175. https://doi.org/10.1115/1.2899483.

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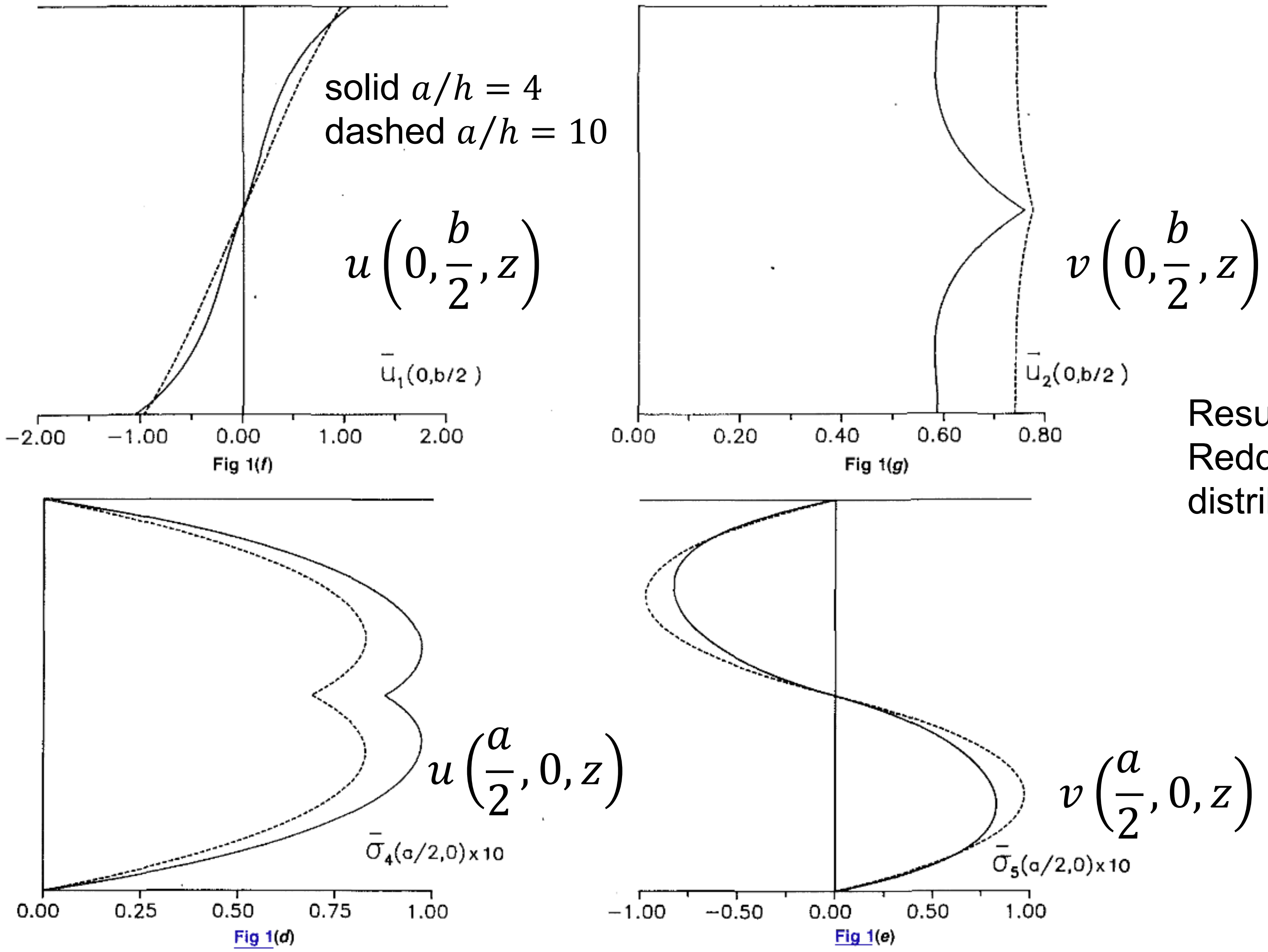
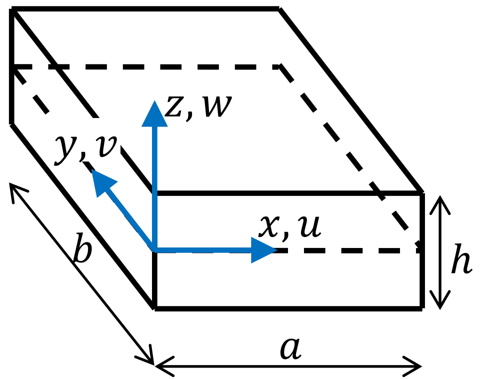
Limitations of ESL theories





Results from Savoia and Reddy [6] to illustrate the 3D distribution of displacements.

