

Beam-like models for the post-buckling behaviour of wingbox structures

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ABSTRACT

Design and optimisation of wing structures are often constrained by buckling, even though thin-walled components, particularly stiffened panels, retain significant load-carrying capacity beyond their initial buckling point. Enabling the design for a safe use of this post-buckling regime requires a computationally efficient methodology that can be integrated into design and optimisation processes. This work provides a representation of one or more post-buckled wingbox regions through the stiffness matrix of a one-dimensional beam model. The approach is motivated by the observation that high-aspect-ratio wings exhibit global deformation patterns that closely resemble beam modes. The three-dimensional wingbox is mapped onto a one-dimensional representation using a cross-section modeller, while the reduction in stiffness induced by panel buckling is estimated using a semi-analytical post-buckling method and incorporated into the cross-sectional stiffness. This leads to a computationally efficient formulation suitable for repeated evaluations within design and optimisation loops. Results for several wingbox configurations show that the predicted buckling loads and post-buckling trends are in good agreement with Abaqus simulations in the early post-buckling regime. The Abaqus response becomes more compliant once local sagging and area reduction occur, which are not yet captured by the current model. Nevertheless, the proposed methodology accurately predicts the overall post-buckling behaviour within the range relevant for design, where wings are not expected to operate deeply into the post-buckling regime.

1. Introduction

Aerospace structures are typically thin-walled and lightweight, making it critical to strike a balance between structural weight and failure limits. In many designs, buckling is treated as a failure limit, as it can coincide with final failure, such as in single load-path critical structures. However, significant load-carrying capacity may still exist after buckling and before final failure, particularly for plate-like components [1, 2]. Experimental and theoretical studies further demonstrate that thin plates can follow a stable post-buckling deformation path and sustain increasing loads well beyond the buckling point before ultimate failure occurs [3, 4]. Stiffened plates, in particular, are known to sustain increasing loads in the post-buckling regime, with numerous experimental studies showing that local skin buckling occurs well before collapse and that the ultimate failure load can significantly exceed the initial buckling load [5–8]. Recognising this behaviour has led to a shift in aerospace structural design philosophy, in which the post-buckling regime is intentionally exploited to improve structural efficiency, allowing controlled post-buckling under ultimate load conditions rather than treating buckling as a strict failure limit [9, 10].

The literature shows that buckling often emerges as the first active constraint in wing design and optimisation. The buckling response was used as a constraint for sizing a flexible aircraft in a nonlinear aeroelastic framework developed by Cesnik and Su [11]. Oliveri et al. [12] designed a unitised wingbox demonstrator with variable angle tow, in which buckling of the skin was only considered at the maximum design load. Kenway et al. [13] used the NASA Common Research Model in an optimisation study where buckling was a limiting constraint for the upper skin. Stodieck et al. [14] used buckling as an optimisation constraint in their work. A multilevel aeroelastic tailoring framework developed by Othman et al. [15], aimed to minimise the weight

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of a wing box by imposing buckling as one of the constraints. Sodja et al. [16] aimed to minimise structural mass while considering multiple constraints including cruise shape, control effectiveness, maximum strain, and buckling. They compared a tailored composite wing design to a reference design, and for both designs, buckling was an active constraint.

Operating beyond initial buckling can enable further use of the unexploited load-carrying capacity of structural components, potentially leading to lighter-weight structural designs. To achieve this in practice, it is essential to incorporate post-buckling behaviour through a computationally efficient approach that is suitable for use within design and optimisation frameworks. Even the inclusion of buckling constraints alone can significantly increase the computational cost of structural optimisation, limiting scalability in practical design studies [17]. When nonlinear post-buckling analysis is coupled directly with optimisation schemes, the need for repeated nonlinear equilibrium and stability evaluations leads to prohibitive computational cost, making such approaches impractical for use within design and optimisation frameworks [18–20]. Accordingly, existing optimisation approaches typically rely on reduced-order initial post-buckling formulations, such as asymptotic Koiter-based methods, or surrogate models to alleviate the computational cost of full nonlinear analyses [18, 21].

