On the Local Convergence of the Third-Order Newton Iteration

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***Abstract*—We present a local convergence analysis of the third-order Newton iteration for unconstrained optimization recently proposed by Ahmadi, Chaudhry, and Zhang. For the single-variable case, we derive the local error equation, establish cubic convergence, and obtain an explicit expression for the asymptotic error constant. A numerical example confirms the theoretical predictions. The obtained convergence order is consistent with the classical local convergence theory of Traub for one-point iterative methods.**

I. Introduction

Consider the unconstrained optimization problem

$$\min_{x\in\mathbb{R}} f(x),$$

where $f : \mathbb{R} \to \mathbb{R}$ is assumed to be sufficiently smooth. A stationary point α satisfies

$$f'(\alpha) = 0.$$

Recently, Ahmadi, Chaudhry, and Zhang [1] introduced higher-order Newton methods obtained by solving local polynomial models exactly. In the one-dimensional third-order case, their method admits the closed-form iteration

$$\begin{aligned} x_{k+1} &= x_k + \delta_k, \\ \delta_k &= -\frac{2f''(x_k)}{f'''(x_k)} - \frac{\sqrt[3]{12f'(x_k)f''(x_k)f'''(x_k) - 8\big(f''(x_k)\big)^3}}{f'''(x_k)} \end{aligned} \tag{1}$$

provided that $f'''(x_k) \neq 0$.

Throughout this paper, we adopt the notation $a := f'(x_k)$, $b := f''(x_k)$, and $c := f'''(x_k)$, so that (1) can be written compactly as

$$\delta_k = -\frac{2b}{c} - \frac{\sqrt[3]{12abc - 8b^3}}{c}.$$

The principal objective of this paper is to establish the local convergence properties of the third-order closed-form Newton iteration (1). In particular, we derive its order of convergence to a non-degenerate stationary point α satisfying $f'(\alpha) = 0$ by developing an exact algebraic analysis of the cube-root term, thereby avoiding the asymptotic approximations employed in conventional Taylor-series treatments.

II. Main Results

The key observation is that δ_k, although given in closed form via a cube root, is in fact characterized *exactly* by a polynomial equation.

Lemma 1 (The radical-elimination identity)**.** *Let $a, b, c \in \mathbb{R}$ with $c \neq 0$, and let δ_k be defined by (1) (with a, b, c in place of f', f'', f'''). Then δ_k satisfies*

$$c^2\delta_k^3 + 6bc\,\delta_k^2 + 12b^2\delta_k + 12ab = 0. \tag{2}$$

Proof. Isolating the radical in (1) gives

$$c\delta_k + 2b = -\sqrt[3]{12abc - 8b^3}.$$

Cubing both sides yields

$$(c\delta_k + 2b)^3 = -(12abc - 8b^3) = 8b^3 - 12abc.$$

Expanding the left-hand side via $(X+Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$ with $X = c\delta_k$, $Y = 2b$,

$$c^3\delta_k^3 + 6bc^2\delta_k^2 + 12b^2c\delta_k + 8b^3 = 8b^3 - 12abc.$$

The term $8b^3$ cancels from both sides, leaving

$$c^3\delta_k^3 + 6bc^2\delta_k^2 + 12b^2c\delta_k = -12abc.$$

Dividing by $c \neq 0$ gives (2). □

Equation (2) contains no radical and is the object on which all subsequent asymptotic analysis is performed.

Remark 1 (Uniqueness of the real branch near a stationary point)*.* Cubing is not, in general, an invertible operation, so it is worth confirming that (2) does not introduce spurious real solutions in the regime of interest. Evaluate (2) at $x_k = \alpha$, where $a = f'(\alpha) = 0$: the equation factors as

$$\delta_k\left(c^2\delta_k^2 + 6bc\delta_k + 12b^2\right) = 0,$$

whose roots are $\delta_k = 0$ together with

$$\delta_k = \frac{-6bc \pm \sqrt{36b^2c^2 - 48b^2c^2}}{2c^2} = \frac{-6bc \pm \sqrt{-12b^2c^2}}{2c^2}.$$

Provided $b \neq 0$ and $c \neq 0$, the discriminant $-12b^2c^2$ is strictly negative, so these two additional roots are complex conjugates. Hence, in a neighborhood of a non-degenerate stationary point ($f''(\alpha) \neq 0$), $\delta_k = 0$ is the *only* real root of (2), and by continuity of the roots of a polynomial in its coefficients, the real branch through $\delta_k = 0$ remains isolated and real-analytic for x_k near α. This justifies positing a convergent power-series expansion for δ_k in Section II-A.

