Objective generalized covariant derivative with respect to time in time-varying curvilinear coordinates

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Abstract

This paper develops an objective generalized covariant derivative with respect to time from the perspective of space-time duality. It is formulated in arbitrary time-varying curvilinear coordinate systems and serves as a temporal counterpart of the spatial covariant derivative. Reference bases generated by the Eulerian-Lagrangian tensor are used to construct the temporal covariant derivative. The resulting unified component form contains temporal connection terms determined by the Eulerian-Lagrangian tensor, whereas the spatial connection terms involving Christoffel symbols cancel systematically in the corresponding tensor-entity expansion. Hence, this covariant temporal derivative can be expressed without spatial Christoffel symbols even in time-varying curvilinear coordinates. Classical objective rates, including the Oldroyd, Cotter-Rivlin, Green-Naghdi, and Jaumann rates, are unified and interpreted within this framework. In particular, the Green-Naghdi rate shows that such a Christoffel-free component structure is not restricted to Lie-type objective rates, but can also be obtained for a non-Lie-type rate. The formulation clarifies the common covariant structure of Lie-type and non-Lie-type objective rates, and extends the construction from ordinary tensor components to generalized components.

Keywords: tensor analysis, objectivity, generalized covariance, objective covariant derivative with respect to time, space-time

1 Introduction

The concept of covariance lies at the core of Ricci's covariant differential calculus [1–3]. In recent years, the concept of covariance has undergone further development. Yin [4–6] introduced the notion of generalized covariance, in which the term "generalized" refers to the extension from ordinary tensor components to generalized components including covariant and contravariant base vectors (as recalled in Subsection 8.1). Furthermore, Yin [7, 8] extended covariant derivatives from the space field to the time field through the principle of "space-time duality". Specifically, Yin defined generalized covariant derivatives with respect to time for tensor components and base vectors within both Eulerian [7] and Lagrangian [8] coordinate systems. However, due to the limited generality of these coordinate systems, the covariant derivatives proposed by Yin have not been connected to the concept of "objectivity" [9] proposed in large deformation continuum mechanics. It is important to note that the "principle of objectivity" serves as a fundamental basis for constitutive analysis [9–15]. Without establishing a link to the principle of objectivity, the applicability of "the covariance based on space-time duality" and "generalized covariance" within continuum mechanics remains significantly constrained.

The question raised in this paper is whether there exists a system that is dual to the spatial covariant derivative and can formulate time derivatives satisfying the condition of objectivity. Establishing such a system would connect objective derivatives with the spatial covariant derivative, thereby reducing the conceptual and analytical difficulties associated with objective derivatives and facilitating their further development and application. Furthermore, the "generalized covariance" of the spatial covariant derivative [4–6] can then be extended to the study of objective derivatives. More broadly, this question also concerns the relationship among objectivity, covariance, and Lie derivatives.

When Truesdell and Noll [9] defined the concept of objectivity in their treatise in 1965, they did not introduce coordinate systems, which posed certain challenges in expressing objectivity using the terminology of coordinate systems and covariance. However, this indicates from another perspective that the notion of objectivity fundamentally stems from physical principles rather than being derived from mathematical formalisms or presuppositions. Marsden and Hughes [16] systematically expounded the relationship between covariance and objectivity from the perspective of the Lie derivative and explicitly stated that "all so-called objective rates of second-order tensors are in fact Lie derivatives". Subsequently, the relationship among objective derivatives, Lie derivatives, and covariance was further discussed by other scholars [17–20]. In 2024, Kolev and Desmorat [21] systematically pointed out that **objective rates do not all fall within the class of Lie derivatives and their linear combinations**. Kolev and Desmorat demonstrated that under a natural locality assumption, objective rates can be understood as covariant derivatives on the manifold formed by all Riemannian metrics defined on the body [21].

Based on the idea of "space-time duality", this paper places particular emphasis on the physical mechanism for the generation of objective derivatives and on their explicit component expressions in general time-varying curvilinear coordinate systems. Although Lie-type objective rates in time-varying curvilinear coordinate systems have been investigated [16, 17, 22], **for such time-varying curvilinear systems, explicit component representations of non-Lie-type objective rates, such as the Green-Naghdi rate, do not appear to have been established. More fundamentally, when such rates are expressed in general time-varying curvilinear coordinate systems, it remains open whether the resulting spatial connection terms cancel completely, yielding component representations free of spatial Christoffel symbols.** This paper provides an affirmative answer to this question.

The framework proposed in this paper also clearly reflects the relationship between covariance and the invariance of tensor entities, thereby facilitating both representation and application. Specifically, from the perspective of space-time duality, this paper develops a temporal covariant-derivative structure corresponding to the spatial covariant derivative and extends it to arbitrary time-varying curvilinear coordinate systems. This covariant derivative is termed the objective covariant derivative with respect to time. In this form, the spatial connection terms involving Christoffel symbols are systematically canceled in the expansion of the tensor entity, whereas the temporal connection terms compensate for the non-covariance of the ordinary time derivative. Moreover, the results for tensor components are then extended to generalized components, and an objective combined-type covariant derivative with respect to time is constructed by means of a conjugate correction.

From a physical point of view, objectivity means that no particular reference frame is privileged; in the formulation of this paper, this idea is expressed through the relative derivatives induced by Eulerian-Lagrangian tensors (cross-time tensors; see Section 4 for details).

Within the framework of this paper, the Oldroyd rate, which is a Lie derivative, the Cotter-Rivlin rate, the Green-Naghdi rate, and related objective rates can be interpreted through the basic temporal covariant-derivative structure. In particular, the Green-Naghdi rate represents a class of objective rates that are not of Lie type. The common features and differences between such non-Lie-type rates and Lie-type rates, both in physical meaning and in covariant form, can therefore be made explicit in the present framework. The Jaumann rate, in turn, can be incorporated into the same algebraic framework through the objective combined-type covariant derivative with respect to time.

The structure of this paper is as follows. Sections 2–5 introduce and refine several conceptual frameworks, thereby facilitating the description of Truesdell's notion of objectivity within the context of covariance centered on coordinate systems and the Ricci transformation. In Section 6, within rectangular coordinate systems, we finalize the formulation of the objective covariant derivative with respect to time and derive the corresponding tensor entity that satisfies Truesdell's definition of objectivity. Section 7 extends the analysis to general curvilinear coordinate systems, where we derive the general expression for the objective covariant derivative with respect to time. In Sections 8–10, the results on tensor components are extended to generalized components, and the explicit forms of the related objective rates, as well as their relation to Lie derivatives, are discussed.

2 Four types of coordinate systems

The encounter of covariance and objectivity hinges on the extension of stationary coordinate systems to the active coordinate systems. Although the stationary coordinate system is more commonly known, the general active coordinate system is less frequently used. This section aims to provide a systematic overview of four types of coordinate systems to support their appropriate use in subsequent analyses.

- **Fixed rectangular coordinate systems**: Cartesian rectangular coordinate systems that remain stationary relative to the inertial reference system.

In classical tensor analysis, the "fixed rectangular coordinate system" is also referred to as the "Eulerian rectangular coordinate system".

- **Rotating rectangular coordinate systems**: Coordinate systems that undergo rigid body rotation relative to fixed rectangular coordinate systems. In this paper, it is assumed that the rotation angle of the rotating rectangular system is a continuously varying and infinitely differentiable function of time.

In the field of computational mechanics, the "corotational rectangular coordinate system" [23] is frequently employed, which represents a specific instance of rotating rectangular coordinate systems, wherein the coordinate axes rotate in synchrony with the motion of the object being analyzed. However, in the general case of rotating rectangular systems, the rotational motion of the system relative to the inertial frame can follow an arbitrary trajectory, as illustrated on the right-hand side of Fig. 1.

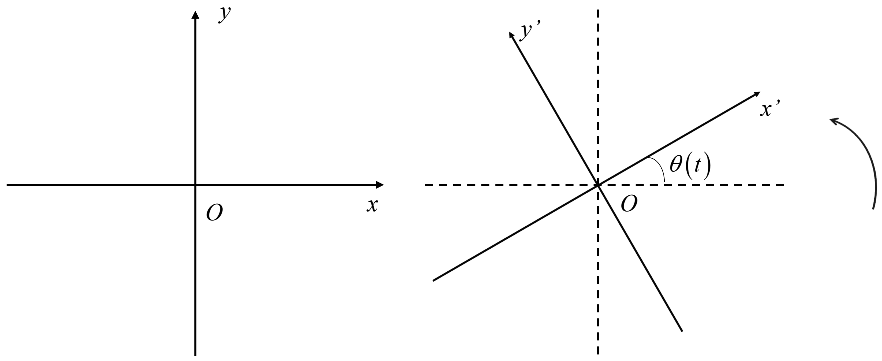

Fig. 1 the fixed rectangular coordinate system (left) and the rotating rectangular coordinate system (right)

- **Fixed curvilinear coordinate systems**: Curvilinear coordinate systems that remain stationary relative to the inertial reference system.

In classical tensor analysis, the "fixed curvilinear coordinate system" is also referred to as the "Euler curvilinear coordinate system".

- **Time-varying curvilinear coordinate systems**: Curvilinear coordinate systems that move with respect to an inertial frame and vary continuously over time.

In the case of a general time-varying curvilinear coordinate system, the evolution of the coordinate system is arbitrary and independent of the motion and deformation of the object being analyzed, as illustrated on the right-hand side of Fig. 2. In fact, a fixed curvilinear coordinate system may be viewed as a special case of a time-varying curvilinear coordinate system, where the motion and deformation of the coordinate system relative to the inertial frame reduce to zero.

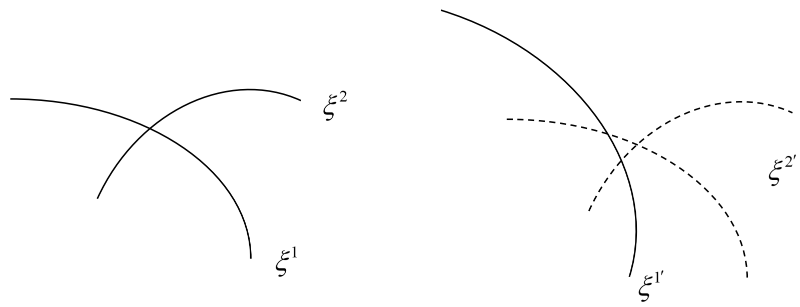

Fig. 2 the fixed curvilinear coordinate system (left) and the time-varying curvilinear coordinate system (right)

Based on the aforementioned definitions, we have generalized the concept from the well-known fixed coordinate system and embedded coordinate system to the broader framework of the active coordinate system. Accordingly, objectivity can be rigorously formulated through covariance across coordinate systems.

3 Covariant base vectors of the four types of coordinate systems

The covariant base vector is a fundamental concept in classical tensor analysis, and it is induced by the natural coordinate system. To facilitate further discussion, covariant base vectors of all four coordinate systems outlined in the preceding section are formally introduced.