The limitations of high-fidelity nonlinear post-buckling analyses and the reliance on reduced-order approximations motivate the need for alternative modelling strategies that are computationally efficient while remaining suitable for wing-level structural analysis. For slender wing structures, such strategies naturally involve reduced-order structural representations capable of capturing the dominant global response while accounting for local effects in an averaged sense. Beam-based models therefore provide a natural framework for incorporating post-buckling effects at the wingbox level within a computationally efficient formulation. Beam formulations have been developed in the literature as reduced-order representations for slender structural components such as wings, forming a hierarchy of models with increasing kinematic richness. Classical Euler–Bernoulli and Timoshenko beam theories constitute lower-order members of this hierarchy, while refined beam formulations introduce enriched cross-sectional kinematics to capture shear deformation, warping, and other local effects [22, 23]. Within this hierarchical setting, the Carrera Unified Formulation provides a general framework for generating beam theories of arbitrary order, with classical models recovered as special cases and higher-order expansions enabling an increasingly detailed representation of three-dimensional structural behaviour [22, 24]. The enhanced fidelity offered by such higher-order beam models has been shown to improve the prediction of cross-sectional deformation and stress distributions in thin-walled and non-classical geometries, albeit at the cost of a rapidly increasing number of degrees of freedom as the kinematic order is enriched [24, 25]. In contrast, lower-order beam descriptions, particularly Timoshenko-type formulations, retain the dominant global bending, shear, and torsional response without explicitly enriching the kinematic field, and therefore remain attractive within the broader hierarchy of beam models [23, 26]. An alternative means of incorporating complex cross-sectional behaviour within such reduced-order beam models is provided by cross-sectional pre-integration techniques, in which detailed two-dimensional analyses are used to compute generalised stiffness matrices employed in one-dimensional beam formulations [26, 27]. These beam-based approaches inherently rely on beam-like kinematic assumptions and typically assume constant or slowly varying cross-sections along the span, which delineates their range of applicability [24, 27].

The main contribution of the present study is a methodology to obtain the stiffness matrix of one-dimensional Timoshenko beam elements with locally post-buckled regions. It is herein observed that the proposed methodology correlates well with the detailed shell-based model into the post-buckling regime, until cross-sectional sagging develops. Four isotropic rectangular wingbox cases are used in this investigation, consisting of a lower and higher aspect ratio, respectively 10 and 20, each having a thinner or thicker spar. The proposed nonlinear algorithm evaluates the local reduction in stiffness occurring upon buckling using a perturbation-based semi-analytical approach [28] and is integrated into the process, leading to an efficient approach that suits design and optimisation purposes.

The paper is organised as follows: Section 2 outlines the proposed methodology, including the one-dimensional representation of a three-dimensional wingbox, the semi-analytical post-buckling formulation, and the interaction between different components. A verification case is also presented. Section 3 presents and discusses the results, including the limitations of the proposed approach. Finally, Section 4 summarises the main conclusions.

2. Methodology

The three-dimensional response is reduced to a one-dimensional Timoshenko beam description through cross-sectional stiffness properties, as illustrated in Fig. 1. Local post-buckling effects of wing panels are incorporated in an averaged sense by modifying the effective cross-sectional stiffness following buckling, rather than resolving local nonlinear deformation fields explicitly. The semi-analytical post-buckling formulation proposed by Elalfy et al. [28] estimates the stiffness reduction associated with local panel buckling and integrates this effect into the beam-level representation, enabling efficient analysis suitable for design and optimisation applications.

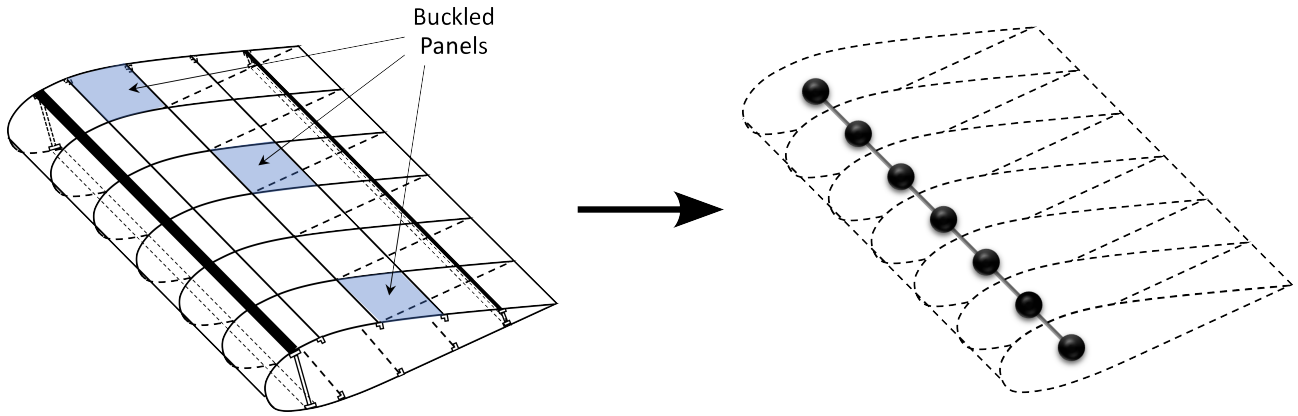


Figure 1: One-dimensional representation of three-dimensional buckled wingbox.