[6] M. Savoia and J. N. Reddy. A variational approach to three-dimensional elasticity solutions of laminated composite plates. Journal of Applied Mechanics, 1992. https://doi.org/10.1115/1.2899483

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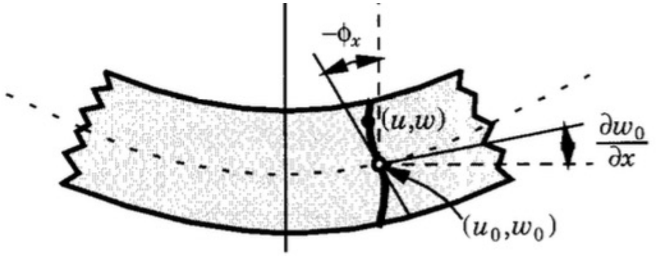
Zig-Zag (ZZ) theories for laminated structures

Back to the Figure (TSDT) by Reddy [4]:

TSDT



We can see that ESL theories are kinematically homogeneous [5], despite being able to describe transverse shear and normal strains, including transverse warping of cross section.

The open literature refers to “Zig-Zag” (ZZ) theories as those proposing a piecewise form of transverse stresses, whereas the piecewise form of displacement fields are often referred to as Interlaminar Continuity (IC) theories [5].

Carrera [5] refers to both cases as “Zig-Zag” theories.

Koiter (1959) states [5] that: “A refinement of Love’s first approximation theory is indeed meaningless, in general, unless the **effects of transverse shear and normal stresses** are taken into account at the same time”.

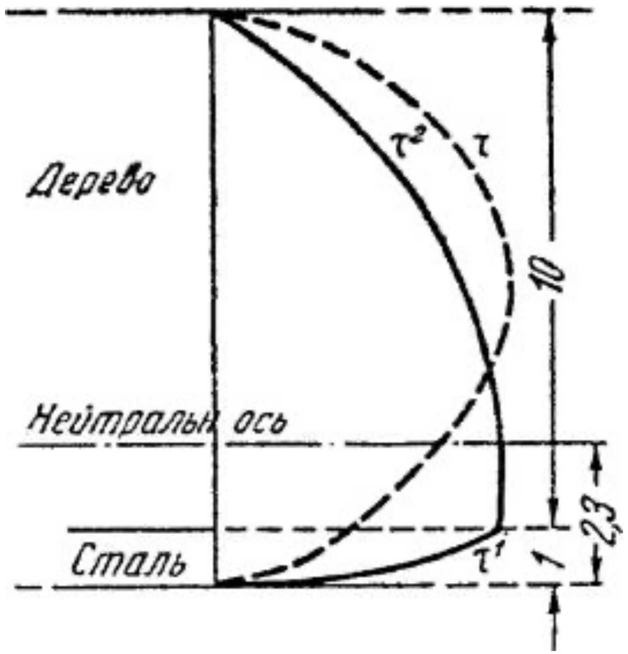
[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.
[5] E. Carrera. Historical review of Zig-Zag theories for multilayered plates and shells. Applied Mechanics Reviews, 2003. https://doi.org/10.1115/1.1557614

Zig-Zag (ZZ) theories

Zig-Zag theories aim to address Koiter's recommendation, thus, being a theory that:

- Removes the assumption that $\varepsilon_{zz} = 0$
- Enables a piecewise discretization with C_z^0 continuity along the z direction

According to Carrera [5], Lekhnitskii (1935) was the first to describe the later called Zig-Zag effect for both in-plane and through-the-thickness displacements:



Note in this Figure that the C_z^1 continuity that would have been produced by the TSDT is an approximation for multilayered structures, which intrinsically only have C_z^0 continuity.

This Figure from Lekhnitskii (1935) concerns a two-layered structure consisting of low (top) and high (bottom) stiffness materials.