The covariant base vectors of a fixed rectangular coordinate system are a set of standard orthogonal bases and remain invariant over time. If x^i denotes the fixed rectangular coordinates, the covariant base vectors are defined as

$$\boldsymbol{e}_i \triangleq \frac{\partial \boldsymbol{r}}{\partial x^i}. \tag{1}$$

The covariant base vectors of a rotating rectangular coordinate system are also an orthonormal set, while they evolve over time as the system rotates. If coordinates of the rotating coordinate system are denoted by x^j, the covariant base vectors are defined as

$$\boldsymbol{e}_j(t) \triangleq \frac{\partial \boldsymbol{r}}{\partial x^j}(t). \tag{2}$$

For the convenience of subsequent discussion, we pay particular attention to the covariant base vectors of the rotating rectangular coordinate system at the initial moment, which are denoted as

$$\boldsymbol{e}_j(t_0) \triangleq \frac{\partial \boldsymbol{r}}{\partial x^j}(t_0) \triangleq \boldsymbol{e}_{j_0}. \tag{3}$$

The covariant base vectors at the current time $\boldsymbol{e}_i$ and those at the initial time $\boldsymbol{e}_{j_0}$ can constitute a "hybrid" basis $\boldsymbol{e}_i\boldsymbol{e}_{j_0}$, which plays a significant role in the representation of second-order tensors under certain conditions.

The covariant base vectors of a fixed curvilinear coordinate system are a set of "constant" base vectors, where "constant" means "not varying with time". If the coordinates of the fixed coordinate system are denoted by θ^i, the covariant base vectors are defined as

$$\boldsymbol{g}_i \triangleq \frac{\partial \boldsymbol{r}}{\partial \theta^i}. \tag{4}$$

The covariant base vectors of a time-varying curvilinear coordinate system are a set of base vectors that evolve over time. If the coordinates of the time-varying system are denoted by θ^j, then the covariant base vectors can be defined as

$$\boldsymbol{g}_j(t) \triangleq \frac{\partial \boldsymbol{r}}{\partial \theta^j}(t). \tag{5}$$

Similar to Eq. (3), we denote the covariant base vectors of the time-varying curvilinear coordinate system at the initial moment as

$$\boldsymbol{g}_j(t_0) \triangleq \frac{\partial \boldsymbol{r}}{\partial \theta^j}(t_0) \triangleq \boldsymbol{g}_{j_0}. \tag{6}$$

4 Eulerian-Lagrangian and Eulerian-Eulerian tensor entities

The "two-point tensor" is a classical concept in continuum mechanics [24]. Based on this, researchers have further classified the tensors in continuum mechanics into four types: Eulerian-Eulerian tensors, Eulerian-Lagrangian tensors, Lagrangian-Eulerian type tensors, and Lagrangian-Lagrangian tensors [23, 25]. In the following, we provide a further examination of the meanings of Eulerian-Eulerian tensors and Eulerian-Lagrangian tensors.

Consider a rotating rectangular coordinate system. Generally, a second-order tensor $\boldsymbol{T}$ can be decomposed within a rotating rectangular coordinate system as

$$\boldsymbol{T}(t) = T_{ij}(t)\,\boldsymbol{e}_i(t)\,\boldsymbol{e}_j(t) \triangleq T_{ij}\boldsymbol{e}_i\boldsymbol{e}_j. \tag{7}$$

Since the intrinsic properties of a tensor entity are independent of the choice of the frames, the tensor can also be decomposed using the hybrid basis introduced in the preceding section as

$$\boldsymbol{T}\left(t\right)=T_{ij_0}\left(t\right)\boldsymbol{e}_i\left(t\right)\boldsymbol{e}_j\left(t_0\right)\triangleq T_{ij_0}\boldsymbol{e}_i\boldsymbol{e}_{j_0}. \tag{8}$$

The second base vector $\boldsymbol{e}_{j_0}$ in the hybrid frame is the covariant base vector of the rotating rectangular coordinate system at the initial time, as defined in Eq. (3). The components T_{ij} presented in Eq. (7) can be referred to as the **Eulerian-Eulerian components** of the second-order tensor $\boldsymbol{T}$, while T_{ij_0} in Eq. (8) are referred to as the **Eulerian-Lagrangian components** of tensor $\boldsymbol{T}$. In a similar fashion, the Lagrangian-Eulerian components of the tensor can also be defined.

For clarity and ease of understanding, we take the deformation gradient tensor $\boldsymbol{F}$ in the field of continuum mechanics as a representative example for analysis, whose definition is

$$\mathrm{d}\boldsymbol{x}=\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X}. \tag{9}$$

$\mathrm{d}\boldsymbol{X}$ is a fixed "snapshot" at the initial moment that cannot evolve over time, while $\mathrm{d}\boldsymbol{x}$ is a time-varying vector. Therefore, Eq. (9) can also be expressed as

$$\mathrm{d}\boldsymbol{x}\left(t\right)=\boldsymbol{F}\left(t\right)\cdot\mathrm{d}\boldsymbol{X}\left(t_0\right). \tag{10}$$

Suppose an observer is situated within a rotating rectangular coordinate system and moves together with it. Then, since the vector $\mathrm{d}\boldsymbol{X}$ only exists at the initial moment, the observer can only obtain its components under the covariant base vectors of the rotating rectangular coordinate system at the initial moment, which are the components $\mathrm{d}X_{i_0}$. In contrast, for the vector $\mathrm{d}\boldsymbol{x}$, the observer can obtain its components under the covariant base vectors of the system at the current moment, which are the components $\mathrm{d}x_i$. By incorporating Eq. (9), it becomes evident that from the observer's perspective, only the components F_{ij_0} of the deformation gradient tensor $\boldsymbol{F}$ can be directly obtained—that is the Eulerian-Lagrangian components of tensor $\boldsymbol{F}$. Furthermore, if there exists an external observer located in a fixed (inertial) frame, who can provide the transformation relationship between the rotating and fixed rectangular coordinate systems throughout the entire motion, it becomes possible to obtain the Eulerian-Eulerian components F_{ij} of tensor $\boldsymbol{F}$.

In fact, both types of components of the deformation gradient tensor play significant roles in the field of computational mechanics, where a "global coordinate system" is usually employed—one that remains relatively stationary with respect to the inertial frame—and this system serves the role of the aforementioned external observer.

In the absence of reliance on a fixed rectangular coordinate system, if an observer situated in a rotating rectangular coordinate system can only obtain the Eulerian-Lagrangian components of a second-order tensor $\boldsymbol{T}$, then the tensor entity $\boldsymbol{T}$ is referred to as an **Eulerian-Lagrangian tensor entity**. The deformation gradient tensor $\boldsymbol{F}$ serves as the most typical example.

Similarly, within a rotating rectangular coordinate system and without reference to a fixed one, if an observer can only access the Eulerian-Eulerian components of a second-order tensor $\boldsymbol{T}$, then $\boldsymbol{T}$ is classified as an **Eulerian-Eulerian tensor entity**. A canonical example is the Cauchy stress tensor $\boldsymbol{\sigma}$.

From the above analysis, it can be seen that on the surface, the "Eulerian-Eulerian" and "Eulerian-Lagrangian" classifications pertain to distinctions in tensor components. However, at a more fundamental level, the characteristics of the Eulerian-Lagrangian tensor stem from the "time-travel" nature of the tensor entity itself. Therefore, we refer to the Eulerian-Lagrangian tensor entity as a "**cross-time tensor entity**". "cross-time" is an intrinsic property of the tensor entity and remains invariant regardless of the coordinate systems and the choice of frames. The core functionality of a cross-time tensor lies in its ability to perform a "time-travel" transformation—mapping a physical quantity, which exists solely at an initial time point and therefore cannot evolve with time, to a physical quantity existing at the current time, evolving dynamically with the progression of time. This functionality plays a significant role in the construction in Section 6.

5 Truesdell's objectivity

This paper focuses on the objectivity of Eulerian-Eulerian tensors. It is important to note that Truesdell's definition of tensor objectivity was originally formulated with reference to Eulerian-Eulerian tensors.

The central concept in tensor objectivity lies in the transformation of tensors across different reference frames. According to Truesdell, for each moving reference frame, there exists a corresponding set of base vectors that are stationary relative to that frame. These base vectors enable the transformation of tensors between distinct moving frames [9]. Notably, Truesdell's formulation of objectivity relies solely on the notion of base vectors, without invoking coordinate systems. In contrast, the present paper employs covariant base vectors derived from coordinate systems. When tensor components are transformed across these covariant base vectors, the principle of covariance becomes relevant. Under this framework, Truesdell's definition of objectivity can be expressed accordingly.

If the components of a second-order tensor in two arbitrary rotating rectangular coordinate systems satisfy the corresponding coordinate transformation relations, i.e.,

$$T_{i'j'}\left(t\right) = \frac{\partial x^{i'}}{\partial x^{m}}\left(t\right)\frac{\partial x^{j'}}{\partial x^{n}}\left(t\right)T_{mn}\left(t\right), \tag{11}$$

where $T_{i'j'}$ and T_{mn} are the components in the two arbitrary rotating rectangular coordinate systems, then the tensor $\boldsymbol{T}$ is said to satisfy **objectivity**. This definition can be easily extended to tensors of arbitrary order. Since the objectivity pertains exclusively to orthogonal transformations, it may also be referred to as **orthogonal objectivity**. The collection of all tensors that satisfy the classical orthogonal objectivity constitutes a proper subset of the set of all tensors.

By calculating the particle derivative of both sides of the transformation relation Eq. (11), it becomes evident that the particle derivatives of the components of an objective tensor no longer conform to the coordinate transformation relation (11). Consequently, these derivatives are no longer components of an objective tensor, meaning they fail to constitute a tensor entity that satisfies objectivity. This explanation is presented in terms of components. When expressed in terms of tensor entities, it implies that the particle derivative of a classical objective tensor entity, although still a tensor entity, does not satisfy the condition of objectivity.

The construction in this paper is guided by the requirement of the classical orthogonal objectivity, as defined by Truesdell. Further investigations into the different levels of objectivity will be addressed in subsequent studies.

6 Objective covariant derivative with respect to time for tensor components within rectangular coordinate systems

From this section, based on the concept of space-time duality, we start to construct tensor entities that satisfy orthogonal objectivity, whose components are referred to as **objective covariant derivatives with respect to time**. Our construction methodology closely parallels that of spatial covariant derivatives.

The particle derivative of an objective tensor no longer preserves objectivity. The fundamental reason for this lies in the fact that the particle derivative represents a physical quantity that is dependent on the inertial frame of reference. From the standpoint of the principle that all reference frames are equivalent, such inertial-frame-dependent quantities are inherently devoid of physical meaning. In order to construct tensor entities that maintain classical objectivity, it becomes necessary to introduce a new type of derivative that eliminates this dependence on the inertial frame. Accordingly, we define a "relative derivative" with respect to a set of "reference base vectors."