2.1. Cross-sectional Stiffness Matrix Formulation

As proposed by Ferede and Abdalla [29], the stiffness properties of composite beam-like structures can be reduced to a 6×6 stiffness matrix, compatible with first-order shear deformable Timoshenko beam elements. In the present work, a mathematically consistent formulation of the cross-sectional stiffness matrix is developed, starting from a mixed displacement-stress representation and yielding a closed-form 6×6 Timoshenko stiffness matrix $\mathbf{K}_{cs}$. Starting from the cross-sectional kinematic description, the strain field ϵ is expressed in compact form in terms of the displacement vector $\mathbf{q}$ and its derivative with respect to the beam coordinate $\mathbf{q}'$:

$$\epsilon = \mathbf{B}_0 \mathbf{q} + \mathbf{B}_1 \mathbf{q}', \quad (1)$$

where $\mathbf{B}_0$ and $\mathbf{B}_1$ are kinematic operators. This leads to the following expression for the strain energy per unit depth:

$$U = \frac{1}{2} \int_A (\mathbf{B}_0 \mathbf{q} + \mathbf{B}_1 \mathbf{q}')^\top \mathbf{C} (\mathbf{B}_0 \mathbf{q} + \mathbf{B}_1 \mathbf{q}') dA, \quad (2)$$

where $\mathbf{C}$ is the constitutive matrix and A is the cross-sectional area.

By applying a variational principle to the strain energy functional and defining the Lagrangian functional accordingly, the Euler-Lagrange equations introduce the generalised momenta $\mathbf{p}$, leading to a first-order system written as:

$$\mathbf{x}' = \mathcal{A} \mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}, \quad (3)$$

where $\mathcal{A}$ is the system matrix and the generalised momenta $\mathbf{p}$ correspond to the tractions acting over the cross-section.

According to Saint-Venant's principle, only stress states that persist away from the load-introduction region are physically admissible. In the present formulation, these persistent solutions are obtained by considering the invariant subspace associated with the zero eigenvalue:

$$\mathcal{A} \phi = \mathbf{0}, \quad (4)$$

where ϕ is a vector in the null space of $\mathcal{A}$.

Since the system matrix $\mathcal{A}$ is non-normal, the eigenvectors associated with the zero eigenvalue do not, in general, span the full invariant subspace. As a result, the Saint–Venant solution space is complemented by generalised eigenvectors ψ , which are defined recursively through:

$$\mathcal{A}\psi = \phi. \quad (5)$$

A solution of the first-order system associated with the zero eigenvalue can be expressed as a linear combination of the corresponding eigenvectors and generalised eigenvectors. Accordingly, the state vector $\mathbf{x}$ is expressed as a linear combination of the persistent Saint–Venant modes, with $\mathbf{c}$ as the corresponding coefficients. Using this representation, the generalised momenta $\mathbf{p}$ can be expressed in terms of the same coefficients, as derived in Appendix A. As a result, the force and moment resultants over the cross-section can be written as:

$$\mathbf{f} = \Phi_1 \mathbf{c}. \quad (6)$$

An energy-consistent definition of the cross-sectional stiffness then follows as:

$$\mathbf{K}_{cs} = \Phi_1 \Phi_3^{-1} \Phi_1^T, \quad (7)$$

where Φ_1 and Φ_3 are matrices constructed from the Saint–Venant modes and the constitutive matrix, as defined in Appendix A. The complete derivation of the cross-sectional stiffness matrix is provided in Appendix A, where a self-contained and fully traceable formulation is developed. Appendix B summarises the rigid-body relations repeatedly used to simplify the derivation.

2.2. Semi-analytical perturbation-based local post-buckling

A perturbation-based analysis [28] is used to estimate the post-buckling stiffness of buckled panels over the cross-section, such as those illustrated in Fig. 1. The methodology is developed using a semi-analytical approach based on the Rayleigh–Ritz method and the perturbation formulation originally proposed by [30]. Hierarchical polynomials are employed to approximate the displacement field, improving computational efficiency by eliminating the need for numerical integration [28]. Furthermore, they offer enhanced flexibility for modelling different boundary conditions, unlike trigonometric functions, which are inherently limited to representing simply supported cases.

Within this framework, the plate displacements are approximated using a Rayleigh–Ritz expansion of the form

$$\begin{aligned} w(x, y) &= \sum_{i=2}^N a_i \phi_i^w \\ u(x, y) &= -\lambda e_{x0} x - \lambda \frac{g_{xy0}}{2} y + \sum_{i=2}^N b_i \phi_i^u \\ v(x, y) &= -\lambda e_{y0} y - \lambda \frac{g_{xy0}}{2} x + \sum_{i=2}^N c_i \phi_i^v, \end{aligned} \quad (8)$$

where $w(x, y)$ denotes the out-of-plane displacement, $u(x, y)$ and $v(x, y)$ are the in-plane displacements. The coefficients a_i , b_i , and c_i are Ritz parameters, with ϕ_i^w , ϕ_i^u , and ϕ_i^v being the corresponding shape functions. The quantities e_{x0} , e_{y0} , and g_{xy0} are the prescribed uniform in-plane edge strains in the x - and y -directions and the in-plane shear strain, respectively, while λ is a load parameter scaling these applied strains.