[5] E. Carrera. Historical review of Zig-Zag theories for multilayered plates and shells. Applied Mechanics Reviews, 2003. https://doi.org/10.1115/1.1557614

Zig-Zag (ZZ) theories

The Reissner Mixed Variational Theorem (RMVT) was the only one developed in the West [5], which permits satisfying completely and a priori the C_z^0 for both the displacement (IC theory) and transverse stresses (ZZ theory.

It is assumed two independent fields, one for displacements:

$$\boldsymbol{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

and one for transverse stresses:

$$\boldsymbol{\sigma_n} = \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \\ \sigma_{zz} \end{Bmatrix}$$

[5] E. Carrera. Historical review of Zig-Zag theories for multilayered plates and shells. Applied Mechanics Reviews, 2003. https://doi.org/10.1115/1.1557614

Zig-Zag versus Layerwise theories

Using a combination of global and local variables and satisfying both stress and displacement continuity conditions leads to theories with a fixed number of variables that are called “Zig-Zag” [7].

When only displacement continuity conditions are enforced at ply interfaces, the kinematic assumption is called “layerwise” [7].

[7] Abrate, S., & Di Sciuva, M. (2018). 1.16 Multilayer Models for Composite and Sandwich Structures. In Comprehensive Composite Materials II (pp. 399–425). Elsevier. https://doi.org/10.1016/b978-0-12-803581-8.09885-4.

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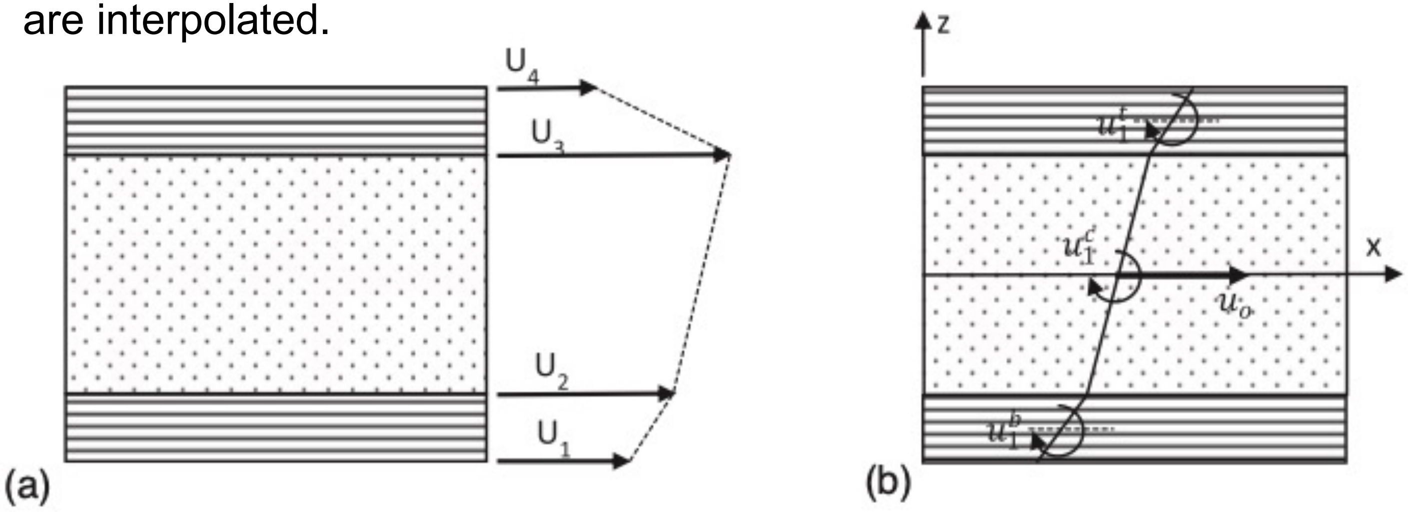
Layerwise Theories

In the original Layerwise theories [8,9,4] the primary variables are the displacements at the ply interfaces including the top and bottom surfaces. At any arbitrary point through the thickness, the displacements are interpolated.



h-refinement: add more control points through the thickness

p-refinement: use higher-order Lagrange (not Hermite!) interpolation functions between control points

Example of a sandwich beam: (a) axial displacements at interface as primary variables; (b) axial displacement at neutral axis and rotation of the normal as primary variables.

[4] Reddy, J. N. (2003). Mechanics of Laminated Composite Plates and Shells. CRC Press. https://doi.org/10.1201/b12409.