6.1 Notation conventions and some preparations within rectangular coordinate systems

Throughout Section 6, the following conventions are adopted unless otherwise stated: Let x^i and $x^{i'}$ denote the coordinates in the fixed and rotating rectangular coordinate systems, respectively; $\boldsymbol{e}_i$ represents the covariant base vectors in the fixed rectangular coordinate system, while $\boldsymbol{e}_{i'}$ represents the covariant base vectors in the rotating rectangular coordinate system. Then in fixed and rotating rectangular coordinate systems, there is, respectively

$$
\begin{cases}
\boldsymbol{e}_i \triangleq \dfrac{\partial \boldsymbol{r}}{\partial x^i}, \\
\boldsymbol{e}_{i'} \triangleq \dfrac{\partial \boldsymbol{r}}{\partial x^{i'}}.
\end{cases} \tag{12}
$$

Thus, the base vectors of the rotating and fixed rectangular coordinate systems satisfy

$$
\boldsymbol{e}_{i'} \triangleq \frac{\partial \boldsymbol{r}}{\partial x^{i'}} = \frac{\partial \boldsymbol{r}}{\partial x^j}\frac{\partial x^j}{\partial x^{i'}} = \frac{\partial x^j}{\partial x^{i'}}\boldsymbol{e}_j. \tag{13}
$$

Introducing $\partial x^j / \partial x^{i'} = Q_{ji'}$, we have

$$
\begin{aligned}
\boldsymbol{e}_{i'} &= Q_{ji'}\boldsymbol{e}_j, \\
\boldsymbol{e}_j &= Q_{i'j}\boldsymbol{e}_{i'}.
\end{aligned} \tag{14}
$$

Taking the dot product of both sides of Eq. (14) with the base vectors, we obtain

$$
\begin{aligned}
Q_{ij'} &= \boldsymbol{e}_i \cdot \boldsymbol{e}_{j'}, \\
Q_{i'j} &= \boldsymbol{e}_{i'} \cdot \boldsymbol{e}_j.
\end{aligned} \tag{15}
$$

The commutative law of the dot product in Eq. (15) yields that

$$
\begin{aligned}
Q_{ij'} &= Q_{j'i}, \\
Q_{i'j} &= Q_{ji'}.
\end{aligned} \tag{16}
$$

With this property, for the sake of convenience in notation, we can uniformly place the primed indices at the front, and there exist

$$
\begin{aligned}
Q_{i'k}Q_{j'k} = Q_{ki'}Q_{j'k} = \frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^k} = \delta_{ij}, \\
Q_{k'i}Q_{k'j} = Q_{ik'}Q_{k'j} = \frac{\partial x^i}{\partial x^{k'}}\frac{\partial x^{k'}}{\partial x^j} = \delta_{ij}.
\end{aligned} \tag{17}
$$

For the components u_j and $u_{i'}$ of an arbitrary vector $\boldsymbol{u}$, which denote the components within fixed and rotating rectangular coordinate systems respectively, there are

$$
\begin{aligned}
u_{i'} &= Q_{i'j}u_j, \\
u_j &= Q_{ji'}u_{i'}.
\end{aligned} \tag{18}
$$

6.2 The reference base vector

For an Eulerian-Lagrangian regular tensor $\boldsymbol{A}$, leveraging its "cross-time" property, we can obtain a set of time-dependent **reference base vectors** through the inner product of the tensor $\boldsymbol{A}$ with the base vectors of a fixed rectangular coordinate system, i.e.

$$\boldsymbol{E}_i = \boldsymbol{A} \cdot \boldsymbol{e}_i = A_{mi} \boldsymbol{e}_m. \tag{19}$$

Transforming the base vectors of the fixed rectangular coordinate system into those of the rotating rectangular coordinate system, we can obtain the representation of $\boldsymbol{E}_i$ under the base vectors of the rotating rectangular coordinate system, i.e.

$$\boldsymbol{E}_i = A_{ni} \boldsymbol{e}_n = A_{ni} Q_{m'n} \boldsymbol{e}_{m'} = A_{m'i} \boldsymbol{e}_{m'} = A_{m'j_0'} Q_{j_0'i} \boldsymbol{e}_{m'}, \tag{20}$$

In the above equation, $A_{m'j_0'}$ denotes the Eulerian-Lagrangian components of $\boldsymbol{A}$ in the rotating rectangular coordinate system. $Q_{j_0'i}$ signifies the coordinate transformation coefficient between the initial configuration of the rotating rectangular coordinate system and the current configuration of the fixed rectangular coordinate system.

$$Q_{j_0'i} = \frac{\partial x^{j'}(t_0)}{\partial x^i} = \boldsymbol{e}_{j'}(t_0) \cdot \boldsymbol{e}_i. \tag{21}$$

It can be inferred from the definition in Eq. (21) that $Q_{j_0'i}$ is constant over time.

For the convenience of subsequent operations, it is also feasible to decompose the base vectors of both fixed and rotating rectangular coordinate systems under reference base vectors. Multiplying both sides of Eq. (19) by the components $\left(A^{-1}\right)_{in}$ of tensor $\boldsymbol{A}^{-1}$ yields that

$$\left(A^{-1}\right)_{in} \boldsymbol{E}_i = \boldsymbol{e}_n. \tag{22}$$

Multiplying both the left and right ends of Eq. (20) by $(A^{-1})_{k_0'n'} Q_{k_0'i}$ yields that

$$(A^{-1})_{k_0'n'} Q_{k_0'i} \boldsymbol{E}_i = (A^{-1})_{k_0'n'} A_{m'k_0'} \boldsymbol{e}_{m'} = \boldsymbol{e}_{n'}. \tag{23}$$

6.3 The relative derivative

We introduce the **relative derivative**, denoted by $\overset{\circ}{(\ )}$, to characterize the rate of change of a physical quantity relative to a set of reference base vectors over time. By convention, the particle derivative in an inertial frame is represented by $\dot{(\ )}$ or $\frac{\mathrm{d}}{\mathrm{d}t}$, the two symbols equivalent in meaning in this paper. The operational rules for calculating

the relative derivative of a physical quantity relative to the reference base vectors are presented as follows.
(i) The relative time derivatives of tensor components are equal to their particle derivatives. Considering the vector component u_i as an example, there is

$$\overset{\circ}{u}_i = \dot{u}_i. \tag{24}$$

(ii) The relative derivatives of the covariant base vectors can be derived by first calculating their particle derivatives and subsequently omitting the terms arising from the reference base vector's particle derivative. Incorporating Eq. (22), we derive that the relative derivatives of the base vectors of fixed rectangular coordinate systems are the following.

$$\overset{\circ}{\boldsymbol{e}}_i = \overset{\cdot}{\overbrace{(A^{-1})_{ji}}}\boldsymbol{E}_j = \overset{\cdot}{\overbrace{(A^{-1})_{ji}}}A_{kj}\boldsymbol{e}_k. \tag{25}$$

In the second equality of the above equation, we applied Eq. (19) to replace the reference base vectors $\boldsymbol{E}_j$ with the covariant base vectors $\boldsymbol{e}_k$.
Correspondingly, considering that $Q_{k_0'j}$ remains constant over time, the relative derivatives of the base vectors of the rotating rectangular coordinate system can be derived from Eq. (23) as follows:

$$\overset{\circ}{\boldsymbol{e}}_{i'} = \overset{\cdot}{\overbrace{(A^{-1})_{k_0'i'}}}Q_{k_0'j}\boldsymbol{E}_j = \overset{\cdot}{\overbrace{(A^{-1})_{k_0'i'}}}Q_{k_0'j}A_{m'n_0'}Q_{n_0'j}\boldsymbol{e}_{m'} = \overset{\cdot}{\overbrace{(A^{-1})_{j_0'i'}}}A_{k'j_0'}\boldsymbol{e}_{k'}. \tag{26}$$

In the second equality of the above equation, we applied Eq. (20) to replace the reference base vectors $\boldsymbol{E}_j$ with the covariant base vectors $\boldsymbol{e}_{m'}$.
(iii) The operation of relative derivatives satisfies the Leibniz rule.

These three fundamental properties guarantee the computational feasibility of relative derivatives and accurately capture their physical meaning.

6.4 The perspective of space-time symmetry

It is important to note that in Eqs. (25) and (26), the relative derivatives of the base vectors $\boldsymbol{e}_i$ of the fixed rectangular coordinate systems and $\boldsymbol{e}_{i'}$ of the rotating rectangular coordinate systems are decomposed respectively under the base vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_{i'}$ themselves. In other words, the results of the relative derivatives are decomposed under the operational objects of the relative derivatives. In fact, this is precisely an imitation of the construction process of the spatial covariant derivative.

Recall the definition formula of the Christoffel symbol within the general curvilinear coordinate systems, i.e.

$$\frac{\partial \boldsymbol{g}_i}{\partial \theta^j} \triangleq \Gamma_{ij}^k \boldsymbol{g}_k, \tag{27}$$

where θ^j is the general curvilinear coordinate. The above equation implies that there are three key points in the construction process of the spatial covariant derivative.
(i) Calculating the spatial derivative of the covariant base vector.
(ii) The derivative obtained is decomposed under the covariant base vectors themselves.
(iii) The decomposition result of the derivative in different coordinate systems needs to be written in a unified form, that is the Christoffel symbol Γ_{ij}^k. In classical tensor analysis, the Christoffel symbol in a general curvilinear system can be calculated through the components g_{ij} of the metric tensor.

After that, the process of defining the covariant derivative by combining the ordinary partial derivative of the tensor entity with the Christoffel symbol terms becomes straightforward, where the covariance of the resulting derivative is automatically ensured.

Based on Eqs. (25) and (26), the first two key points of the spatial covariant derivative are effectively captured and fulfilled. We now turn our attention to the third key aspect. Notably, because the base vectors in a fixed coordinate system remain constant over time, the Eulerian-Lagrangian and Eulerian-Eulerian components within such a system are equal. Consequently, the expression on the far right of Eq. (26) can be seen as a unified formulation applicable to both fixed and rotating rectangular coordinate systems. This implies that the third key point of the spatial covariant derivative is also effectively captured. Furthermore, the physical reasoning behind this unified form is relatively straightforward. Since the derivative is calculated relative to the reference base vectors, inertial and non-inertial systems are treated equivalently throughout the calculation. As a result, the fixed rectangular coordinate system tied to an inertial frame does not possess any privileged status. The relative derivative calculation thereby establishes complete equivalence between fixed and rotating rectangular coordinate systems. This equivalence, as reflected in the unified mathematical form, signifies a seamless integration of physical concepts and mathematical representation.

From the preceding discussion, it can be seen that the coefficients of the base vectors on the right-hand side of Eq. (26), $\overset{\centerdot}{\overbrace{(A^{-1})}}_{j_0' i'} A_{k' j_0'}$, correspond to the Christoffel symbols in the time field. Accordingly, we can define the **Christoffel symbols in the time field** within the rotating rectangular coordinate system as follows:

$$\tilde{\Gamma}_{k' i'} \triangleq A_{k' j_0'} \overset{\centerdot}{\overbrace{(A^{-1})}}_{j_0' i'}. \tag{28}$$

In the fixed rectangular coordinate system, which can be regarded as a special case of the rotating rectangular coordinate system, Eq. (28) remains valid. To differentiate the temporal Christoffel symbol from its spatial counterpart, a wavy line is employed. Consequently, Eq. (26) can be reformulated as follows:

$$\overset{\circ}{\boldsymbol{e}}_{i'} = \tilde{\Gamma}_{k'i'}\boldsymbol{e}_{k'}. \tag{29}$$

Eqs. (27) and (29) exhibit a high degree of spatio-temporal symmetry.

