A Family of Alternating Equilibrium For Pair-valued Bi-matrix Games On The Unit Square

by

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Abstract

We define and prove the existence of a family of alternating equilibrium for pair-valued bi-matrix games on the unit square.

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1. Introduction: This note is based on the fact that any real number u in the closed interval [0, 1] can be uniquely represented as $\sum_{k=1}^{\infty}\frac{u_k}{10^k}$, where $u_k \epsilon\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for all positive integers 'k' and $\{k: u_k \neq 0\}$ is "***minimal***", i.e., for *no other* representation of u say $u = \sum_{k=1}^{\infty}\frac{u_k^{'}}{10^k}$ the set $\{k|u_k^{'} \neq 0\}$ is a "*proper subset*" of $\{k: u_k \neq 0\}$. For instance $1 = \sum_{k=1}^{\infty}\frac{9}{10^k}$, $\frac{10}{99} = \sum_{k=1}^{\infty}\frac{1}{10^{2k-1}} + \sum_{k=1}^{\infty}\frac{0}{10^{2k}}$, $0.5 = \frac{5}{10} + \sum_{k=2}^{\infty}\frac{0}{10^k}$, ... and so on. While $\frac{4}{10} + \sum_{k=2}^{\infty}\frac{9}{10^k} = 0.5$, the representation of 0.5 that is consistent with our desired type of "*minimality*" is $0.5 = \frac{5}{10} + \sum_{k=2}^{\infty}\frac{0}{10^k}$.

Given any positive integer K, let $►_K$ be the *operation* on the unit square [0, 1]×[0, 1] defined as follows: for every pair of real numbers u, v in the interval [0, 1], with $u = \sum_{k=1}^{\infty}\frac{u_k}{10^k}$ and $v = \sum_{k=1}^{\infty}\frac{v_k}{10^k}$, let $u►_K v$ denote the real number $\sum_{k=1}^{K}\frac{u_k}{10^{2k-1}} + \sum_{h=1}^{\infty}\frac{1}{10^{hK}}[\sum_{k=hK+1}^{(h+1)K}\frac{u_k}{10^k} + \sum_{k=(h-1)K+1}^{(hK}\frac{v_k}{10^k}]$, which also belongs to the closed interval [0, 1]. $u►_K v$ are alternating blocks, alternating between u's and v's, with each block having K terms.

The h^{th} block of u's for $h \geq 2$, which is $\sum_{k=(h-1)K+1}^{hK}\frac{u_k}{10^k}$ is transformed under $u►_K v$ to the $(2h - 1)^{th}$ block in the alternating series. Thus, for $j\epsilon\{1, \ldots, K\}$ the $((h - 1)K + j)^{th}$ term in the h^{th} block of u's for $h \geq 2$ gets transformed into the $(2(h - 1)K + j)^{th}$ term in the $(2h - 1)^{th}$ block in the alternating series. Thus, for $j\epsilon\{1, \ldots, K\}$, $\frac{u_{(h-1)K+j}}{10^{(h-1)K+j}}$ has to be pre-multiplied by $\frac{1}{10^{(h-1)K}}$. Note that $2(h - 1)K + j = 2hK - 2K + j$.

The h^{th} block of v's for $h \geq 1$, which is $\sum_{k=(h-1)K+1}^{hK} \frac{v_k}{10^k}$ is transformed under $u \blacktriangleright_K v$ to the $(2h)^{th}$ block in the alternating series. Thus, for $j\epsilon\{1, \ldots, K\}$ the $((h - 1)K + j)^{th}$ term in the h^{th} block of v's for $h \geq 1$ gets transformed into the $((2h - 1)K + j)^{th}$ term in the $(2h)^{th}$ block in the alternating series. Note that $(2h - 1)K + j \neq 2hK - 2K + j$.Thus, for $j\epsilon\{1, \ldots, K\}$, $\frac{v_{(h-1)K+j}}{10^{(h-1)K+j}}$ has to be pre-multiplied by $\frac{1}{10^{hK}}$.