Based on the assumed displacement field, the governing equations are obtained by formulating the total potential energy as follows:

$$\begin{aligned} \Pi(u, v, w) &= \frac{1}{2} \int_0^A \int_0^B \left[A_{11} (\epsilon_x^0)^2 + 2A_{12} \epsilon_x^0 \epsilon_y^0 + A_{22} (\epsilon_y^0)^2 + A_{66} (\gamma_{xy}^0)^2 \right. \\ &\quad \left. + D_{11} \kappa_x^2 + 2D_{12} \kappa_x \kappa_y + 2D_{16} \kappa_x \kappa_{xy} + 2D_{26} \kappa_y \kappa_{xy} + D_{22} \kappa_y^2 + D_{66} \kappa_{xy}^2 \right] dy dx, \end{aligned} \quad (9)$$

where ϵ_x^0 , ϵ_y^0 , and γ_{xy}^0 denote the mid-plane membrane strains, κ_x , κ_y , and κ_{xy} are the corresponding plate curvatures, and A_{ij} and D_{ij} represent the in-plane and bending stiffness coefficients, respectively.

Minimisation of the total potential energy with respect to the Ritz coefficients yields a set of nonlinear equilibrium equations. Following the perturbation approach adopted in [28, 30], the applied edge strains and the transverse Ritz coefficients are expanded about the buckling point as

$$\begin{aligned} e_{x0} &= e_{x0}^{(0)} + \epsilon^2 e_{x0}^{(2)} + \dots & e_{y0} &= e_{y0}^{(0)} + \epsilon^2 e_{y0}^{(2)} + \dots \\ g_{xy0} &= g_{xy0}^{(0)} + \epsilon^2 g_{xy0}^{(2)} + \dots & a_i &= \epsilon a_i^{(1)} + \epsilon^2 a_i^{(2)} + \dots \end{aligned} \quad (10)$$

Here, (0) and (2) denote the prebuckling and second-order postbuckling contributions, while $a_i^{(1)}$ represents the linear buckling eigenvector. The post-buckled in-plane stiffnesses: A_{11}^{PB} , A_{12}^{PB} , A_{21}^{PB} , A_{22}^{PB} and A_{66}^{PB} , are then obtained by relating the second-order perturbed membrane resultants: $\bar{N}_x^{(2)}$, $\bar{N}_y^{(2)}$ and $\bar{N}_{xy}^{(2)}$; to the corresponding second-order edge strains $e_{x0}^{(2)}$, $e_{y0}^{(2)}$ and $g_{xy0}^{(2)}$, as shown in Eq. 11. The definitions of these second-order parameters are detailed in Elalfy et al. [28].

$$A_{11}^{PB} = \frac{\bar{N}_x^{(2)}}{e_{x0}^{(2)}}, \quad A_{12}^{PB} = \frac{\bar{N}_x^{(2)}}{e_{y0}^{(2)}}, \quad A_{21}^{PB} = \frac{\bar{N}_y^{(2)}}{e_{x0}^{(2)}}, \quad A_{22}^{PB} = \frac{\bar{N}_y^{(2)}}{e_{y0}^{(2)}}, \quad A_{66}^{PB} = \frac{\bar{N}_{xy}^{(2)}}{g_{xy0}^{(2)}}. \quad (11)$$

The boundary conditions of stiffened panels within a wingbox structure cannot be accurately modelled using ideal boundary conditions. In practice, panel edges are neither perfectly simply supported nor clamped and have been modelled in the literature using intermediate boundary conditions to better represent the actual structural constraints [31, 32]. Differences in stiffness between the skin, ribs, stiffeners, and spars affect the degree of allowed rotation along the panel edges. For example, a very stiff spar may significantly restrict rotation, causing the panel to behave nearly as if it were clamped, although not fully clamped. To account for this, the present study investigates two limiting cases: simply supported edges, giving a lower and therefore conservative bound on the buckling load, and clamped edges, giving an upper bound. The use of hierarchical polynomials allows this boundary condition variation to be implemented efficiently within the same modelling framework [28].

In the context of the present work, a key output of the post-buckling formulation is the post-buckled in-plane stiffness matrix, denoted $\mathbf{A}_{PB}$. This matrix captures the stiffness reduction of the plate after buckling, and its elements are the post-buckled in-plane stiffnesses of Eq. 11. It will later play a central role in modifying the cross-sectional constitutive matrix, $\mathbf{C}$ of Eq. 2 as shown in the next section.

