Fraction-valued Equilibrium for Fraction-valued Bi-matrix Games

by

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Abstract

Using the well-known result about existence of equilibrium for bi-matrix games, we prove that a fraction-valued bi-matrix game has a fraction-valued equilibrium. Our proof is direct, and simpler than the one using the Lemke-Howson algorithm.

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1. **Introduction:** Most bi-matrix games considered in the social sciences have payoffs that are rational numbers (if not integers). Rational numbers are what are known more widely as fractions. Integers are fractions whose denominators are 1. In this note, we refer to such bi-matrix games as fraction-valued bi-matrix games.

A fraction-valued strategy for a player in a fraction-valued bi-matrix game is a probability mass function on the set of actions available to the player, such that the probability of every action is a fraction. A fraction-valued equilibrium for a fraction-valued bi-matrix game is a pair of fraction-valued strategies such that having chosen the strategies, no player can be better-off by unilaterally deviating from its chosen strategy.

Thanks to George Dantzig's (Dantzig (1951)) seminal contribution that proves the "equivalence" between the set of equilibrium of a matrix game and the the set of pairs of solutions of a related solvable linear programming problem and its dual, it can be easily shown that every "fraction-valued matrix game" has a fraction-valued equilibrium. More contemporary treatments of Dantzig's result is available in Adler (2013) and von Stengel (2024).

The result: "*every fraction-valued bi-matrix game has a fraction-valued equilibrium*", follows directly from the Lemke-Howson algorithm (see Lemke and Howson (1964)). In this note, we prove the same, using the well known result that every bi-matrix game has an equilibrium. Our proof is direct, and simpler than the one based on Lemke-Howson algorithm.

For basic concepts related to bi-matrix games one may refer to Chandrasekaran (nd). In this note, we refer to a solution of a system of linear equations as a "*basic solution*", if the columns in the coefficient matrix corresponding to non-zero coordinates of the solution are linearly independent

2. **The Framework of Fraction-valued Bi-matrix Games:** For positive integers m, n ≥ 2, a **bi-matrix game** is a pair (A, B)$\in\mathbb{R}^{m\times n}\times\mathbb{R}^{m\times n}$ where $\mathbb{R}^{m\times n}$ is the set of all real-valued m×n matrices. Thus, an m-dimensional real-valued column vector is a point in $\mathbb{R}^{m\times 1}$.

Notation: Given positive integers r, s and C$\in\mathbb{R}^{r\times s}$, for i$\in${1, …, r}let C_i denote the i^{th} row of C, for j$\in${1, …, s} let C^j denote the j^{th} column of C and for (i, j)$\in${1, …, r}×{1, …, s} let c_{ij} denote the $(i, j)^{th}$ entry of C. Let $C^T\in\mathbb{R}^{s\times r}$ denote the transpose of C. ▌

Let $\mathbb{Q}$ denote the set of all rational numbers (fractions) and $\mathbb{Q}^{m\times n}$ be the set of all "fraction-valued" m×n matrices.

For a positive integer r and i$\in${1, …, r}, let $E^{(r,i)}$ be the r-dimensional column vector whose i^{th} coordinate is 1 and all other coordinates. Let $E^{(r,i)}$ is the *r-dimensional i^{th} unit coordinate vector* and the r-dimensional column vector $E^{(r)} = \sum_{i=1}^{r} E^{(r,i)}$ is the *r-dimensional sum vector*.

For our purposes here, we will refer to a pair (A, B)$\in\mathbb{Q}^{m\times n}\times\mathbb{Q}^{m\times n}$ as a ***fraction-valued bi-matrix game***.

For any positive integer r, let $\Delta^{r-1} = \{\xi\in\mathbb{R}^r_+ \mid \sum_{j=1}^{r}\xi_j = 1\}$.

A **strategy for the row player** is a point in Δ^{m-1}, and a **strategy for the column player** is a point in Δ^{n-1}.

A **strategy profile** is a pair (x, y) where x is a strategy for the row player and y is a strategy for the column player.

A **fraction-valued strategy for the row player** is a point in $\Delta^{m-1} \cap \mathbb{Q}^m_+$ and a **fraction-valued strategy for the column player** is a point in $\Delta^{n-1} \cap \mathbb{Q}^n_+$.

