# A numerical procedure based on the Backward Euler method for two point boundary value ordinary differential equation (ODE) systems with applications to a model in flight mechanics

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**Abstract**

This short communication develops a numerical procedure suitable for a large class of ordinary differential equation systems found in models in physics and engineering. More specifically, we have utilized the standard Backward Euler method combined with a MAT-LAB function for obtaining a method suitable for a class of two point boundary value ODE non-linear systems. In the last section, we apply the method to a model in flight mechanics.

Keywords: Two point boundary value ODE system, Backward Euler method, model in flight mechanics.

MSC: 65L10, 65L12

## 1 Indroduction

In this article we develop an application of the Backward Euler method adapted for a two point boundary value ODE (ordinary differential equation) system to a model in flight mechanics.

The model in flight mechanics here addressed may be found in [1].

Details on the Sobolev spaces involved may be found in [2].

Related results to those concerning the generalized method of lines, may be found in the here referenced books of 2011 and 2020, [3, 4] respectively.

For a reference on standard finite differences schemes, please see [5].

Finally, related results concerning the fourth order Runge-Kutta method have been presented in [6].

## 2 The main numerical method

Consider the first order system of ordinary differential equations given by

$$\frac{du_j}{dt} = f_j(\{u_l\}), \text{ on } [0, t_f] \; \forall j \in \{1, 2, 3\},$$

with the boundary conditions

$$u_1(0) = 0, \;\; u_1(t_f) = u_f = 11000.$$

Here $\mathbf{u} = \{u_l\} \in W^{1,2}([0, t_f], \mathbb{R}^3)$ and $f_j$ are functions at least of $C^1$ class, $\forall j \in \{1, 2, 3\}$.

Concerning a finite differences scheme, we denote the number of nodes on $[0, t_f]$ by $m_8$ and set $d = t_f/m_8$.

Based on the Backward Euler method, we propose the following algorithm.

In finite differences, (indeed we have made it through an appropriate MAT-LAB function), we define $t_f = t_0$ (here $t_0$, the final time, it is also a unknown variable to be minimized through the minimization of the functional $J$ to be specified in the next lines),

$$u_1(0) = 0, \; u_2(0) = G_0, \; u_3(0) = V_0;$$

Fixing $k_{10} = 15$, $m_8 = 10000$, $d = m_8/t_f = m_8/t_0$ we calculate $(u_1)_1, (u_2)_1$ and $(u_3)_1$ through the following iterations,

Set $(u_{01})_1 = u_1(0)$, $(u_{02})_1 = u_2(0)$, $(u_{03})_1 = u_3(0)$.

For $k_3$ from 1 to $k_{10}$ do

1. $(u_j)_1 = u_j(0) + f_j((u_{01})_1, (u_{02})_1, (u_{03})_1) \; d$, $\forall j \in \{1, 2, 3\}$,
2. $(u_{0j})_1 = (u_j)_1$, $\forall j = \{1, 2, 3\}$.

   end;

for $k$ from 2 to $m_8$ do

$(u_{0j})_k = (u_j)_{k-1}$, $\forall j \in \{1, 2, 3\}$.

for $k_3$ from 1 to $k_{10}$ do

1. $(u_j)_k = (u_j)_{k-1} + f_j((u_{01})_k, (u_{02})_k, (u_{03})_k) \; d$, $\forall j \in \{1, 2, 3\}$,
2. $(u_{0j})_k = (u_j)_k$, $\forall j = \{1, 2, 3\}$.

   end;

end;

Finally, having obtained

$$\{(u_j)_k(G_0, V_0, t_0)\},$$

we obtain $(G_0, V_0, t_0)$ by minimizing the functional

$$J(G_0, V_0, t_0).$$

where

$$J(G_0, V_0, t_0) = 25^2 t_0^2 + 30^2((u_1)_{m_8}(G_0, V_0, t_0) - 11000)^2 + (u_2)_{m_8}(G_0, V_0, t_0) + 500^2 \sum_{j=1}^{m_8-2} \left( \frac{((u_1)_{j+2} - 2(u_1)_{j+1} + (u_1)_j)}{d^2} \right)^2. \quad (1)$$