2.3. Algorithm for Post-buckling Representation in Beam Models

As schematically shown in Fig. 2, the three main components involved in the proposed algorithm are: cross-section modeller, perturbation-based post-buckling analysis, and co-rotational nonlinear beam solver. The proposed methodology starts by feeding the geometry and material properties into the cross-section modeller to calculate the 6×6 Timoshenko stiffness matrices $\mathbf{K}_{cs}$ at all node locations. These stiffness matrices are the inputs for the co-rotational nonlinear beam solver, which calculates forces and displacements at each node. Normally, when no local post-buckling is accounted for, the nonlinear beam solver keeps iterating step-by-step until the full applied load is attained.

To include post-buckling, the proposed workflow requires that all panels must be checked for buckling, at each converged step of the Newton–Raphson scheme. The converged forces and displacements are fed back into the cross-section modeller, which in turn retrieves the stress and strain distribution throughout all defined cross-sections. The retrieved stress distribution is then used by the semi-analytical perturbation-based post-buckling analysis to check if buckling occurs. When no panel buckles, the Newton–Raphson proceeds with the next load increment. However, when buckling occurs at any region over the defined cross-sections, the nonlinear beam solver steps back to a load level and corresponding displacement state associated with the lowest buckling eigenvalue, considering all panel pockets in all defined cross-sections over the wingbox. In fact, this load level corresponds to the first local buckling load of the wing. At this point, the cross-section modeller reevaluates the stress distribution at the critical load point. Given the stresses at the critical load point and the critical panel dimensions, the perturbation-based post-buckling analysis calculates the post-buckled stiffness of the buckled regions ($\mathbf{A}_{PB}$ of Eq. 11). The updated stiffness of the buckled panel is then accounted for in

the cross-section modeller, with a new point on the load-displacement curve obtained by the nonlinear beam solver. This procedure is repeated iteratively, identifying subsequent buckled panels, updating their stiffness, and reflecting these changes in the cross-section modeller accordingly.

The present nonlinear finite element solver uses a co-rotational approach in which small elastic deformations are superimposed on beam rigid rotations [33]. De Breuker [33] implemented the beam solver using the Newton–Raphson root-finding algorithm, in which the load-displacement curve is obtained in a step-wise manner.

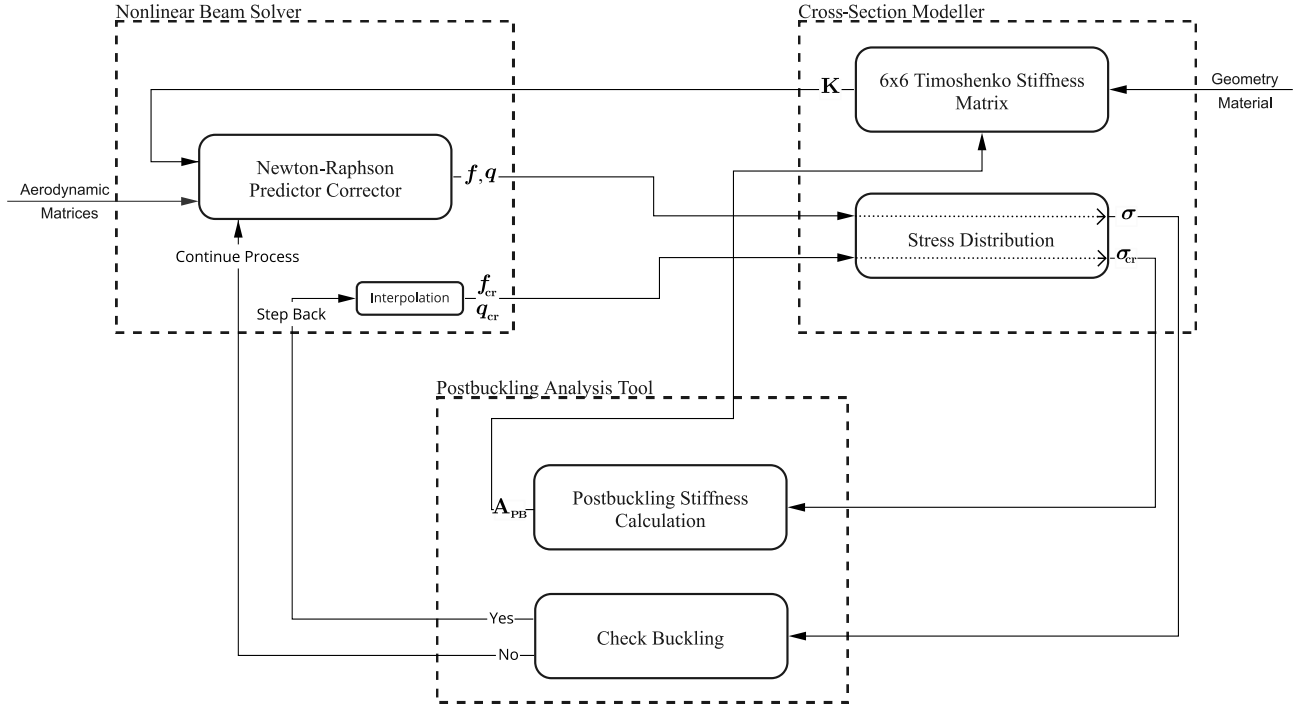


Figure 2: Workflow of representing post-buckling using low-fidelity model.