A strategy profile (x, y) is an **equilibrium (strategy profile)** for the fraction-valued bi-matrix game (A, B) if for all (i, j)ϵ{1, …, m}×{1, ..., n}: $A_i y \leq x^T Ay$ and $x^T B^j \leq y$.

The following result is an immediate consequence of theorem 2 in Chandrasekaran (nd).

Theorem 1: There exists an equilibrium for the fraction-valued bi-matrix game (A, B).

We are concerned here with the following concept.

An equilibrium strategy profile (x, y) such that x is a fraction-valued strategy profile for the row player and y is a fraction-valued strategy profile for the column player is a **fraction-valued equilibrium strategy profile**.

3. **Fraction-valued Equilibrium for Fraction-valued Bi-matrix Games:** Let (A, B) be a fraction-valued bi-matrix game. Before we proceed to our main result let us recall the following two well-known results (available in https://drive.google.com/file/d/1MQx8DKtqv3vTj5VqPNw4wzi2Upf7JfCm/view?usp=sharing and/or https://www.academia.edu/44541645/The_essential_appendix_on_Linear_Programming).

Given positive integers r, s, let C$\epsilon\mathbb{R}^{r\times s}$ and let d be an s-dimensional column vector.

Result 1: Suppose that the columns of C are linearly independent. Then the columns of the s×s matrix C^TC are also linearly independent.

Thus, if the columns of C are linearly independent, then $(C^TC)^{-1}$ exists and the s×r matrix $(C^TC)^{-1}C^T$ denoted C^- is said to be the **pseudo-inverse** of C.

Note: If C is a fraction-valued matrix, the entries of C^- can be obtained from the entries of C by applying addition, subtraction, multiplication of fractions and division by non-zero fractions. Hence, if C is fraction-valued so is C^-.

Further, if for some x $\epsilon\mathbb{R}^s$, it is the case that Cx = d, then x = C^-d.∎

Result 2 (Non-negative pivot Theorem): If C ≠ 0, d ≠ 0, x$\epsilon\mathbb{R}^s_+$ satisfies Cx = d, then there exists $x^+\epsilon\mathbb{R}^s_+$ such that $Cx^+ = d^+$, $\{j|\ x_j^+ > 0\} \subset \{j|x_j > 0\}$ and the columns $\{C^j|x_j^+ > 0\}$ are linearly independent.

Theorem 2: Let (A, B)$\epsilon\mathbb{Q}^{m\times n} \times \mathbb{Q}^{m\times n}$ be a ***fraction-valued bi-matrix game***. Then, (A, B) has a ***fraction-valued equilibrium***.

Proof: By theorem 1 we know that (A, B) has an equilibrium. Let (x^*, y^*) be such an equilibrium. Let $u^* = x^{*T}Ay^*$ and let $v^* = x^{*T}By^*$. Then, $A_iy^* \leq u^*$ for all $i\epsilon\{1, \ldots, m\}$, $x^{*T}B^j \leq v^*$ for all $j\epsilon\{1, \ldots, n\}$, $\{i|x_i^* > 0\} \subset \{i|A_iy^* = u^*\}$ and $\{j|y_j^* > 0\} \subset \{j|x^{*T}B^j = v^*\}$. Since $(x^*, y^*)\epsilon\Delta^{m-1}\times\Delta^{n-1}$, it must be the case that $\{i|x_i^* > 0\} \neq \phi$ and $\{j|y_j^* > 0\} \neq \phi$. Thus, $\{i|A_iy^* = u^*\} \neq \phi$ and $\{j|x^{*T}B^j = v^*\} \neq \phi$.

If necessary, by interchanging rows and columns, we may without loss of generality suppose $A = \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix}$ where the rows of $A^{(1)}$ correspond to the rows in the array $<A_i|A_iy^* = u^*>$ and $B = (B^{(1)} \quad B^{(2)})$ where the columns of $B^{(1)}$ correspond to the columns in the array $<B^j|x^{*T}B^j = v^*>$.

Let $A^{(1)}$ be a matrix of size $m_1\times n$ where $m_1 \leq m$ and $B^{(1)}$ be a matrix of size $m\times n_1$ where $n_1 \leq n$.