**Remark 2.1.** *Through the minimization of such a functional $J$, we intend to obtain $(G_0, V_0, t_0) \in \mathbb{R}^3$ which minimizes the final time $t_f = t_0$ and approximately satisfy the boundary conditions $u_1(t_f = t_0) = 11000$, $u_2(t_f = t_0) = 0$ and, at the same time, minimizes the integral of the second derivative square of $u_1$ (intending to minimize the trajectory curvature as well) along the time interval $[0, t_f]$.*

The problem is then approximately solved.

# 3 Applications to a flight mechanics model

We present numerical results for the following system of equations, which models the in plane climbing motion of an airplane (please, see more details in [1]).

$$\begin{cases} \dot{h} = V \sin\gamma, \\ \dot{\gamma} = \frac{1}{m_f V}(F \sin[a + a_F] + L) - \frac{g}{V}\cos\gamma, \\ \dot{V} = \frac{1}{m_f}(F\cos[a + a_F]) - D) - g\sin\gamma \\ \dot{x} = V\cos\gamma, \end{cases} \quad (2)$$

with the boundary conditions,

$$\begin{cases} h(0) = h_0, \\ \gamma(0) = \gamma_0 \\ x(0) = x_0 \\ h(t_f) = h_f, \end{cases} \quad (3)$$

where the optimal final time $t_f = t_0$ will be also calculated, $h$ is the airplane altitude, $V$ is its speed, $\gamma$ is the angle between its velocity and the horizontal axis, and finally $x$ denotes the horizontal coordinate position.

For numerical purposes, we assume (data close to those of an Air bus 320)
$m_f = 120,000Kg$, $S_f = 260m^2$, $a = 0.138$ rad, $g = 9.8m/s^2$,

$$\rho(h) = 1.225(1 - 0.0065h/288.15)^{4.225} Kg/m^3,$$

$$a_F = 0.0175,$$

$$(C_L)_a = 5$$

$$(C_D)_0 = 0.0175,$$

$$K_1 = 0.0,$$

$$K_2 = 0.06$$

$$(C_L)_0 = 0,$$

$$C_D = (C_D)_0 + K_1 C_L + K_2 C_L^2,$$
$$C_L = (C_L)_0 + (C_L)_a a,$$
$$L = \frac{1}{2}\rho(h) V^2 C_L S_f,$$
$$D = \frac{1}{2}\rho(h) V^2 C_D S_f,$$
$$F = 240000$$

and where units refer to the International System.

To simplify the analysis, we redefine the variables as below indicated:

$$\begin{cases} h = u_1, \\ \gamma = u_2 \\ V = u_3 \\ x = u_4. \end{cases} \tag{4}$$

Thus, denoting $\mathbf{u} = (u_1, u_2, u_3, u_4) \in U = W^{1,2}([0, t_f]; \mathbb{R}^4)$, the system above indicated may be expressed by

$$\begin{cases} \dot{u}_1 = f_1(\mathbf{u}) \\ \dot{u}_2 = f_2(\mathbf{u}) \\ \dot{u}_3 = f_3(\mathbf{u}) \\ \dot{u}_4 = f_4(\mathbf{u}), \end{cases} \tag{5}$$

where,

$$\begin{cases} f_1(\mathbf{u}) = u_3 \sin(u_2), \\ f_2(\mathbf{u}) = \frac{1}{m_f u_3}(F \sin[a + a_F] + L(\mathbf{u})) - \frac{g}{u_3}\cos(u_2), \\ f_3(\mathbf{u}) = \frac{1}{m_f}(F \cos[a + a_F] - D(\mathbf{u})) - g \sin(u_2) \\ f_4(\mathbf{u}) = u_3 \cos(u_2). \end{cases} \tag{6}$$