Thus, $u \blacktriangleright_K v = \sum_{k=1}^{K} \frac{u_k}{10^{2k-1}} + \frac{1}{10^K}\sum_{k=1}^{K} \frac{v_k}{10^{2k-1}} + \frac{1}{10^K}\sum_{k=K+1}^{2K} \frac{u_k}{10^{2k-1}} + \frac{1}{10^{2K}}\sum_{k=K+1}^{2K} \frac{v_k}{10^{2k-1}} + \frac{1}{10^{2K}}\sum_{k=2K+1}^{3K} \frac{v_k}{10^{2k-1}} + \ldots$ which is a convergent infinite series converging to a point in [0, 1].

Clearly $u \blacktriangleright_K v$ gives some amount of priority to u over v.

This allows us to define preferences over $[0, 1]\times[0, 1]$ which has the function $U^{(K)}:[0, 1]\times[0, 1]\rightarrow[0, 1]$ satisfying $U^{(K)}(u, v) = u \blacktriangleright_K v$ for all $(u, v)\epsilon[0, 1]\times[0, 1]$ as a numerical representation. We refer to this function as the **K-alternating utility function**.

Let $u^{(1)} = \sum_{k=1}^{\infty} \frac{1}{10^k}$, $v^{(1)} = \sum_{k=1}^{\infty} \frac{8}{10^k}$. Thus, $U^{(1)}(u^{(1)}, v^{(1)}) = \sum_{k=1}^{\infty} \frac{1}{10^{2k-1}} + \sum_{k=1}^{\infty} \frac{8}{10^{2k}}$.

Let $u^{(2)} = \frac{1}{10} + \frac{2}{100} + \sum_{k=3}^{\infty} \frac{1}{10^k}$, $v^{(2)} = \sum_{k=1}^{\infty} \frac{7}{10^k}$. Thus, $U^{(1)}(u^{(1)}, v^{(1)}) = \frac{1}{10} + \frac{2}{1000} + \sum_{k=3}^{\infty} \frac{1}{10^{2k-1}} + \sum_{k=1}^{\infty} \frac{7}{10^{2k}}$.

It is easy to see that $U^{(1)}(u^{(1)}, v^{(1)}) > U^{(1)}(u^{(2)}, v^{(2)})$ although $u^{(2)} > u^{(1)}$. Thus, according to lexicographic preferences (see Varian (1978)), $(u^{(2)}, v^{(2)})$ is strictly preferred to $(u^{(1)}, v^{(1)})$.

Since lexicographic preferences cannot be represented by any real valued utility function, preferences represented by a K-alternating utility function on $[0, 1]\times[0, 1]$ can be considered as an alternative that incorporates the "core idea" behind lexicographic preferences and yet- unlike lexicographic preferences- has a reasonable numerical representation. Further, the larger the natural number K is, the better is the approximation of lexicographic preferences by a preference relation which has $U^{(K)}$ as a numerical representation.

2. The Framework of Pair-valued Bi-matrix Games: For positive integers m, n consider an interactive decision making context (game) with two players/decisionmakers who we denote by ‘a’ and ‘b’ respectively, and where player ‘a’ has ‘m’ actions to choose from and ‘b’ has ‘n’ actions to choose from.

A **strategy for player ‘a’** is a point in $\Delta^{m-1} = \{x\epsilon\mathbb{R}^m_+ | \sum_{i=1}^{m} x_i = 1\}$ and a **strategy for player ‘b’** is a point in $\Delta^{n-1} = \{x\epsilon\mathbb{R}^n_+ | \sum_{j=1}^{n} y_j = 1\}$.

A strategy for player ‘a’ is a probability distribution on {1, ..., m}, with each coordinate of the strategy denoting the probability with which player ‘a’ chooses the corresponding action. A strategy for player ‘b’ is a probability distribution on {1, ..., n}, with each coordinate of the strategy denoting the probability with which player ‘b’ chooses the corresponding action.