The spatial Christoffel symbols represent the spatial variation of the covariant base vectors, while the temporal Christoffel symbols describe the temporal evolution of the covariant base vectors. The problems corresponding to spatial and temporal Christoffel symbols exhibit a natural symmetry. Within rectangular coordinate systems, the covariant base vectors change only with time and remain constant with respect to spatial coordinates, thereby exhibiting the most transparent form of space-time symmetry.

In rectangular coordinate systems, the two indices of the temporal Christoffel symbols differ only in their order, not in their upper or lower positions. In Section 7, the temporal Christoffel symbols will be extended to curvilinear coordinate systems, where the distinction between upper and lower indices becomes apparent.

6.5 Objective covariant derivative with respect to time for vector components in rectangular coordinate systems

For a vector $\boldsymbol{q}$, within the fixed and rotating rectangular coordinate systems, there are respectively

$$\begin{cases} \boldsymbol{q} = q_i\boldsymbol{e}_i, \\ \boldsymbol{q} = q_{i'}\boldsymbol{e}_{i'}. \end{cases} \tag{30}$$

Calculating the relative derivative of vector $\boldsymbol{q}$ according to Eqs. (25) and (26), we obtain in fixed and rotating rectangular coordinate systems that

$$\begin{cases} \overset{\circ}{\boldsymbol{q}} = \overset{\circ}{q}_i\boldsymbol{e}_i + q_i\overset{\circ}{\boldsymbol{e}}_i = \left[\overset{\circ}{q}_i + q_j\overset{\cdot}{\overbrace{(A^{-1})_{kj}}}A_{ik}\right]\boldsymbol{e}_i, \\ \overset{\circ}{\boldsymbol{q}} = \overset{\circ}{q}_{i'}\boldsymbol{e}_{i'} + q_{i'}\overset{\circ}{\boldsymbol{e}}_{i'} = \left[\overset{\circ}{q}_{i'} + q_{j'}\overset{\cdot}{\overbrace{(A^{-1})_{k'_0 j'}}}A_{i'k'_0}\right]\boldsymbol{e}_{i'}. \end{cases} \tag{31}$$

Following the definition process of the spatial covariant derivative, we define the **objective covariant derivatives with respect to time** for the vector components in fixed and rotating coordinate systems respectively, as follows:

$$
\begin{cases}
\nabla_{t(o)} q_i \triangleq \mathring{q}_i + q_j \overset{\boldsymbol{\cdot}}{\overbrace{(A^{-1})_{kj}}} A_{ik}, \\
\nabla_{t(o)} q_{i'} \triangleq \mathring{q}_{i'} + q_{j'} \overset{\boldsymbol{\cdot}}{\overbrace{(A^{-1})_{k'_0 j'}}} A_{i' k'_0},
\end{cases}
\tag{32}
$$

where (o) in the subscript denotes "objective". Since the first equation represents a special case of the second equation within a fixed rectangular coordinate system, the second equation can be considered as the general form of the objective covariant derivative with respect to time in any rectangular coordinate system. Based on Definition (28), the objective covariant derivative with respect to time in a rectangular coordinate system can be expressed as

$$
\nabla_{t(o)} q_{i'} \triangleq \mathring{q}_{i'} + q_{j'} \tilde{\Gamma}_{i'j'}. \tag{33}
$$

According to Eq. (24), the relative derivatives of tensor components are equal to their particle derivatives, therefore Eq. (33) can be expressed as

$$
\nabla_{t(o)} q_{i'} \triangleq \dot{q}_{i'} + q_{j'} \tilde{\Gamma}_{i'j'}. \tag{34}
$$

Review the definition formula of the spatial covariant derivative (θ^i denotes general curvilinear coordinates), i.e.

$$
\nabla_m q^i \triangleq \frac{\partial q^i}{\partial \theta^m} + q^j \Gamma^i_{jm}. \tag{35}
$$

It can be seen that Eqs. (34) and (35) exhibit a high degree of symmetry.

We can gain a deeper understanding of this space-time duality. In the spatial covariant derivative, the choice of coordinate system is inherently subjective, and the ordinary partial derivative reflects this subjectivity, making it unsuitable for objective descriptions. The Christoffel symbol serves as an "identifier" for a specific coordinate system, thereby compensating for the inherent subjectivity of the ordinary partial derivative. Similarly, in the temporal covariant derivative (34), the selection of a reference frame is also subjective. Consequently, the particle derivative of the tensor component alone cannot be considered objective. The temporal Christoffel symbol $\tilde{\Gamma}_{i'j'}$, representing the physical changes observed within a specific reference frame, acts as an identifier for that frame and thus offsets the subjectivity associated with it.

It follows from the coordinate independence of the entity in Eq. (31) that since the base vectors of fixed and rotating rectangular coordinate systems satisfy the covariant transformation relation (14), $\nabla_{t(o)} q_{i'}$, as the components of the entity $\mathring{\boldsymbol{q}}$, also satisfy the covariant relationship (i.e., the coordinate transformation rule), i.e.

$$\nabla_{t(o)} q_{i'} = Q_{i'm}(t) \nabla_{t(o)} q_m. \tag{36}$$

Obviously, it can then be demonstrated that the objective covariant derivatives with respect to time satisfy the coordinate transformation relation between two arbitrarily rotating rectangular coordinate systems. We can obtain the complete form of the entity composed of the objective covariant derivatives with respect to time and the corresponding base vectors, that is,

$$\left[\nabla_{t(o)} q_i\right] \boldsymbol{e}_i = \left[\overset{\circ}{q}_i + q_j \overset{\cdot}{\overbrace{(A^{-1})_{k_0 j}}} A_{ik_0}\right] \boldsymbol{e}_i = \dot{\boldsymbol{q}} + \boldsymbol{A} \cdot \overset{\cdot}{\overbrace{\boldsymbol{A}^{-1}}} \cdot \boldsymbol{q}. \tag{37}$$

According to the definition of objectivity in Eq. (11), the entity $\dot{\boldsymbol{q}} + \boldsymbol{A} \cdot \overset{\cdot}{\overbrace{\boldsymbol{A}^{-1}}} \cdot \boldsymbol{q}$ is an objective tensor entity. The physical meaning of this tensor entity is the relative derivative of the vector $\boldsymbol{q}$ relative to the reference base vectors.

6.6 Extension to tensors

For a second-order Eulerian-Eulerian tensor $\boldsymbol{B}$, following the previous process, it can be defined in arbitrary rotating rectangular coordinate systems that

$$\nabla_{t(o)} B_{i'j'} \triangleq \overset{\circ}{B}_{i'j'} + B_{m'j'} \overset{\cdot}{\overbrace{(A^{-1})_{k_0' m'}}} A_{i'k_0'} + B_{i'm'} \overset{\cdot}{\overbrace{(A^{-1})_{k_0' m'}}} A_{j'k_0'}. \tag{38}$$

It can also be obtained that the objective covariant derivatives with respect to time for the second-order tensor components satisfy the covariant relationship between two arbitrarily rotating rectangular coordinate systems, i.e., they satisfy the coordinate transformation relationship

$$\nabla_{t(o)} B_{i'j'} = Q_{i'm}(t)\, Q_{j'n}(t)\, \nabla_{t(o)} B_{mn}. \tag{39}$$

The combination of the objective covariant derivatives with respect to time and base vectors yields an objective tensor entity

$$\left[\nabla_{t(o)} B_{i'j'}\right] \boldsymbol{e}_{i'} \boldsymbol{e}_{j'} = \dot{\boldsymbol{B}} + \boldsymbol{A} \cdot \overset{\cdot}{\overbrace{\boldsymbol{A}^{-1}}} \cdot \boldsymbol{B} + \boldsymbol{B} \cdot (\boldsymbol{A} \cdot \overset{\cdot}{\overbrace{\boldsymbol{A}^{-1}}})^T, \tag{40}$$

whose physical meaning is the relative time derivative of the second-order tensor $\boldsymbol{B}$ relative to the second-order reference base $\boldsymbol{E}_i \boldsymbol{E}_j$.

Similar forms and conclusions can be extended to tensors of arbitrary order.

7 Objective covariant derivative with respect to time for tensor components within curvilinear coordinate systems

7.1 Objective covariant derivative with respect to time for tensor components within time-varying curvilinear coordinate systems

Within the time-varying curvilinear coordinate system, we can still follow the processes in Subsections 6.3 and 6.5 to derive that

$$\begin{aligned}
\overset{\circ}{\boldsymbol{q}} &= \overset{\circ}{q^i}\boldsymbol{g}_i + q^i\overset{\circ}{\boldsymbol{g}}_i = \left[\overset{\circ}{q^i} + A^i_{\cdot k_0}\frac{\mathrm{d}\left(A^{-1}\right)^{k_0}_{\cdot j}}{\mathrm{d}t}q^j\right]\boldsymbol{g}_i \triangleq \nabla_{t(o)}q^i\boldsymbol{g}_i,\\
\overset{\circ}{\boldsymbol{q}} &= \overset{\circ}{q}_i\boldsymbol{g}^i + q_i\overset{\circ}{\boldsymbol{g}^i} = \left[\overset{\circ}{q}_i + A_{ik_0}\frac{\mathrm{d}\left(A^{-1}\right)^{k_0 j}}{\mathrm{d}t}q_j\right]\boldsymbol{g}^i \triangleq \nabla_{t(o)}q_i\boldsymbol{g}^i.
\end{aligned} \tag{41}$$

Here, $A^i_{\cdot k_0}$ denotes the component of the tensor entity $\boldsymbol{A}$ under the tensor basis $\boldsymbol{g}_i\boldsymbol{g}^{k_0}$, where $\boldsymbol{g}_i$ and $\boldsymbol{g}^{k_0}$ are the covariant base vector at the current position of the material point and the contravariant base vector at its initial position, respectively. The third equality in each line of Eq. (41) gives the definition of the covariant derivative with respect to time $\nabla_{t(o)}q^i$ or $\nabla_{t(o)}q_i$. According to the approach in Subsection 6.3, the "relativity" in the "relative derivative" operation resides exclusively in the relative derivative of the base vector. For the vector components q^i and q_i, the relative derivative is equal to the particle derivative, that is,

$$\begin{aligned}
\overset{\circ}{q^i} &= \dot{q^i}\\
\overset{\circ}{q}_i &= \dot{q}_i.
\end{aligned} \tag{42}$$