2.4. High-fidelity Shell Element Reference Model

A higher-fidelity finite element model based on shell elements is implemented in Abaqus [34] to verify both the onset of local buckling, and the global structural response of the wingbox following the post-buckling of individual panels. This model serves as a reference for assessing the accuracy and limitations of the proposed approach, particularly for capturing the stiffness degradation due to subsequent local buckling events, and the overall load-displacement behaviour of the wing.

The wingbox geometry is created using Engineering Sketch Pad (ESP) [35, 36], enabling easy parametric modifications. As shown in Fig. 3, the wingbox is a non-tapered rectangular box, with its dimensions and rib distribution inspired by the work of François et al. [37]. The geometry is exported to a STEP file using the pyCAPS Python library [38] and subsequently imported into Abaqus for meshing and analysis. The following variables define the wing geometry: chord W , height H , half-span L , thickness for the skin t_{skin} , and spar t_{spar} .

Each case is defined by a rectangular, untapered wingbox geometry with a fixed cross-sectional shape. The span is divided into nine bays using evenly spaced ribs, as shown in Fig. 3. Table 1 summarises the key properties for the cases presented in this section, where two high-aspect-ratio ("high-AR") and two low-aspect-ratio ("low-AR") cases are investigated. Two spar thicknesses are investigated, $t_{spar} = 1$ mm and $t_{spar} = 5$ mm, referred to hereafter as the "t1" and "t5" cases. All structural panels are modelled using homogeneous shell sections and meshed with square S4R elements (4-node, reduced integration). A linear elastic, homogeneous material with isotropic in-plane properties is assumed, with the corresponding parameters summarised in Table 2. For each case, the mesh resolution is selected to ensure sufficient element density for capturing all relevant local buckling modes and post-buckling deformations.

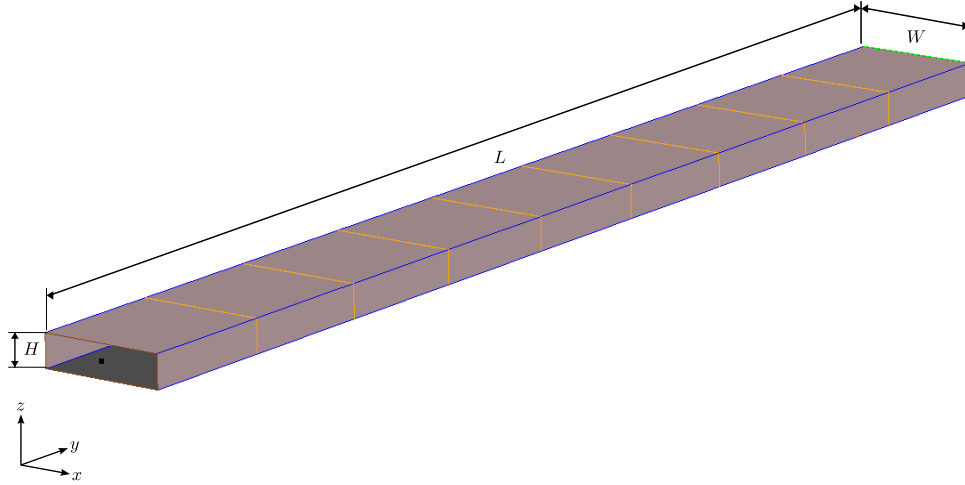


Figure 3: Wingbox CAD in Engineering Sketch Pad.

Table 1

Geometric parameters for each wingbox case.

Case	W [mm]	L [mm]	H [mm]	# Ribs	t_{skin} [mm]	t_{spar} [mm]
High-AR-t5	50.5	1010.0	12.12	8	1.0	5.0
High-AR-t1	50.5	1010.0	12.12	8	1.0	1.0
Low-AR-t5	50.5	502.0	12.12	8	1.0	5.0
Low-AR-t1	50.5	502.0	12.12	8	1.0	1.0

Table 2

Isotropic material properties [36].

E [GPa]	G [GPa]	ν	ρ [kg/m ³]
73.00	27.44	0.33	2.795×10^3

The wingbox is clamped at the root, and a concentrated vertical force is applied at the tip to induce buckling and drive the nonlinear response. First, a linear buckling analysis is performed to identify critical loads and mode shapes. A subsequent static Riks analysis is then carried out around the buckling point to capture the post-buckling behaviour of the structure. The proposed framework is implemented in the in-house wingbox analysis and optimisation tool Proteus [33]. The same verification cases are also analysed using this framework.