[7] Abrate, S., & Di Sciuva, M. (2018). 1.16 Multilayer Models for Composite and Sandwich Structures. In Comprehensive Composite Materials II (pp. 399–425). Elsevier. https://doi.org/10.1016/b978-0-12-803581-8.09885-4.

[8] Reddy, J. N. (1987). A generalization of two-dimensional theories of laminated composite plates. Communications in Applied Numerical Methods, 3(3), 173–180. https://doi.org/10.1002/cnm.1630030303.

[9] Robbins, D. H., Jr., & Reddy, J. N. (1993). Modelling of thick composites using a layerwise laminate theory. International Journal for Numerical Methods in Engineering, 36(4), 655–677. https://doi.org/10.1002/nme.1620360407.

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Exercises

Plate deflection under bending

- Bending analysis, reproduce the result of Reddy Fig. 11.4.3
- Compare CLPT, FSDT, TSDT

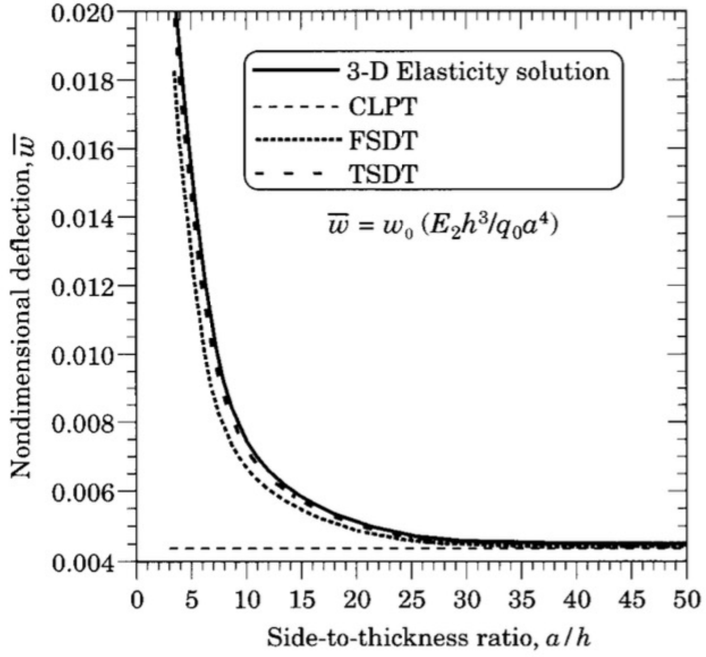

Figure 11.4.3: Plots of nondimensionalized center transverse deflection versus side-to-thickness ratio of a symmetric cross-ply (0/90/90/0) laminate under sinusoidally distributed load (Material 1).

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Plate buckling

- Compare CLPT, FSDT, TSDT
- Based on the example `ex/buckling_plate_FSDT.ipynb`
- Extend this example for CLPT and TSDT, check your results before moving next
- Compare these three ESL theories for

$$2 \leq \frac{a}{h} \leq 2000$$

- Keep $a/b = 4$
- Use same isotropic material properties of the base example
- Create a plot of Nxx_{crit} versus a/h for the three ESL theories

Extra reading material

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Partial Derivatives from Cartesian to Natural Coordinates

If $x(\xi,\eta)$ and $y(\xi,\eta)$ it comes that $\xi(x,y)$ and $\eta(x,y)$

Such that:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$\xi(x,y)$ and $\eta(x,y)$ usually do not have analytical expressions

Therefore, the values for $\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$ are calculated numerically

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FSDT – Strains in Natural Coordinates

$$\delta U = \iint_{xy} \left(\boldsymbol{N}^\top \begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix} + \boldsymbol{M}^\top \begin{Bmatrix} \delta\kappa_{xx} \\ \delta\kappa_{yy} \\ \delta\kappa_{xy} \end{Bmatrix} + \boldsymbol{Q}^\top \begin{Bmatrix} \delta\gamma_{yz} \\ \delta\gamma_{xz} \end{Bmatrix} \right) dxdy$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x}$$

Same idea for transverse shear and rotational strains

$x(\xi,\eta)$ and $y(\xi,\eta)$ are known, but not $\xi(x,y)$ and $\eta(x,y)$

How to calculate?

$$\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$$

The Jacobian Matrix

Given that:

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \qquad \text{and} \qquad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} \qquad \qquad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

In matrix form:

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \boldsymbol{J} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \qquad \text{and} \qquad \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \boldsymbol{J}^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