As demonstrated in Appendix A, in time-varying curvilinear coordinate systems there also exists

$$\left[\nabla_{t(o)}q^i\right]\boldsymbol{g}_i = \left[\nabla_{t(o)}q_i\right]\boldsymbol{g}^i = \dot{\boldsymbol{q}} + \boldsymbol{A}\cdot\overset{\cdot}{\overbrace{\left(\boldsymbol{A}^{-1}\right)}}\cdot\boldsymbol{q}. \tag{43}$$

Since the base vectors of the time-varying coordinate systems satisfy the coordinate transformation relationship and the index-raising and index-lowering relations, i.e.,

$$
\begin{aligned}
\boldsymbol{g}_{i'}(t) &= \beta_{i'}^{j}(t)\,\boldsymbol{g}_j(t)\,,\\
\boldsymbol{g}^{i'}(t) &= \beta_{j}^{i'}(t)\,\boldsymbol{g}^j(t)\,,\\
\boldsymbol{g}^{i}(t) &= g^{ij}(t)\,\boldsymbol{g}_j(t)\,,\\
\boldsymbol{g}_{i}(t) &= g_{ij}(t)\,\boldsymbol{g}^j(t)\,.
\end{aligned}
\tag{44}
$$

Considering the coordinate independence of the tensor entity in Eq. (43), we can directly obtain that in time-varying coordinate systems, $\nabla_{t(o)}q^i$ and $\nabla_{t(o)}q_i$ also satisfy the coordinate transformation rules as well as the index-raising and index-lowering relations, that is, the component parts in Eq. (41) satisfy that

$$
\begin{aligned}
\nabla_{t(o)}q^{i'} &= \beta_{m}^{i'}(t)\,\nabla_{t(o)}q^m,\\
\nabla_{t(o)}q_{i'} &= \beta_{i'}^{m}(t)\,\nabla_{t(o)}q_m,\\
\nabla_{t(o)}q_{i} &= g_{im}(t)\,\nabla_{t(o)}q^m,\\
\nabla_{t(o)}q^{i} &= g^{im}(t)\,\nabla_{t(o)}q_m.
\end{aligned}
\tag{45}
$$

The general form of Eq. (29) in time-varying curvilinear coordinate systems is that

$$
\begin{aligned}
\overset{\circ}{\boldsymbol{g}}_i &= \tilde{\Gamma}^{k}_{\cdot i}\boldsymbol{g}_k = \tilde{\Gamma}_{ki}\boldsymbol{g}^k,\\
\overset{\circ}{\boldsymbol{g}}{}^i &= \tilde{\Gamma}^{ki}\boldsymbol{g}_k = \tilde{\Gamma}_{k}^{\cdot i}\boldsymbol{g}^k,
\end{aligned}
\tag{46}
$$

where the definition formulas of $\tilde{\Gamma}^{i}_{\cdot j}$ and $\tilde{\Gamma}_{i}^{\cdot j}$ are that

$$
\begin{aligned}
\tilde{\Gamma}^{i}_{\cdot j} &\triangleq A^{i}_{\cdot k_0}\frac{\mathrm{d}\left(A^{-1}\right)^{k_0}_{\cdot j}}{\mathrm{d}t} = A^{ik_0}\frac{\mathrm{d}\left(A^{-1}\right)_{k_0 j}}{\mathrm{d}t},\\
\tilde{\Gamma}_{i}^{\cdot j} &\triangleq A_{ik_0}\frac{\mathrm{d}\left(A^{-1}\right)^{k_0 j}}{\mathrm{d}t} = A_{i}^{\cdot k_0}\frac{\mathrm{d}\left(A^{-1}\right)_{k_0 j}}{\mathrm{d}t}.
\end{aligned}
\tag{47}
$$

The reason why the second equal sign holds in each equation is that the Lagrangian-Lagrangian component $g_{k_0m_0}$ or $g^{k_0m_0}$ of the metric tensor is independent of time and can therefore freely enter and exit the particle derivative, achieving the raising and lowering of the index k_0. With the help of the Christoffel symbol in the time domain defined by Eq. (47), the component part of Eq. (41) is written as

$$\nabla_{t(o)} q^i = \overset{\circ}{q^i} + \tilde{\Gamma}^i_{\cdot j} q^j,$$
$$\nabla_{t(o)} q_i = \overset{\circ}{q_i} + \tilde{\Gamma}_i^{\cdot j} q_j. \tag{48}$$

That is the definition formula of the objective covariant derivative with respect to time for vector components within time-varying coordinate systems.

For second-order tensor components, there is

$$\begin{aligned}
\nabla_{t(o)} B^{ij} &\triangleq \frac{\mathrm{d}B^{ij}}{\mathrm{d}t} + B^{mj}\tilde{\Gamma}^i_{\cdot m} + B^{im}\tilde{\Gamma}^j_{\cdot m},\\
\nabla_{t(o)} B_{ij} &\triangleq \frac{\mathrm{d}B_{ij}}{\mathrm{d}t} + B_{mj}\tilde{\Gamma}_i^{\cdot m} + B_{im}\tilde{\Gamma}_j^{\cdot m},\\
\nabla_{t(o)} B^i_{\cdot j} &\triangleq \frac{\mathrm{d}B^i_{\cdot j}}{\mathrm{d}t} + B^m_{\cdot j}\tilde{\Gamma}^i_{\cdot m} + B^i_{\cdot m}\tilde{\Gamma}_j^{\cdot m},\\
\nabla_{t(o)} B_i^{\cdot j} &\triangleq \frac{\mathrm{d}B_i^{\cdot j}}{\mathrm{d}t} + B_m^{\cdot j}\tilde{\Gamma}_i^{\cdot m} + B_i^{\cdot m}\tilde{\Gamma}^j_{\cdot m}.
\end{aligned} \tag{49}$$

Similar forms and conclusions can be extended to tensors of arbitrary order. In contrast to the classical spatial covariant derivative, an elegant duality between space and time can be observed.

7.2 Disappearance of the spatial Christoffel symbols

Eqs. (41) to (43) give that

$$\left[\dot{q}^i + A^i_{\cdot k_0}\frac{\mathrm{d}\left(A^{-1}\right)^{k_0}_{\cdot j}}{\mathrm{d}t}q^j\right]\boldsymbol{g}_i = \dot{\boldsymbol{q}} + \boldsymbol{A}\cdot\overset{\cdot}{\overbrace{\left(\boldsymbol{A}^{-1}\right)}}\cdot\boldsymbol{q},$$
$$\left[\dot{q}_i + A_{ik_0}\frac{\mathrm{d}\left(A^{-1}\right)^{k_0 j}}{\mathrm{d}t}q_j\right]\boldsymbol{g}^i = \dot{\boldsymbol{q}} + \boldsymbol{A}\cdot\overset{\cdot}{\overbrace{\left(\boldsymbol{A}^{-1}\right)}}\cdot\boldsymbol{q}. \tag{50}$$

Taking the first equation as an example, it should be noted that in curvilinear coordinate systems, the two terms on the right-hand side are not respectively equal to the two terms on the left-hand side. In a fixed curvilinear coordinate system, the base vectors at a material point vary with the motion of the material point, and the particle derivatives of the base vectors are therefore nonzero. Thus, there is [26]

$$\frac{\mathrm{d}\boldsymbol{g}_i}{\mathrm{d}t} = \Gamma^k_{im} v^m \boldsymbol{g}_k. \tag{51}$$

Therefore, the difference between the first terms on the two sides of the equality is precisely the term $q^k\Gamma^i_{km}v^m\boldsymbol{g}_i$ induced by the particle derivative of the base vector. A similar situation also occurs for the second terms. In time-varying curvilinear coordinate systems, the situation becomes even more complicated.

Nevertheless, it is satisfying to observe that the two sides of the equality in Eq. (50) are equal as whole entities. This indicates that, in the first equation, the term $q^k\Gamma^i_{km}v^m\boldsymbol{g}_i$ generated by the particle derivative $\dot{\boldsymbol{q}}$ is canceled by the extra term generated by $\boldsymbol{A}\cdot\dot{\overbrace{\boldsymbol{A}^{-1}}}\cdot\boldsymbol{q}$. The second equation is similar. Therefore, in curvilinear coordinate systems, the calculation of the vector $\dot{\boldsymbol{q}}+\boldsymbol{A}\cdot\dot{\overbrace{\boldsymbol{A}^{-1}}}\cdot\boldsymbol{q}$ does not require any spatial Christoffel symbols, which brings great convenience to the computation. In time-varying curvilinear coordinate systems, the overall situation is similar, except that the canceled terms are more complicated. The derivation in Appendix A applies to arbitrary time-varying curvilinear coordinate systems, from which the cancellation mechanism of the extra terms can be clearly observed.

From the perspective of covariance behavior, neither $\dot{q}^i$ nor $A^i_{\cdot k_0}\frac{\mathrm{d}\left(A^{-1}\right)^{k_0}_{\cdot j}}{\mathrm{d}t}q^j$ (i.e., $\tilde{\Gamma}^i_{\cdot j}q^j$) satisfies the coordinate transformation relation between arbitrary time-varying curvilinear coordinate systems. However, their sum is covariant. Therefore, the term $\tilde{\Gamma}^i_{\cdot j}q^j$ plays a role similar to that of the connection term in the spatial covariant derivative: by means of its own non-covariance, it cancels the non-covariant part in the ordinary derivative term. Moreover, in the temporal covariant derivative proposed in this paper, it completely replaces the spatial connection appearing in the spatial covariant derivative. This once again demonstrates the reasonableness of referring to $\tilde{\Gamma}^i_{\cdot j}$ as the "Christoffel symbol in the time field", and also reflects the structural symmetry between the temporal and spatial covariant derivatives.

The analysis of this subsection can be extended to tensors of arbitrary order.

8 From restricted covariance to generalized covariance

8.1 Generalized components and the objective generalized covariant derivative with respect to time

Yin's research [4] indicates that any quantity satisfying the Ricci transformation (i.e. the coordinate transformation and the index raising and lowering) belongs to **generalized components** and thus admits a **generalized covariant derivative**. Furthermore, the definition formula for the generalized covariant derivative is identical in form to that of the classical covariant derivative [4]. This line of thought is continued in this paper. Accordingly, Eq. (48) is also the definition formula for the objective generalized covariant derivative with respect to time of a generalized component.