A pair (x, y) where x is a strategy for player ‘a’ and y is a strategy for player ‘b’ is said to be a **strategy profile**.

Let $\mathbb{R}^{m\times n}$ be the set of all m×n matrices.

Given $C\epsilon\mathbb{R}^{m\times n}$ and (i, j)ϵ{1, …, m}×{1, ..., n}, let C_i denote the i^{th} row of C and C^j denote the j^{th} column of C.

For our purposes here, a member of $\mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ will be referred to as a **bi-matrix game**.

Given a bi-matrix game (A, B) a strategy profile (x, y) is said to be an **equilibrium strategy profile for (A, B)**, if for all (i, j)ϵ{1, …, m}×{1, ..., n}: $A_iy \leq x^TAy$ and $x^TB^j \leq x^TBy$.

Let $\mathbb{R}^{(2,\, m\times n)}$ denote the set of all m×n matrices whose entries are pairs of real number.

If $V\epsilon\mathbb{R}^{(2,\, m\times n)}$, then for all (i, j)ϵ{1, …, m}×{1, ..., n}, the typical notations are as follows: $(i, j)^{th}$ entry of V is denoted by $(v_{ij}^{(1)}, v_{ij}^{(2)})$, the i^{th} row of V is denoted by V_i and the j^{th} column of V is denoted by V^j.

For our purposes here, a member of $\mathbb{R}^{(2,\, m\times n)} \times \mathbb{R}^{(2,\, m\times n)}$ will be referred to as a **pair-valued bi-matrix game**.

This definition of a pair-valued bi-matrix game is available in Lahiri (2026a, 2026b) and subsequent work base on them.

Notation 1: Given a pair-valued bi-matrix game (V(a), V(b)), for rϵ{1, 2} and αϵ{a, b}, let $V^{(r)}(\alpha)$ be the m×n real-valued matrix whose $(i, j)^{th}$ term for (i, j)ϵ{1, …, m}×{1, ..., n} is $v_{ij}^{(r)}(\alpha)$. The i^{th} row of $V^{(r)}(\alpha)$ for iϵ{1, …, m} is denoted by $V_i^{(r)}(\alpha)$ and the j^{th} column of $V^{(r)}(\alpha)$ for jϵ{1, …, n} is denoted by$V^{(r)j}(\alpha)$. ▌

We refer to a pair-valued bi-matrix game (V(a), V(b)) satisfying $v_{ij}^{(r)}(\alpha)\epsilon[0, 1]$ for all $r\epsilon\{1, 2\}$, $\alpha\epsilon\{a, b\}$ and $(i, j)\epsilon\{1, \ldots, m\}\times\{1, ..., n\}$ as a **pair-valued bi-matrix game (based) on the unit square.**

Given the pair valued bi-matrix game on the unit square (V(a), V(b)) and any positive ineger K, let $(\vec{V}(K)(a), \vec{V}(b))$ be the bi-matrix game such that for all $\alpha\epsilon\{a, b\}$ and $(i, j)\epsilon\{1, \ldots, m\}\times\{1, ..., n\}$, the $(i, j)^{th}$ term of $\vec{V}(K)(\alpha)$ is the real number

$v_{ij}^{(1)}(\alpha) \blacktriangleright_K v_{ij}^{(2)}(\alpha)$.

Thus, every entry in $\vec{V}(K)(\alpha)$, gives some kind of priority to the first coordinate over the second coordinate in the corresponding entry in V(α).

A strategy profile (x, y) is said to a **K-alternating equilibrium** for the pair-valued bi-matrix game on the unit square if (x, y) is an equilibrium strategy profile for the bi-matrix game $(\vec{V}(K)(a), \vec{V}(K)(b))$.

3. Existence of Alternating equilibrium: The proof of the following theorem follows immediately from the existence of equilibrium strategy profile for bi-matrix games (see theorem 2 in Chandrasekaran (nd.)).

Theorem 1: For every positive integer K, the pair-valued bi-matrix game on the unit square (V(a), V(b)) has a K-alternating equilibrium.

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