Where:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{J}^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

Note the numerical calculation of $\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$

Matrix $\boldsymbol{J}$ is the Jacobian

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Transformation of Variables

$$\delta U = \iint_{xy} \left(\boldsymbol{N}^\top \begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix} + \boldsymbol{M}^\top \begin{Bmatrix} \delta\kappa_{xx} \\ \delta\kappa_{yy} \\ \delta\kappa_{xy} \end{Bmatrix} + \boldsymbol{Q}^\top \begin{Bmatrix} \delta\gamma_{yz} \\ \delta\gamma_{xz} \end{Bmatrix} \right) dxdy$$

$$dxdy = |\boldsymbol{dx} \times \boldsymbol{dy}|$$

$$\boldsymbol{dx} = \frac{\partial x}{\partial \xi}\boldsymbol{d\xi} + \frac{\partial x}{\partial \eta}\boldsymbol{d\eta}$$

$$\boldsymbol{dy} = \frac{\partial y}{\partial \xi}\boldsymbol{d\xi} + \frac{\partial y}{\partial \eta}\boldsymbol{d\eta}$$

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$dxdy = \begin{vmatrix} \boldsymbol{\xi} & \boldsymbol{\eta} & \boldsymbol{1} \\ \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & 0 \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & 0 \end{vmatrix} = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \right) d\xi d\eta \quad = \det \boldsymbol{J} d\xi d\eta$$

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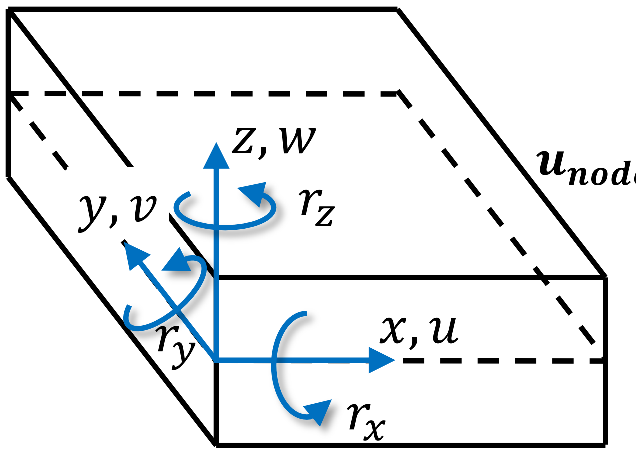
Rotations in 3D for ESL theories

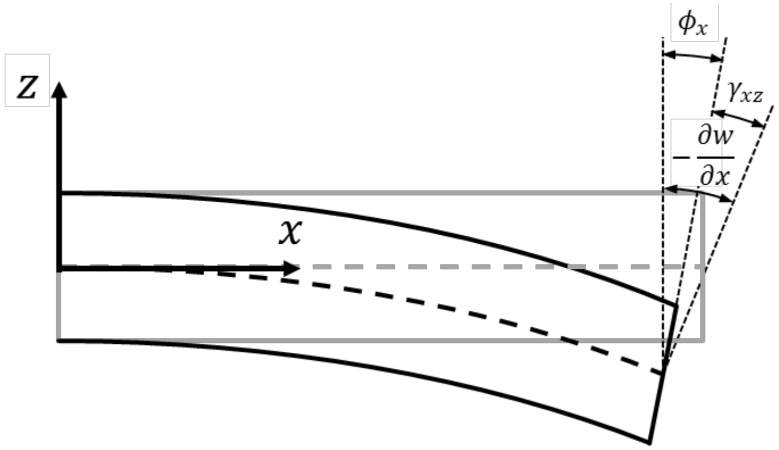


$$\boldsymbol{u}_{node} = \begin{Bmatrix} u \\ v \\ w \\ r_x \\ r_y \\ r_z \end{Bmatrix}$$

$$r_x(x,y) = -\phi_y(x,y)$$
$$r_y(x,y) = \phi_x(x,y)$$
$$r_z \equiv drilling\ DOF$$

It is the standard to use these 3 translational and 3 rotational degrees-of-freedom in commercial software (Nastran, Abaqus etc).