The covariant base vector $\boldsymbol{g}_i$ is the most typical generalized component. In the following, we will analyze the objective covariant derivative with respect to time of the covariant base vector in arbitrary time-varying coordinate systems. By substituting $\boldsymbol{g}_i$ as a generalized component into the second equation of Eq. (48), the following result is obtained:

$$
\begin{aligned}
\nabla_{t(o)} \boldsymbol{g}_i &\triangleq \mathring{\boldsymbol{g}}_i + \tilde{\Gamma}_{i}^{\cdot j} \boldsymbol{g}_j \\
&= \tilde{\Gamma}_{\cdot i}^{k} \boldsymbol{g}_k + \tilde{\Gamma}_{i}^{\cdot j} \boldsymbol{g}_j \\
&= \left(\tilde{\Gamma}_{\cdot i}^{k} + \tilde{\Gamma}_{i}^{\cdot k} \right) \boldsymbol{g}_k \\
&= \left[A_{\cdot j_0}^{k} \overbrace{\left(A^{-1}\right)_{\cdot i}^{j_0}}^{\cdot} + A_{i}^{\cdot j_0} \overbrace{\left(A^{-1}\right)_{j_0}^{\cdot k}}^{\cdot} \right] \boldsymbol{g}_k .
\end{aligned} \tag{52}
$$

Generally, $\nabla_{t(o)} \boldsymbol{g}_i$ is not equal to zero. When $\boldsymbol{A}$ is an orthogonal tensor, using $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$ we can derive that

$$
\nabla_{t(o)} \boldsymbol{g}_i = \mathbf{0}. \tag{53}
$$

For the contravariant base vector $\boldsymbol{g}^i$, similarly, $\nabla_{t(o)} \boldsymbol{g}^i$ is not equal to zero in general. When $\boldsymbol{A}$ is an orthogonal tensor, there is

$$
\nabla_{t(o)} \boldsymbol{g}^i = \mathbf{0}. \tag{54}
$$

For both the restricted and generalized components, the covariant derivative is formulated based on the Ricci transformation. Furthermore, **the physical meaning of the Ricci transformation is that the transformations of the components and the base vectors match each other, thereby ensuring the combination of components and base vectors results in a coordinate-independent physical entity**. The actual object of the covariant derivative is not the components themselves, but rather the underlying entity represented by these components.

In Yin's research, in a flat space—whether in the spatial or temporal field—the generalized covariant derivative of the covariant base vector $\boldsymbol{g}_i$ is identically zero [4, 7, 8]. The authors of this paper believe that this conclusion can be interpreted from the perspective of the metric tensor entity. The covariant base vector, together with the contravariant base vector, constitutes the metric tensor entity $\boldsymbol{G}$, that is,

$$
\boldsymbol{G} = \boldsymbol{g}_i \boldsymbol{g}^i. \tag{55}
$$

In Yin's research, the generalized covariant derivatives of the covariant base vector $\boldsymbol{g}_i$ reflect the spatial derivative and the particle time derivative of the metric tensor entity $\boldsymbol{G}$. In a flat space, the metric tensor $\boldsymbol{G}$ is constant; therefore, both its spatial derivative and its particle derivative with respect to time are zero. Consequently, the generalized covariant derivatives of the covariant base vector are also zero.

However, in the research of this article, the objective covariant derivative with respect to time of base vectors reflects the relative derivative of the metric tensor entity

$\boldsymbol{G}$ relative to the reference frame. When the tensor $\boldsymbol{A}$ is not orthogonal, the selected reference frame not only rotates but also stretches and deforms. In a relative sense, the metric tensor entity $\boldsymbol{G}$ is no longer constant, leading to a non-zero covariant derivative. When $\boldsymbol{A}$ is an orthogonal tensor, the reference frame undergoes only rotation without deformation. In that case, since the metric tensor $\boldsymbol{G}$ is isotropic, it remains invariant in this relative framework, and consequently the generalized covariant derivatives of both the covariant and contravariant base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ remain zero.

8.2 The objective combined-type generalized covariant derivative with respect to time

From a physical point of view, the metric tensor entity should remain invariant. Consequently, we aim to adjust it in a manner that compensates for the variation in the metric tensor, yielding results that are both physically sound and computationally convenient. This consideration serves as the motivation for the extension presented in this section.

In the preceding sections, the temporal derivative of a physical quantity relative to the "reference frame generated by the cross-time tensor $\boldsymbol{A}$" is termed the relative derivative, denoted by $\overset{\circ}{(\ )}$. According to the definitions in Section 4, the inverse transpose $\boldsymbol{A}^{-T}$ of an Eulerian-Lagrangian tensor entity (cross-time tensor entity) $\boldsymbol{A}$ is also an Eulerian-Lagrangian tensor entity (cross-time tensor entity), because the Eulerian-Lagrangian components of $\boldsymbol{A}^{-T}$ can be derived from the Eulerian-Lagrangian components of $\boldsymbol{A}$. Consequently, we introduce the **conjugate relative derivative**, defined as the temporal derivative of a physical quantity relative to the "reference frame generated by the tensor $\boldsymbol{A}^{-T}$", denoted by $\overset{\bar{\circ}}{(\ )}$. The associated temporal covariant derivative generated from the conjugate relative derivative is termed the **objective conjugate covariant derivative with respect to time**, denoted by $\bar{\nabla}_{t(o)}$. By replacing $\overset{\circ}{(\ )}$ with $\overset{\bar{\circ}}{(\ )}$ and the tensor $\boldsymbol{A}$ with $\boldsymbol{A}^{-T}$ in the definition (48) of the objective covariant derivative with respect to time, we obtain the defining formula (56) for the objective conjugate covariant derivative with respect to time.

$$\begin{aligned}\bar{\nabla}_{t(o)} q^i &= \overset{\bar{\circ}}{q^i} + \bar{\bar{\Gamma}}^{i}_{\cdot j} q^j, \\ \bar{\nabla}_{t(o)} q_i &= \overset{\bar{\circ}}{q_i} + \bar{\bar{\Gamma}}_{i}^{\cdot j} q_j,\end{aligned} \tag{56}$$

where $\bar{\bar{\Gamma}}^{i}_{\cdot j}$ and $\bar{\bar{\Gamma}}_{i}^{\cdot j}$ are the conjugate Christoffel symbols in the time field. Their definitions are obtained by replacing $\boldsymbol{A}$ of Eq. (47) with $\boldsymbol{A}^{-T}$, i.e.,

$$
\begin{aligned}
\bar{\tilde{\Gamma}}^{i}_{\cdot j} &\triangleq \left(A^{-T}\right)^{i}_{\cdot k_0} \frac{\mathrm{d}\left(A^{T}\right)^{k_0}_{\cdot j}}{\mathrm{d}t} = \left(A^{-T}\right)^{ik_0} \frac{\mathrm{d}\left(A^{T}\right)_{k_0 j}}{\mathrm{d}t}, \\
\bar{\tilde{\Gamma}}_{i}^{\cdot j} &\triangleq \left(A^{-T}\right)_{ik_0} \frac{\mathrm{d}\left(A^{T}\right)^{k_0 j}}{\mathrm{d}t} = \left(A^{-T}\right)_{i}^{\cdot k_0} \frac{\mathrm{d}\left(A^{T}\right)_{k_0}^{\cdot j}}{\mathrm{d}t}.
\end{aligned} \tag{57}
$$

We define the arithmetic mean of the objective covariant derivative and the objective conjugate covariant derivative with respect to time as the **objective combined-type covariant derivative with respect to time**. For the generalized components, its definition formula is

$$
\begin{aligned}
\nabla_{t(o,c)} q^i &= \frac{1}{2}\left(\nabla_{t(o)} q^i + \bar{\nabla}_{t(o)} q^i\right) = \frac{1}{2}\left[\left(\overset{\circ}{q^i} + \overset{\bar{\circ}}{q^i}\right) + \left(\tilde{\Gamma}^{i}_{\cdot j} + \bar{\tilde{\Gamma}}^{i}_{\cdot j}\right) q^j\right], \\
\nabla_{t(o,c)} q_i &= \frac{1}{2}\left(\nabla_{t(o)} q_i + \bar{\nabla}_{t(o)} q_i\right) = \frac{1}{2}\left[\left(\overset{\circ}{q_i} + \overset{\bar{\circ}}{q_i}\right) + \left(\tilde{\Gamma}_{i}^{\cdot j} + \bar{\tilde{\Gamma}}_{i}^{\cdot j}\right) q_j\right],
\end{aligned} \tag{58}
$$

where c in the subscript denotes "combined". It follows that the objective combined-type covariant derivative with respect to time also exhibits covariance under coordinate transformations. When restricted to tensor components specifically, the relative derivative and the conjugate relative derivative are both equal to the particle derivative, so the equations reduce to

$$
\begin{aligned}
\nabla_{t(o,c)} q^i &= \frac{1}{2}\left(\nabla_{t(o)} q^i + \bar{\nabla}_{t(o)} q^i\right) = \dot{q^i} + \frac{1}{2}\left(\tilde{\Gamma}^{i}_{\cdot j} + \bar{\tilde{\Gamma}}^{i}_{\cdot j}\right) q^j, \\
\nabla_{t(o,c)} q_i &= \frac{1}{2}\left(\nabla_{t(o)} q_i + \bar{\nabla}_{t(o)} q_i\right) = \dot{q_i} + \frac{1}{2}\left(\tilde{\Gamma}_{i}^{\cdot j} + \bar{\tilde{\Gamma}}_{i}^{\cdot j}\right) q_j.
\end{aligned} \tag{59}
$$

In this case, the combination of the objective combined-type covariant derivative and the base vector yields a tensor entity, as shown in Eq. (60).

$$
\left[\nabla_{t(o,c)} q^i\right] \boldsymbol{g}_i = \left[\nabla_{t(o,c)} q_i\right] \boldsymbol{g}^i = \dot{\boldsymbol{q}} + \frac{1}{2}\left[\boldsymbol{A} \cdot \dot{\overbrace{\boldsymbol{A}^{-1}}} \cdot \boldsymbol{q} + \boldsymbol{A}^{-T} \cdot \left(\dot{\boldsymbol{A}^{T}}\right) \cdot \boldsymbol{q}\right]. \tag{60}
$$

Substituting the covariant base vector $\boldsymbol{g}_i$ as a generalized component into the second equation of Eq. (58), we obtain that

$$
\nabla_{t(o,c)} \boldsymbol{g}_i = \frac{1}{2}\left(\tilde{\Gamma}^{k}_{\cdot i} + \tilde{\Gamma}_{i}^{\cdot k} + \bar{\tilde{\Gamma}}^{k}_{\cdot i} + \bar{\tilde{\Gamma}}_{i}^{\cdot k}\right) \boldsymbol{g}_k. \tag{61}
$$

According to the definition formulas (47) and (57), it can be noticed that $\tilde{\Gamma}^{k}_{\cdot i}$ and $\bar{\tilde{\Gamma}}^{\cdot k}_{i}$ cancel each other, and similarly $\tilde{\Gamma}^{\cdot k}_{i}$ and $\bar{\tilde{\Gamma}}^{k}_{\cdot i}$ cancel each other. Consequently, we obtain

$$\nabla_{t(o,c)}\boldsymbol{g}_i = \mathbf{0}. \tag{62}$$

Likewise, we have

$$\nabla_{t(o,c)}\boldsymbol{g}^i = \mathbf{0}. \tag{63}$$

The objective generalized combined-type covariant derivatives with respect to time for the covariant base vector $\boldsymbol{g}_i$ and contravariant base vector $\boldsymbol{g}^i$ are both zero, which is an elegant result. Therefore, the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ can be freely moved into or out of the objective combined-type covariant derivative with respect to time without generating additional terms. Physically, this result means that in the relative sense of the combination of the relative derivative $\overset{\circ}{(\ )}$ and the conjugate relative derivative $\overset{\bar{\circ}}{(\ )}$, the metric tensor entity $\boldsymbol{G}$ remains a constant tensor entity.