A. Local Convergence Analysis

Let α satisfy $f'(\alpha) = 0$ and assume $f \in C^4$ in a neighborhood of α with $f''(\alpha) \neq 0$ (non-degeneracy). Define the error $e_k := x_k - \alpha$ and the derivative values

$$b_0 := f''(\alpha), \qquad c_0 := f'''(\alpha), \qquad d_0 := f''''(\alpha).$$

By Taylor's theorem,

$$a = f'(x_k) = b_0 e_k + \tfrac{1}{2} c_0 e_k^2 + \tfrac{1}{6} d_0 e_k^3 + O(e_k^4), \tag{3a}$$
$$b = f''(x_k) = b_0 + c_0 e_k + \tfrac{1}{2} d_0 e_k^2 + O(e_k^3), \tag{3b}$$
$$c = f'''(x_k) = c_0 + d_0 e_k + O(e_k^2). \tag{3c}$$

Theorem 1 (Cubic convergence)**.** *Under the stated hypotheses, there is a neighborhood of α in which the iteration (1) converges to α with order three:*

$$e_{k+1} = C\, e_k^3 + O(e_k^4), \qquad C = \frac{f'''(\alpha)^2 - 2f''(\alpha)f''''(\alpha)}{12\, f''(\alpha)^2}. \tag{4}$$

Proof. By Remark 1, $\delta_k = \delta_k(e_k)$ is real-analytic near $e_k = 0$ with $\delta_k(0) = 0$, so write

$$\delta_k(e_k) = q_1 e_k + q_2 e_k^2 + q_3 e_k^3 + O(e_k^4), \tag{5}$$

where q_1, q_2, q_3 are constants to be determined.

Substituting (3a)–(3c) and (5) into the exact polynomial identity (2), collecting like powers of e_k, and retaining terms through $O(e_k^3)$ gives

$$\begin{aligned} 0 &= 12b_0^2(q_1+1)e_k \\ &\quad + \Big(12b_0^2 q_2 + 6b_0c_0q_1^2 + 24b_0c_0q_1 + 18b_0c_0\Big)e_k^2 \\ &\quad + \Big(12b_0^2 q_3 + 12b_0c_0q_2 + 12(c_0^2 + b_0d_0)q_1 + 12b_0c_0q_1q_2 \\ &\quad + 6(b_0d_0 + c_0^2)q_1^2 + c_0^2q_1^3 + 6c_0^2 + 8b_0d_0\Big)e_k^3 + O(e_k^4). \end{aligned} \tag{6}$$

Matching coefficients of successive powers of e_k yields

$$e_k^1 : 12b_0^2(q_1+1) = 0, \tag{7a}$$
$$e_k^2 : 12b_0^2q_2 + 6b_0c_0q_1^2 + 24b_0c_0q_1 + 18b_0c_0 = 0, \tag{7b}$$
$$\begin{aligned} e_k^3 :\, &12b_0^2q_3 + 12b_0c_0q_2 + 12(c_0^2 + b_0d_0)q_1 + 12b_0c_0q_1q_2 \\ &+ 6(b_0d_0 + c_0^2)q_1^2 + c_0^2q_1^3 + 6c_0^2 + 8b_0d_0 = 0. \end{aligned} \tag{7c}$$

Equation (7a) gives

$$q_1 = -1.$$

Substituting this into (7b) yields

$$12b_0^2q_2 + (6 - 24 + 18)b_0c_0 = 0,$$

and hence

$$q_2 = 0.$$

Finally, substituting $q_1 = -1$ and $q_2 = 0$ into (7c) gives

$$12b_0^2q_3 - c_0^2 + 2b_0d_0 = 0,$$

so that

$$q_3 = \frac{c_0^2 - 2b_0d_0}{12b_0^2}.$$

Since $e_{k+1} = e_k + \delta_k(e_k)$, substituting (5) together with $q_1 = -1$, $q_2 = 0$, and the above expression for q_3 yields

$$e_{k+1} = (1+q_1)e_k + q_2e_k^2 + q_3e_k^3 + O(e_k^4) = q_3e_k^3 + O(e_k^4).$$

Hence

$$e_{k+1} = \frac{c_0^2 - 2b_0d_0}{12b_0^2}\, e_k^3 + O(e_k^4),$$

which is precisely (4) upon substituting back $b_0 = f''(\alpha)$, $c_0 = f'''(\alpha)$, and $d_0 = f''''(\alpha)$. □