We solve this last system for the following boundary conditions:

$$\begin{cases} h(0) = 0 \; m, \\ x(0) = 0 \; m, \\ h(t_f) = 11000 \; m. \end{cases} \tag{7}$$

Observe that having calculated a solution $(h, \gamma, V)$ we may easily calculate $x$ through the fourth equation, that is,

$$x(t) = x(0) + \int_0^t V(t) \cos(\gamma(t)) \; dt$$

Thus, for a solution, we may consider just the first three equations in the three variables $(h, \gamma, V)$. We have numerically obtained $t_f = t_0 = 662.9544s$ and for the solutions for $h, \gamma, V$, please see the figures 1, 2 and 3, respectively. We have set $m_8 = 10000$ nodes.

Here we present the software in MAT-LAB entitled "FlightMech08Sept2026A" through which such numerical results have been obtained.

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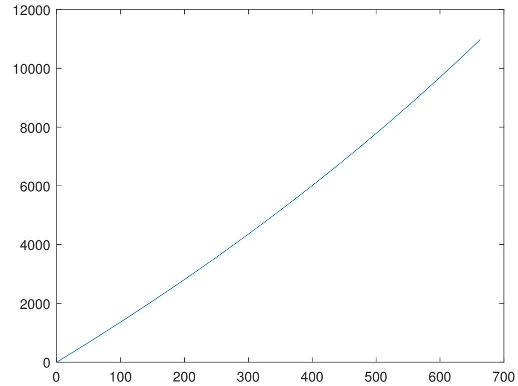


Figure 1: The solution $h$ (in m) for $t_f = 662.9544s$.

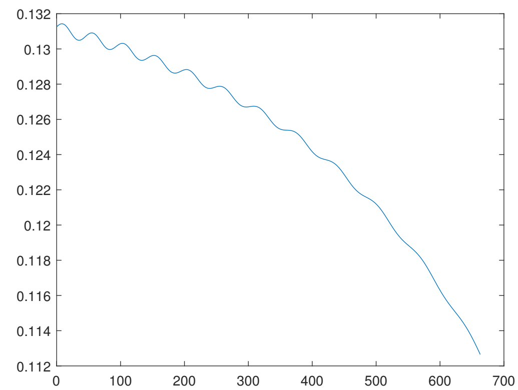


Figure 2: The solution $\gamma$ (in rad) for $t_f = 662.9544s$.

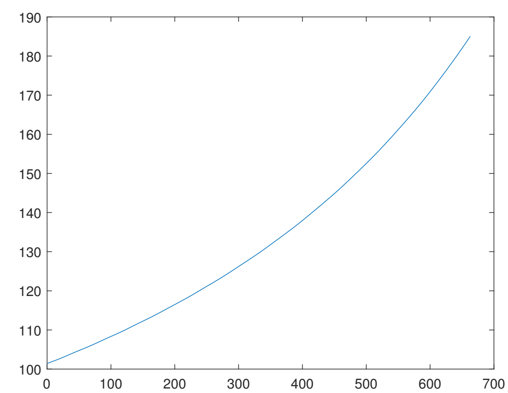


Figure 3: The solution $V$ (in m/s) for $t_f = 662.9544s$.

1. clear all global m8 d tf CL CD ho go vo Sf mf a aF F h v g W5 G K8 b25 k10 to
   m8=10000;
   K8=0;
   mf=120000;
   Sf=260;
   G=9.81;
   a=0.138;
   aF=0.0175;
   F=240000;
   k10=10;
   CLa=5;
   CDo=0.0175;K1=0.0;
   K2=0.06;
   CLo=0;
   CL=CLo+CLa*a;
   $CD = CDo + K1 * CL + K2 * CL^2$;
   lb(1,1)=70;
   ub(1,1)=310;
   lb(2,1)=0.05;
   ub(2,1)=0.45;
   lb(3,1)=450;
   ub(3,1)=800;
   xo(1,1)=120;
   xo(2,1)=0.15;
   xo(3,1)=670;
   b14=1;
   k3=1;
   while $(b14 > 10^{-4})$ && $(k3 < 5)$
   k3
   k3=k3+1;
   k10=k10+5;
   b12=1;
   k1=1;
   while $(b12 > 10^{-3.2})$ && $(k1 < 10)$
   k1
   k3
   k1=k1+1;
   X=lsqnonlin('funFlightMech08Sept2026A',xo,lb,ub);
   b12=max(abs(X-xo));

xo=X;
h(m8/2,1);
end;
b14=max(abs(b25));
end;
for i=1:m8
x8(i,1)=i*d;
end;
plot(x8,h)