9 Specific examples

The most typical Eulerian-Lagrangian tensor is the deformation gradient tensor $\boldsymbol{F}$. Within a rotating rectangular coordinate system, the Eulerian-Lagrangian components of $\boldsymbol{F}$ can be utilized to derive the Eulerian-Lagrangian components of the rotation tensor $\boldsymbol{R}$ (which is obtained from the polar decomposition of the deformation gradient tensor), as well as the Eulerian-Lagrangian components of the inverse transpose of the deformation gradient tensor, $\boldsymbol{F}^{-T}$. Consequently, based on the definition provided in Section 4, both $\boldsymbol{R}$ and $\boldsymbol{F}^{-T}$ qualify as Eulerian-Lagrangian tensor entities.

For simplicity, we use the vector, i.e., the first-order tensor, to present several specific forms of the corresponding entities of the objective covariant derivative with respect to time.

For the vector entity corresponding to the objective covariant derivative with respect to time, i.e., $\dot{\boldsymbol{q}} + \boldsymbol{A} \cdot \overset{\bullet}{\overbrace{\boldsymbol{A}^{-1}}} \cdot \boldsymbol{q}$ (Eqs. (37) and (43)), by taking $\boldsymbol{A}$ as the specific Eulerian-Lagrangian tensors (i.e., cross-time tensors), several commonly used objective derivatives are obtained as follows.

1. Taking $\boldsymbol{A}$ as the deformation gradient tensor $\boldsymbol{F}$, we obtain

$$\dot{\boldsymbol{q}} + \boldsymbol{F} \cdot \overset{\bullet}{\overbrace{\boldsymbol{F}^{-1}}} \cdot \boldsymbol{q} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{q} = \dot{\boldsymbol{q}} - \boldsymbol{l} \cdot \boldsymbol{q} = \overset{\circ O}{\boldsymbol{q}}. \tag{64}$$

The above equation gives the Oldroyd derivative of a first-order tensor, where $\boldsymbol{l} = \boldsymbol{v}\nabla = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$ denotes the velocity gradient tensor. The reference base vectors associated with the Oldroyd derivative are obtained by transforming the fixed standard orthogonal base vectors through the deformation gradient tensor $\boldsymbol{F}$. That means the reference base vectors evolve in synchronization with the material line elements of the deforming body. As discussed in Subsection 8.1, the Oldroyd-type objective covariant derivatives with respect to time of the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ are generally non-zero, physically meaning that in the relative sense of the Oldroyd derivative, the metric tensor entity varies due to the stretching or angular changes of the material line elements.

It follows from Eq. (50) that the contravariant component expression of the Oldroyd derivative in time-varying curvilinear coordinate systems is

$$\nabla_{t(o)} q^i = \dot{q}^i + F^i_{\cdot k_0} \frac{\mathrm{d}(F^{-1})^{k_0}_{\cdot j}}{\mathrm{d}t} q^j = \frac{\mathrm{d}q^i}{\mathrm{d}t} - \frac{\partial v^i}{\partial \theta^j} q^j, \tag{65}$$

where θ^j denotes the general curvilinear coordinates. No term depending on the components of the metric tensor appears in this expression. In arbitrary time-varying curvilinear coordinate systems, v^i represents the time derivative of the position coordinates of a material point. In particular, when the coordinate system is taken as a fixed curvilinear coordinate system, v^i is the velocity of the material point observed in the inertial frame.

The Oldroyd derivative is a Lie derivative [16]. The absence of spatial Christoffel symbols in Eq. (65) is a property of the Lie derivative and also a result that can be obtained within the framework of this paper.

2. Taking $\boldsymbol{A}$ as the rotation tensor $\boldsymbol{R}$, we obtain

$$\dot{\boldsymbol{q}} + \boldsymbol{R} \cdot \dot{\boldsymbol{R}}^T \cdot \boldsymbol{q} = \dot{\boldsymbol{q}} - \boldsymbol{\Omega} \cdot \boldsymbol{q} = \overset{\circ G}{\boldsymbol{q}}. \tag{66}$$

The above equation gives the Green-Naghdi derivative of a first-order tensor, where $\boldsymbol{\Omega} = \dot{\boldsymbol{R}} \cdot \boldsymbol{R}^T$ denotes the angular velocity tensor [27]. The reference base vectors associated with the Green-Naghdi derivative are obtained by transforming the fixed standard orthonormal base vectors via the rotation tensor $\boldsymbol{R}$. As discussed in Subsection 8.1, the Green-Naghdi-type objective covariant derivative with respect to time of the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ are both zero, physically meaning that in the relative sense of the Green-Naghdi derivative, the metric tensor remains invariant under material deformation.

It follows from Eq. (50) that the contravariant component expression of the Green-Naghdi derivative in time-varying curvilinear coordinate systems is

$$\nabla_{t(o)} q^i = \frac{\mathrm{d}q^i}{\mathrm{d}t} + R^i_{\cdot k_0} \frac{\mathrm{d}(R^{-1})^{k_0}_{\cdot j}}{\mathrm{d}t} q^j. \tag{67}$$

The covariance of Eq. (67) under transformations between arbitrary time-varying curvilinear coordinate systems is nontrivial. As shown in Subsection 7.2, neither term on the right-hand side separately satisfies the required coordinate-transformation law; only their sum is covariant.

To the best of our knowledge, in arbitrary time-varying curvilinear coordinate systems, the above explicit component expression of the Green-Naghdi derivative has not appeared in the literature. The absence of spatial Christoffel symbols in Eq. (67) indicates that the computation of the Green-Naghdi derivative in arbitrary time-varying curvilinear coordinate systems does not require any spatial Christoffel symbols; this conclusion has likewise not been reported in the literature.

The Green-Naghdi derivative is not a Lie derivative [21]. However, from Subsection 7.2 and Appendix A, it can be seen that the cancellation mechanisms of the spatial Christoffel symbols in the Green-Naghdi derivative and the Oldroyd derivative in arbitrary time-varying curvilinear coordinate systems are unified.

3. Taking $\boldsymbol{A}$ as $\boldsymbol{F}^{-T}$, the inverse transpose tensor of $\boldsymbol{F}$, we obtain

$$\dot{\boldsymbol{q}} + \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^{T} \cdot \boldsymbol{q} = \dot{\boldsymbol{q}} + \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right)^{T} \cdot \boldsymbol{q} = \dot{\boldsymbol{q}} + \boldsymbol{l}^{T} \cdot \boldsymbol{q} = \overset{\circ C}{\boldsymbol{q}}. \tag{68}$$

The above equation gives the Cotter-Rivlin derivative of a first-order tensor. As discussed in Subsection 8.2, the Cotter-Rivlin derivative and the Oldroyd derivative are conjugate relative derivatives; that is, the effects of the material line element's deformation on the reference base vectors associated with these two derivatives are opposite in nature.

It follows from Eq. (50) that the covariant component expression of the Cotter-Rivlin derivative in time-varying curvilinear coordinate systems is

$$\nabla_{t(o)} q_i = \dot{q}_i + F^{j}_{\cdot k_0} \frac{\mathrm{d}(F^{-1})^{k_0}_{\cdot i}}{\mathrm{d}t} q_j = \frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial v^j}{\partial \theta^i} q_j, \tag{69}$$

where the meaning of θ^i and v^j is the same as that in Eq. (65). It should be particularly noted that Eqs. (65) and (69) are not components of the same entity; therefore, they do not satisfy the index-raising and index-lowering relations.

4. For the vector entity corresponding to the objective combined-type covariant derivative with respect to time, i.e., Eq. (60), taking $\boldsymbol{A}$ as the deformation gradient tensor $\boldsymbol{F}$, we obtain

$$\dot{\boldsymbol{q}} + \frac{1}{2}\left[\boldsymbol{F} \cdot \dot{\overbrace{\boldsymbol{F}^{-1}}} + \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^{T}\right] \cdot \boldsymbol{q} = \dot{\boldsymbol{q}} - \frac{1}{2}\left[\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} - \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right)^{T}\right] \cdot \boldsymbol{q} = \dot{\boldsymbol{q}} - \boldsymbol{w} \cdot \boldsymbol{q} = \overset{\circ J}{\boldsymbol{q}}. \tag{70}$$

The above equation gives the Jaumann derivative of a first-order tensor, where $\boldsymbol{w} = (\boldsymbol{l} - \boldsymbol{l}^T)/2$ denotes the spin tensor. As discussed in Subsection 8.2, the Jaumann-type objective covariant derivatives with respect to time of the base vectors $\boldsymbol{g}_i$

and $\boldsymbol{g}^i$ are both equal to zero, physically meaning that in the relative sense of the Jaumann derivative, the metric tensor entity remains invariant under material deformation. Currently, the Jaumann derivative and the Green-Naghdi derivative of the stress tensor are the two most widely adopted co-rotational derivatives in the field of computational mechanics [20, 23].

10 Discussion

Some classical objective derivatives, especially the Oldroyd derivative, have been systematically interpreted by Marsden and Hughes as Lie derivatives [16]. Following the pull-back logic of the Lie derivative, component forms and physical interpretations of the Oldroyd derivative can be obtained [16, 20, 21]. The conclusions obtained are equivalent to those in the case $\boldsymbol{A} = \boldsymbol{F}$ discussed in Section 9 of this paper.

However, the Lie derivative and its linear combinations are not sufficient to cover all objective derivatives [21]. The most typical example is the Green-Naghdi derivative. Since, in general, the rotation tensor $\boldsymbol{R}$ in the polar decomposition is not the tangent map of a diffeomorphism, the Green-Naghdi derivative is not a Lie derivative. This paper attempts to unify the Lie-type objective rates, the Green-Naghdi derivative, and even the relative derivatives whose reference bases are generated by arbitrary physically meaningful cross-time tensors, so as to reveal their physical meanings and present their unified component forms in arbitrary time-varying curvilinear coordinate systems, in which no spatial Christoffel symbols appear. The cancellation mechanism of the spatial Christoffel symbols in arbitrary time-varying curvilinear coordinate systems is also displayed in a unified manner through the derivation in Appendix A.

It should be pointed out that the component expression of the Green-Naghdi derivative in arbitrary time-varying curvilinear coordinate systems still depends on the components of the metric tensor, since the components of $\boldsymbol{R}$ depend on the components of the metric tensor. This metric dependence reflects a structural difference between the Green-Naghdi derivative and the Oldroyd derivative. Therefore, if one adopts the standpoint of conventional Lie-derivative-based covariance arguments, the Green-Naghdi derivative is not on the same footing as the Oldroyd derivative. In fact, the explicit component form and transformation relations of the Green-Naghdi derivative in arbitrary time-varying curvilinear coordinate systems have not been included in previous covariance-based studies. However, within the framework of this paper, the form invariance of Eq. (67) in arbitrary time-varying curvilinear coordinate systems is also a manifestation of covariance. Since Eq. (67) directly reveals that the Green-Naghdi derivative contains no spatial Christoffel symbols in arbitrary time-varying curvilinear coordinate systems, it facilitates the computation of the Green-Naghdi derivative in curvilinear coordinate systems.