3. Results and Discussion

The proposed methodology is assessed against Abaqus reference results across a set of wingbox configurations. First, the predicted linear buckling eigenvalues are compared against Abaqus results. The nonlinear post-buckling load-displacement response is then assessed against Abaqus Riks analyses. Finally, the validity of the beam-level representation is examined in the context of cross-sectional deformation.

3.1. Buckling Eigenvalue Verification

A linear buckling analysis is performed for each case using Abaqus as the first verification step. The results show that the first few buckling modes are all dominated by local buckling behaviour, with deformation concentrated in individual skin panels between ribs. No global bending or torsional modes are observed among the lowest modes, directly supporting the core assumption underlying the semi-analytical approach proposed in [28], which also forms the basis of the current work.

Two boundary condition variants for the local panel model are considered in the comparison: one assuming simply supported (SS) panels and the other assuming fully clamped (C) panels. Both are evaluated against Abaqus linear buckling eigenvalue results, as summarised in Table 3. For all cases, the predictions from Abaqus lie between simply supported and clamped. For the Low-AR-t5 and High-AR-t5 cases, the clamped results are significantly more accurate. This behaviour is consistent with the structural configuration, because the "-t5" cases have the spar thickness high compared to the skin, increasing the rotational restraint at the panel edges and effectively shifting the response toward the clamped boundary condition. However, for cases Low-AR-t1 and High-AR-t1, with the spars having the same thickness as the skin, Abaqus results are further away from the clamped conditions, indicating that the skin is less clamped by the spar, nevertheless far from being simply supported.

Table 3

Comparison of first eigenvalues from Abaqus and proposed local buckling model with different boundary conditions.

Case	Abaqus [N]	Present model (SS) [N]	Present model (C) [N]
High-AR-t5	173.11	92.17	178.74
High-AR-t1	118.81	71.20	138.00
Low-AR-t5	442.17	182.82	454.61
Low-AR-t1	294.48	142.02	353.65

3.2. Nonlinear Post-buckling Load-Displacement Response

A nonlinear Riks analysis is conducted in Abaqus to capture the global load-displacement response of the wingbox beyond buckling. These results serve as a reference for assessing the predictions from the proposed beam model, which iteratively updates the stiffness of buckled panels based on the local response. For each wingbox of Table 1, two loading scenarios are considered: one without and one with a follower tip force.

To support the interpretation of the nonlinear results that follow, a set of spanwise monitoring cross-sections was defined, as shown in Fig. 4. The node sets labelled (CS 1–CS 9) are placed at the mid-bay between ribs, while (CS 10–CS 18) and (CS 20–CS 28) correspond to the first- and third-quarter positions within each bay, respectively. These cross-sections provide a consistent way to examine how the geometry evolves during post-buckling, particularly in regions where the skin is most susceptible to local deformation. The tracked quantities include the deformation shape of each section and its resulting enclosed area, which will later be used to assess cross-sectional changes along the span.

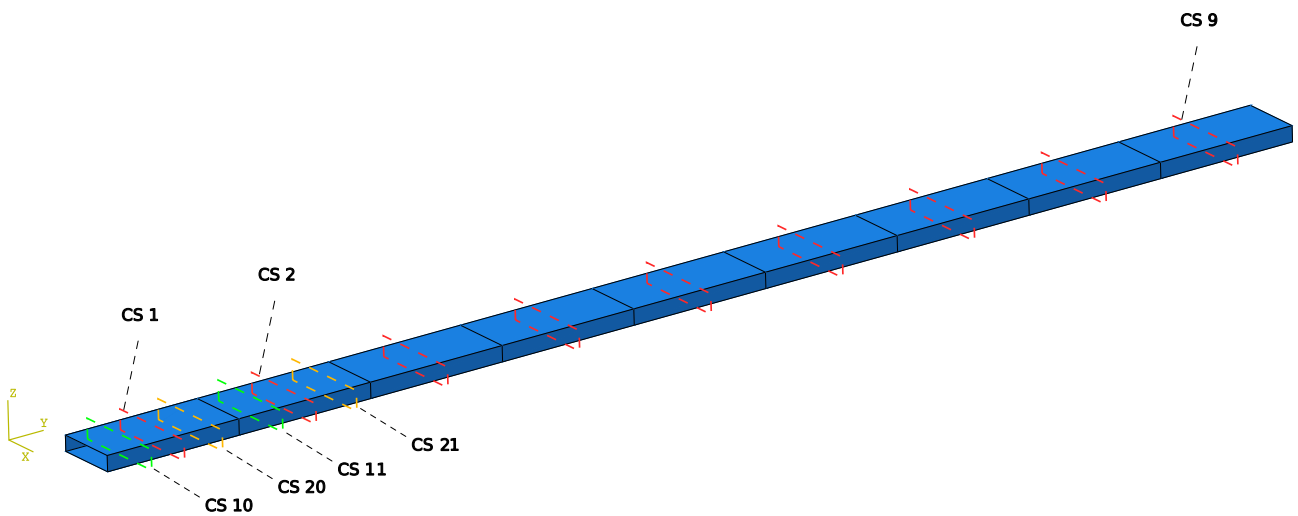


Figure 4: Spanwise cross-sectional locations along the wingbox.