Let A^+ be the $(m + 1)\times n$ matrix $\begin{bmatrix} A^{(1)} \\ A^{(2)} \\ E^{(n)T} \end{bmatrix}$ and let B^+ be the $m\times(n+1)$ matrix $(B^{(1)} \quad B^{(2)} \quad E^{(m)})$.

Thus, $A^+y^* = \begin{bmatrix} A^{(1)}y^* \\ A^{(2)}y^* \\ E^{(n)T}y^* \end{bmatrix} = \begin{bmatrix} u^*E^{(m_1)} \\ A^{(2)}y^* \\ 1 \end{bmatrix}$ and $x^{*T}B^+ = (x^{*T}B^{(1)} \quad x^{*T}B^{(2)} \quad x^{*T}E^{(m)}) = (v^*E^{(n_1)T} \quad x^{*T}B^{(2)} \quad 1)$.

By Result 2, there exists $(x^+, y^+)\epsilon\mathbb{R}_+^m \times \mathbb{R}_+^n$ satisfying:

$\{j|y_j^+ > 0\} \subset \{j|\ y_j^* > 0\}$, $A_iy^+ = u^*$ for all A_i in the matrix $A^{(1)}$, $A_iy^+ = A_iy^* < x^{*T}Ay^*$ for all A_i in the matrix $A^{(2)}$, $\sum_{j=1}^n y_j^+ = 1$, the columns in the array $<A^{+j}|y_j^+ > 0>$ are linearly independent,

&

$\{i|x_i^+ > 0\} \subset \{i|\ x_i^* > 0\}$, $x^{+T}B^j = v^*$ for all B^j in the matrix $B^{(1)}$, $x^{+T}B^j < x^{*T}By^*$ for all B^j in the matrix $B^{(2)}$, $\sum_{i=1}^m x_i^+ = 1$, the rows in the array $<B_i^+|x_i^+ > 0>$ are linearly independent.

If necessary, by interchanging rows and columns, we may without loss of generality assume that $\begin{bmatrix} A^{(1)} \\ A^{(2)} \\ E^{(n)T} \end{bmatrix} = \begin{bmatrix} A^{(1,1)} & A^{(2,1)} \\ A^{(2,1)} & A^{(2,2)} \\ E^{(n_1)T} & E^{(n-n_1)T} \end{bmatrix}$ and $y^+ = \begin{pmatrix} y^{+(1)} \\ y^{+(2)} \end{pmatrix}$, where the first block of columns of $\begin{bmatrix} A^{(1,1)} & A^{(2,1)} \\ A^{(2,1)} & A^{(2,2)} \\ E^{(n_1)T} & E^{(n-n_1)T} \end{bmatrix}$ and $y^{+(1)}$ comprise the coordinates corresponding to $\{j|y_j^+ > 0\}$.

Similarly suppose $\begin{pmatrix} B^{(1)} & B^{(2)} & E^{(m)} \end{pmatrix} = \begin{pmatrix} B^{(1,1)} & B^{(1,2)} & E^{(m_1)} \\ B^{(2,1)} & B^{(2,2)} & E^{(m-m_1)} \end{pmatrix}$ and $x^+ = \begin{pmatrix} y^{+(1)} \\ y^{+(2)} \end{pmatrix}$,

where the first block of rows of $\begin{pmatrix} B^{(1,1)} & B^{(1,2)} & E^{(m_1)} \\ B^{(2,1)} & B^{(2,2)} & E^{(m-m_1)} \end{pmatrix}$ and $x^{+(1)}$ comprise the coordinates corresponding to $\{i | x_i^+ > 0\}$.

Thus, $\begin{bmatrix} A^{(1,1)} \\ E^{(n_1)T} \end{bmatrix} y^{+(1)} = \begin{pmatrix} u^* E^{(m_1)} \\ 1 \end{pmatrix}$, $A_i^{(2,1)} y^{+(1)} < u^*$ for all $i \epsilon \{m_1 + 1, \ldots, m\}$, $y^{+(2)} = 0$,

& $x^{+(1)T} \begin{pmatrix} B^{(1,1)} & E^{(m_1)} \end{pmatrix} = \begin{pmatrix} v^* E^{(n_1)T} & 1 \end{pmatrix}$, $x^{+(1)T} B^{(1,2)j} < v^*$ for all $j \epsilon \{n_1 + 1, \ldots, n\}$, $x^{+(2)} = 0$.