To reflect the generality of the framework proposed in this paper, we may consider the following situation. Assume that there exists an internal variable $\boldsymbol{\alpha}$, which is a symmetric second-order tensor entity with distinct principal values. Consider a physical process in which the state of the internal variable $\boldsymbol{\alpha}$ undergoes pure rotation—i.e., the principal values of the tensor $\boldsymbol{\alpha}$ remain invariant, while only the principal directions rotate. Under this condition, the tensor $\boldsymbol{\alpha}$ can be expressed as

$$\boldsymbol{\alpha} = \boldsymbol{R}_{(\boldsymbol{\alpha})} \cdot \boldsymbol{\alpha}_0 \cdot \boldsymbol{R}_{(\boldsymbol{\alpha})}^T. \tag{71}$$

In the above equation, $\boldsymbol{\alpha}_0$ denotes the tensor $\boldsymbol{\alpha}$ at the initial time instant, which is a "snapshot" and cannot evolve over time. $\boldsymbol{R}_{(\boldsymbol{\alpha})}$ represents the rotation tensor that describes the rotation of the principal directions of the tensor $\boldsymbol{\alpha}$. According to the definition in Section 4, since $\boldsymbol{R}_{(\boldsymbol{\alpha})}$ maps the principal direction vectors at the initial time to those at the current time, $\boldsymbol{R}_{(\boldsymbol{\alpha})}$ qualifies as an Eulerian-Lagrangian tensor entity. Therefore, by taking the tensor $\boldsymbol{A}$ in $\dot{\boldsymbol{q}} + \boldsymbol{A} \cdot \overbrace{\boldsymbol{A}^{-1}}^{\bullet} \cdot \boldsymbol{q}$ as $\boldsymbol{R}_{(\boldsymbol{\alpha})}$, we can obtain the objective derivative entity (72), i.e.,

$$\overset{\circ(\boldsymbol{\alpha})}{\boldsymbol{q}} = \dot{\boldsymbol{q}} + \boldsymbol{R}_{(\boldsymbol{\alpha})} \cdot \dot{\boldsymbol{R}}_{(\boldsymbol{\alpha})}^T \cdot \boldsymbol{q}. \tag{72}$$

The physical meaning of this objective derivative is the relative rate of change of the physical quantity $\boldsymbol{q}$ relative to a co-rotating reference frame—specifically, the frame that rotates together with the internal variable $\boldsymbol{\alpha}$. Clearly, the physical meaning of this derivative is independent of the inertial frame, and thus it is objective. It follows from Eq. (50) that its component expression in arbitrary time-varying curvilinear coordinate systems contains no spatial Christoffel symbols.

Assume that the evolution of this second-order internal variable is not a pure rotation. Then, the cross-time tensor $\boldsymbol{A}$ characterizing its evolution is generally non-orthogonal. In this case, we can apply the method in Subsection 8.2 to perform a conjugate correction, and finally obtain the combined-type covariant derivative with respect to time and its corresponding objective tensor entity, which takes the form given in Eq. (60). It follows from Eqs. (59) and (60) that, in arbitrary time-varying curvilinear coordinate systems, the computation of its components also does not require any spatial Christoffel symbols.

The above example illustrates that, building upon the findings of this study, an objective time derivative can be rigorously formulated directly from any regular and physically meaningful Eulerian-Lagrangian tensor entity. The physical interpretation of this derivative is the relative rate of change of a given physical quantity relative to "a set of reference base vectors generated by the action of this Eulerian-Lagrangian tensor entity on a fixed frame". As a cross-time tensor entity, the Eulerian-Lagrangian tensor inherently defines a mapping from an initial physical quantity to a current one. Consequently, the associated reference base vectors evolve in synchrony with a specific current physical quantity. In multi-field coupled systems, this construction framework can be applied: we can define a set of reference base vectors based on the object's deformation state or on the temporal evolution of another physical quantity—distinct from the quantity whose objective derivative is being calculated—thereby yielding an objective derivative. Such a physically grounded construction and consistent kinematic interpretation are not readily obtained within the conventional Lie-derivative framework.

11 Conclusions

Based on the idea of covariance, this paper develops the concept of temporal covariance from the perspective of space-time duality and defines the objective covariant derivative with respect to time under arbitrary time-varying coordinate systems. Two foundational concepts—the Eulerian-Lagrangian tensor (also referred to as the cross-time tensor) and the relative derivative—serve as theoretical cornerstones for this development. Notably, widely adopted objective rates in continuum mechanics emerge as special cases of this temporal covariant derivative. From the perspective of space-time duality, we further examine the algebraic structure of the objective covariant derivative with respect to time, define the Christoffel symbols in the time field, and clarify their physical meaning as temporal connection terms. We present the unified component form of the objective covariant derivative with respect to time in arbitrary time-varying coordinate systems, in which no spatial Christoffel symbols appear; instead, only the temporal connection terms determined by the cross-time tensor are involved. We also provide the corresponding entity form and, by using the relation between the entity and its components, prove the systematic cancellation of the terms containing spatial Christoffel symbols.

Based on the concept of generalized components, we have extended the objective covariant derivative with respect to time from the case of restricted components to the case of generalized components, examining the objective temporal covariant derivatives of covariant and contravariant base vectors. When the reference frame only rotates without deformation, the generalized objective temporal covariant derivatives of the covariant and contravariant base vectors are both zero. By contrast, when the reference frame undergoes deformation, these derivatives become non-zero, reflecting the relative change of the metric tensor entity. Furthermore, we formulate the objective combined-type temporal covariant derivative, and the combined-type temporal covariant derivatives of covariant and contravariant base vectors are always zero, ensuring the metric consistency.

The proposed form elegantly captures the intrinsic duality between the temporal and spatial covariant derivatives. Moreover, under a unified framework, it clarifies the physical interpretation of the Eulerian-Lagrangian tensors in the objective derivatives, that is, the motion and deformation of the reference frames (or observers). Consequently, Lie-type objective derivatives such as the Oldroyd derivative and non-Lie-type objective derivatives such as the Green-Naghdi derivative can be interpreted and analyzed within the same framework of relative derivatives. We can also generate reference base vectors through any physically meaningful Eulerian-Lagrangian tensor and thereby construct the corresponding objective derivative. This provides a unified physical background for the construction and selection of objective derivatives in specific problems. Moreover, the computation of the objective derivatives constructed in this manner does not require any spatial Christoffel symbols in arbitrary time-varying curvilinear coordinate systems. When constructing the objective temporal covariant derivative, our true starting point is the physical postulate corresponding to Truesdell's objectivity, rather than the specific mathematical form (11). As expected, the resulting construction naturally satisfies the objective and covariant mathematical form.

This physically grounded approach is closer to the essence of the problem, eliminates reliance on unnecessary reference configurations, and achieves logical simplicity.

On the other hand, the findings of this study extend previous work on generalized covariance [4–6] and space-time dual covariance [7, 8], thereby providing a new entry point for their application in continuum mechanics. In the case of the objective combined-type temporal covariant derivatives, the covariant derivatives of the base vectors are always zero, which facilitates the further construction of the algebraic system for the generalized covariant derivative.

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Declarations

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Appendix A

We examine the two terms of the vector entity $\dot{\boldsymbol{q}} + \boldsymbol{A}\cdot\overset{\cdot}{\overbrace{\left(\boldsymbol{A}^{-1}\right)}}\cdot\boldsymbol{q}$ in sequence. For the first term, i.e., the particle derivative of vector $\boldsymbol{q}$, there exists

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\mathrm{d}q^i}{\mathrm{d}t}\boldsymbol{g}_i + q^i\frac{\mathrm{d}\boldsymbol{g}_i}{\mathrm{d}t}. \tag{A1}$$

For the second term, i.e., $\boldsymbol{A}\cdot\overbrace{\left(\boldsymbol{A}^{-1}\right)}^{\cdot}\cdot\boldsymbol{q}$, there exists

$$\begin{aligned}
\boldsymbol{A}\cdot\frac{\mathrm{d}(\boldsymbol{A}^{-1})}{\mathrm{d}t}\cdot\boldsymbol{q} &= -\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\cdot\boldsymbol{A}^{-1}\cdot\boldsymbol{q} \\
&= -\frac{\mathrm{d}\left(A^i_{\cdot l_0}\boldsymbol{g}_i\boldsymbol{g}^{l_0}\right)}{\mathrm{d}t}\cdot\left(A^{-1}\right)^{j_0}_{\cdot m}\boldsymbol{g}_{j_0}\boldsymbol{g}^m\cdot q^k\boldsymbol{g}_k \\
&= -\frac{\mathrm{d}\left(A^i_{\cdot l_0}\boldsymbol{g}_i\right)}{\mathrm{d}t}\left(A^{-1}\right)^{l_0}_{\cdot k}q^k \\
&= -\frac{\mathrm{d}\left(A^i_{\cdot l_0}\right)}{\mathrm{d}t}\left(A^{-1}\right)^{l_0}_{\cdot k}q^k\boldsymbol{g}_i - A^i_{\cdot l_0}\left(A^{-1}\right)^{l_0}_{\cdot k}q^k\frac{\mathrm{d}\boldsymbol{g}_i}{\mathrm{d}t} \\
&= -\frac{\mathrm{d}\left(A^i_{\cdot l_0}\right)}{\mathrm{d}t}\left(A^{-1}\right)^{l_0}_{\cdot k}q^k\boldsymbol{g}_i - q^i\frac{\mathrm{d}\boldsymbol{g}_i}{\mathrm{d}t}.
\end{aligned} \tag{A2}$$

The summation of the two terms yields that

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} + \boldsymbol{A}\cdot\frac{\mathrm{d}(\boldsymbol{A}^{-1})}{\mathrm{d}t}\cdot\boldsymbol{q} = \frac{\mathrm{d}q^i}{\mathrm{d}t}\boldsymbol{g}_i - \frac{\mathrm{d}(A^i_{\cdot l_0})}{\mathrm{d}t}(A^{-1})^{l_0}_{\cdot k}q^k\boldsymbol{g}_i = \left[\frac{\mathrm{d}q^i}{\mathrm{d}t} + A^i_{\cdot l_0}\frac{\mathrm{d}(A^{-1})^{l_0}_{\cdot k}}{\mathrm{d}t}q^k\right]\boldsymbol{g}_i. \tag{A3}$$

Comparing Eq. (A3) and Eq. (41), we obtain that

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} + \boldsymbol{A}\cdot\frac{\mathrm{d}(\boldsymbol{A}^{-1})}{\mathrm{d}t}\cdot\boldsymbol{q} = \nabla_{t(o)}q^i\boldsymbol{g}_i. \tag{A4}$$

Similarly, we can obtain that

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} + \boldsymbol{A}\cdot\frac{\mathrm{d}(\boldsymbol{A}^{-1})}{\mathrm{d}t}\cdot\boldsymbol{q} = \nabla_{t(o)}q_i\boldsymbol{g}^i. \tag{A5}$$

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