***************************************
With the main function entitled ”funFlightMech08Sept2026A”
*******************************************

1. function W5=funFlightMech08Sept2026A(x)
global m8 d tf CL CD ho go vo Sf mf a aF F h v g W5 G K8 k10 b25 to
Ho(1,1)=0;
Vo(1,1)=x(1,1);
Go(1,1)=x(2,1);
to=x(3,1);
d=to/m8;
ho(1,1)=0;
vo(1,1)=120;
go(1,1)=0.15;
k=1;
for k5=1:k10
$r(k) = abs(1.225 * (1 - 0.0065 * Ho(1,1)/288.15)^{(}4.225));$
$L = 1/2 * r(k) * vo(1,1)^2 * CL * Sf;$
$D = 1/2 * r(k) * vo(1,1)^2 * CD * Sf;$
h(k,1)=(Ho(1,1)+vo(k,1)*sin(go(k,1))*d+K8*ho(k,1)*d)/(1+K8*d);
g(k,1)=(Go(1,1)+1/mf*(F*sin(a+aF)+L)/vo(k,1)*d-G/vo(k,1)*cos(go(k,1))*d+K8*go(k,1)*d)/(1+K8*d);
v(k,1)=(Vo(1,1)+1/mf*(F*cos(a+aF)-D)*d-G*sin(go(k,1))*d+K8*vo(k,1)*d)/(1+K8*d);
b25(k,1)=abs(v(k,1)-vo(k,1));
vo(k,1)=v(k,1);
ho(k,1)=h(k,1);
go(k,1)=g(k,1);
end;
for k=2:m8
vo(k,1)=v(k-1,1);
ho(k,1)=h(k-1,1);
go(k,1)=g(k-1,1);

```
for k5=1:k10
r(k) = abs(1.225 * (1 - 0.0065 * ho(k,1)/288.15)^(4.225));
L = 1/2 * r(k) * vo(k,1)^2 * CL * Sf;
D = 1/2 * r(k) * vo(k,1)^2 * CD * Sf;
h(k,1)=(h(k-1,1)+vo(k,1)*sin(go(k,1))*d+K8*ho(k,1)*d)/(1+K8*d);
g(k,1)=(g(k-1,1)+1/mf*(F*sin(a+aF)+L)/vo(k,1)*d-G/vo(k,1)*cos(go(k,1))*d+K8*go(k,1)*d)/(1+K8*d);
v(k,1)=(v(k-1,1)+1/mf*(F*cos(a+aF)-D)*d-G*sin(go(k,1))*d+K8*vo(k,1)*d)/(1+K8*d);
b25(k,1)=abs(v(k,1)-vo(k,1));
vo(k,1)=v(k,1);
ho(k,1)=h(k,1);
go(k,1)=g(k,1);
end;
end;
W5(1,1)=30*(h(m8,1)-11000);
W5(2,1)=g(m8,1);
W5(3,1)=25*to;
for i=1:m8-2
W5(i+3,1) = 500 * (h(i+2,1) - 2 * h(i+1,1) + h(i,1))/d^2;
end;
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# 4 Acknowledgements

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# 5 Conclusion

In this article we have presented a numerical procedure for solving a class of two point boundary value ODE systems with applications to a flight mechanics model.

The numerical method is based on the Backward Euler approach adapted for a two point boundary value ODE system.

In a near future research we intend to address some other linear and non-linear cases.

1. **Conflict of interest declaration: The author declares no conflict of interest concerning this article.**

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