Since the columns of $A^{(1,1)}$ are linearly independent, it must be the case that the columns of $\begin{bmatrix} A^{(1,1)} \\ E^{(n_1)T} \end{bmatrix}$ are linearly independent.

Since the rows of $B^{(1,1)}$ are linearly independent, it must be the case that the rows of $\begin{pmatrix} B^{(1,1)} & E^{(m_1)} \end{pmatrix}$ are linearly independent.

Thus, by Result 1, $\begin{bmatrix} A^{(1,1)} \\ E^{(n_1)T} \end{bmatrix}$ has a pseudo-inverse which we denote by M and $\begin{pmatrix} B^{(1,1)} & E^{(m_1)} \end{pmatrix}^T$ (i.e., the transpose of $\begin{pmatrix} B^{(1,1)} & E^{(m_1)} \end{pmatrix}$) has a pseudo-inverse which we denote by N.

Since, $(A, B) \epsilon \mathbb{Q}^{m \times n} \times \mathbb{Q}^{m \times n}$, M and N must be fraction-valued matrices.

Clearly, $y^{+(1)} = M \begin{pmatrix} u^* E^{(m_1)} \\ 1 \end{pmatrix}$ and $x^{+(1)} = N \begin{pmatrix} v^* E^{(n_1)} \\ 1 \end{pmatrix}$

Consider the formulas:

$z = M \begin{pmatrix} u E^{(m_1)} \\ 1 \end{pmatrix}$, $z \epsilon \mathbb{R}^{n_1}$

&

$w = N \begin{pmatrix} v E^{(n_1)} \\ 1 \end{pmatrix}$, $w \epsilon \mathbb{R}^{n_1}$.

By linearity, the dependence variable in each formula is determined by a continuous function of the independent variable in the same formula.

Thus, for u, $v \epsilon \mathbb{Q}$, sufficiently close to u^*, v^* respectively:

$z = M \begin{pmatrix} u E^{(m_1)} \\ 1 \end{pmatrix} \epsilon \Delta^{(n_1 - 1)} \cap \mathbb{Q}_{++}^{n_1}$ and $A_i^{(2,1)} z < u$ for all $i \epsilon \{m_1 + 1, \ldots, m\}$

&

$w = N \begin{pmatrix} v E^{(n_1)} \\ 1 \end{pmatrix} \epsilon \Delta^{(m_1 - 1)} \cap \mathbb{Q}_{++}^{m_1}$ and $w^T B^{(1,2)j} < v$ for all $j \epsilon \{n_1 + 1, \ldots, n\}$.

Let $y^{(1)} \epsilon \Delta^{(n_1-1)} \cap \mathbb{Q}_{++}^{n_1}$ and $x^{(1)} \epsilon \Delta^{(m_1-1)} \cap \mathbb{Q}_{++}^{m_1}$ satisfy the two conditions above and let $y = \begin{pmatrix} y^{(1)} \\ 0 \end{pmatrix} \epsilon \mathbb{R}^n$, $x = \begin{pmatrix} x^{(1)} \\ 0 \end{pmatrix} \epsilon \mathbb{R}^m$.

Thus, $x^T A y = x^{(1)T} A^{(1,1)} y^{(1)} = x^{(1)T} u E^{(m_1)} = u$ and $x^T B y = x^{(1)T} B^{(1,1)} y^{(1)} = v E^{(n_1)} y^{(1)} = v$.

Further, $A_i y = A_i^{(1,1)} y^{(1)} = u$ for $i \epsilon \{1, \ldots, m_1\}$, $A_i y = A_i^{(2,1)} y^{(1)} < u$ for $i \epsilon \{m_1 + 1, \ldots, m\}$, $x^T B^{(j)} = x^{(1)T} B^{(1,1)j} = v$ for $j \epsilon \{1, \ldots, n_1\}$, $x^T B^{(j)} = x^{(1)T} B^{(1,2)j} < v$ for $j \epsilon \{n_1 + 1, \ldots, n\}$.

Thus, (x, y) is a fraction-valued equilibrium for (A, B). Q.E.D.

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