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$



$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial y} - r_x \\ \frac{\partial w}{\partial x} + r_y \end{Bmatrix}$$

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FSDT Formulation – Kinematic Relations in 3D

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{Bmatrix} = \begin{Bmatrix} u_{0,x} \\ v_{0,y} \\ u_{0,y} + v_{0,x} \\ w_{,y} - r_x \\ w_{,x} + r_y \end{Bmatrix} + z \begin{Bmatrix} r_{y,x} \\ -r_{x,y} \\ r_{y,y} - r_{x,x} \\ 0 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon_0} + z\boldsymbol{\varepsilon}_1$$

Usually, subscript "0" is omitted:

$$\boldsymbol{\varepsilon_0} = \begin{Bmatrix} \varepsilon_{xx0} \\ \varepsilon_{yy0} \\ \gamma_{xy0} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \qquad \boldsymbol{\varepsilon_1} = \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

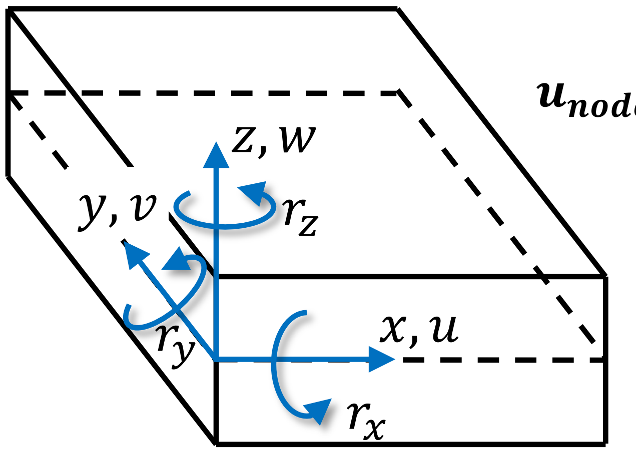
Also, for $\boldsymbol{u}$ the subscript "0" is usually omitted:

$$u(x,y,z) = u(x,y) + zr_y(x,y)$$
$$v(x,y,z) = v(x,y) - zr_x(x,y)$$
$$w(x,y,z) = w(x,y)$$

Drilling in 3D

Reference:
F.M. Adam, A.E. Mohamed, A.E. Hassaballa, Degenerated Four Nodes Shell Element with Drilling Degree of Freedom, IOSR J. Eng. 3 (2013) 10–20. www.iosrjen.org.

$r_z \equiv drilling\ DOF$



$$\boldsymbol{u_{node}} = \begin{Bmatrix} u \\ v \\ w \\ r_x \\ r_y \\ r_z \end{Bmatrix}$$

$$r_z(\xi, \eta) = \sum_{i=1}^{4} N_i(\xi, \eta) r_{z_i}$$

Drilling strain energy:

$$U_{r_z} = \alpha_t G \int_A \left(r_z - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right)^2 dA$$

$\alpha_t \equiv$problem-dependent torsional penalty factor, $G \equiv$ shear modulus

If $\alpha_t = 0$ the shear deformation is decoupled from r_z. A sufficiently high α_t must be used to ensure that the rotation r_z follows the in-plane shear deformation.

For composites and isotropic plates, one could replace $\alpha_t G$ by A_{66}/h, where A_{66} is the ABD shear stiffness term and h the plate thickness.

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Stiffness due to Drilling in 3D

Reference:
F.M. Adam, A.E. Mohamed, A.E. Hassaballa, Degenerated Four Nodes Shell Element with Drilling Degree of Freedom, IOSR J. Eng. 3 (2013) 10–20. www.iosrjen.org.

$r_z \equiv drilling\ DOF$

Variation of the strain energy for the drilling degree-of-freedom:

$$\delta U_{r_z} = 2\alpha_t G \iint_{\xi,\eta} \left(r_z - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) \delta \left(r_z - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right) \det \boldsymbol{J}\, d\xi d\eta$$

$$\delta U_{r_z} = 2\alpha_t G\, \boldsymbol{u}^\top \iint_{\xi,\eta} \left(\boldsymbol{N}^{r_z\top} - \frac{1}{2}\left(\boldsymbol{N}^{v\,\top}_{,x} - \boldsymbol{N}^{u\,\top}_{,y}\right)\right)\left(\boldsymbol{N}^{r_z} - \frac{1}{2}\left(\boldsymbol{N}^{v}_{,x} - \boldsymbol{N}^{u}_{,y}\right)\right) \det \boldsymbol{J}\, d\xi d\eta\, \delta \boldsymbol{u}$$

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