Remark 2 (Optimality in the sense of Traub)*.* Viewing the optimization problem as the root-finding problem $f'(x) = 0$, the iteration (1) is a one-point method that uses the three derivatives: $f'(x_k)$, $f''(x_k)$, and $f'''(x_k)$. A classical theorem of Traub [2] states that a one-point iteration based on n such quantities cannot have order exceeding n. Since Theorem 1 establishes order three, the present method attains this bound and is therefore optimal within the class of one-point methods using only $f'(x_k)$, $f''(x_k)$, and $f'''(x_k)$. Any method with an order of convergence higher than three would necessarily require additional information, such as higher derivatives or evaluations at additional points.

Remark 3 (Automatic cancellation of the quadratic term)*.* The vanishing of the second-order coefficient, $q_2 = 0$, is not imposed a priori; rather, it follows automatically from the order-by-order solution of the exact polynomial identity (2). Since $q_1 = -1$ already eliminates the linear error term, the additional cancellation of the quadratic term is precisely what promotes the method from the generic quadratic convergence of a Newton-type iteration to the cubic convergence established in Theorem 1.

Corollary 1 (Asymptotic error constant)**.** *The asymptotic error constant of the iteration is*

$$C = \lim_{k\to\infty} \frac{e_{k+1}}{e_k^3} = \frac{f'''(\alpha)^2 - 2f''(\alpha)f''''(\alpha)}{12\, f''(\alpha)^2}. \tag{8}$$

The expression is well defined under the standing hypothesis $f''(\alpha) \neq 0$.

III. Numerical Examples

Let $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$, so that $f'(x) = x^3 - x$, $f''(x) = 3x^2 - 1$, $f'''(x) = 6x$, $f''''(x) \equiv 6$. The point $\alpha = 1$ is a non-degenerate stationary point, with

$$b_0 = f''(1) = 2, \qquad c_0 = f'''(1) = 6, \qquad d_0 = f''''(1) = 6,$$

so that Theorem 1 predicts

$$C = \frac{6^2 - 2\cdot 2\cdot 6}{12\cdot 2^2} = \frac{36-24}{48} = \frac{1}{4}.$$

Iterating (1) from $x_0 = 1.3$ (so $e_0 = 0.3$) using 60-digit variable-precision arithmetic and the real (not principal complex) cube root, gives the errors and ratios in Table I.

The ratio e_k/e_{k-1}^3 converges to $C = 1/4$ to five significant figures by the third iterate, and the number of correct digits in e_k triples at each step (1, 3, 9, 25), both hallmarks of third-order convergence, in agreement with Theorem 1.

TABLE I
ERROR SEQUENCE AND SUCCESSIVE RATIOS e_k/e_{k-1}^3 FOR (1) APPLIED TO $f(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$ FROM $x_0 = 1.3$. THE RATIO CONVERGES TO THE PREDICTED CONSTANT $C = 1/4$.

k	$e_k = x_k - \alpha$	e_k/e_{k-1}^3
0	3.0000×10^{-1}	–
1	3.1967×10^{-3}	0.11840
2	8.0889×10^{-9}	0.24762
3	1.3232×10^{-25}	0.24999

IV. CONCLUSIONS

In this paper, we established the local convergence properties of the Ahmadi-Chaudhry-Zhang third-order Newton iteration for unconstrained optimization. By exactly eliminating the cube-root term through an equivalent cubic polynomial identity, we derived the local error equation without relying on asymptotic expansions of the radical. This led to a proof of third-order convergence and an asymptotic error constant, both verified numerically. We emphasize that Theorem 1 is a *local* convergence result in the classical sense of Traub [2]: it guarantees cubic convergence once the iterates enter a sufficiently small neighborhood of a non-degenerate stationary point α, but does not characterize the size of that neighborhood or the global behavior from distant initial points.

REFERENCES

[1] A. A. Ahmadi, A. Chaudhry, and J. Zhang. "Higher-order Newton methods with polynomial work per iteration." *Advances in Mathematics*, vol. 452, 2024, 109808.

[2] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1964.