Figs. 5–12 present the nonlinear response for each case, showing the Abaqus Riks results together with the predictions from the proposed model under simply supported and clamped boundary conditions. The linear

buckling eigenvalues from Abaqus are shown as markers on the load-displacement curves for the cases without a follower force. Deformed wings are shown at different tip-load levels, with contours of the minimum in-plane strain field highlighting regions of compressive and tensile strain during the nonlinear response, thereby making the onset and progression of buckling clearly visible. Additionally, the evolution of the cross-sectional area at selected locations is shown, along with the deformed cross-section shapes directly extracted from the finite element models. The representative cross-sections plotted for each case are selected based on the number of half-waves forming within each bay. This number depends on the skin panel aspect ratio and the skin and spar thicknesses and indicates where the largest local deformations occur. Together, these components allow the geometric changes along the span to be tracked, providing a consistent basis for comparing the proposed beam models against Abaqus results.

For the high-aspect-ratio wings, Figs. 5–8 show that post-buckling develops with two half-waves, such that the skin is more prominently sagging at the cross-section at one-quarter CS 10 and three-quarter CS 20 positions. For the low-aspect-ratio wings, sagging happens in the middle of the bay CS 1, meaning skins with only one half-wave post-buckling. The nonlinear load–displacement curves exhibit a similar trend from the initial loading stage through the early post-buckling regime, with the Abaqus response lying between the simply supported and clamped cases. Deviations emerging at higher load levels are discussed in the following section.

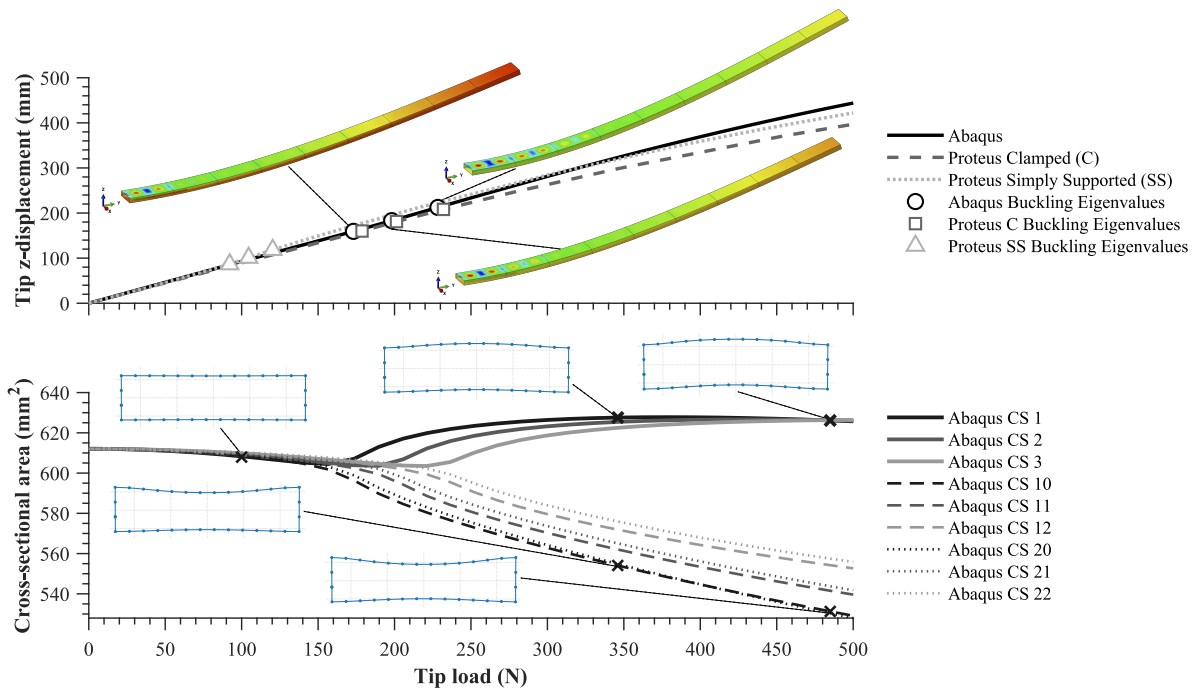


Figure 5: High-AR-t5 without follower force.

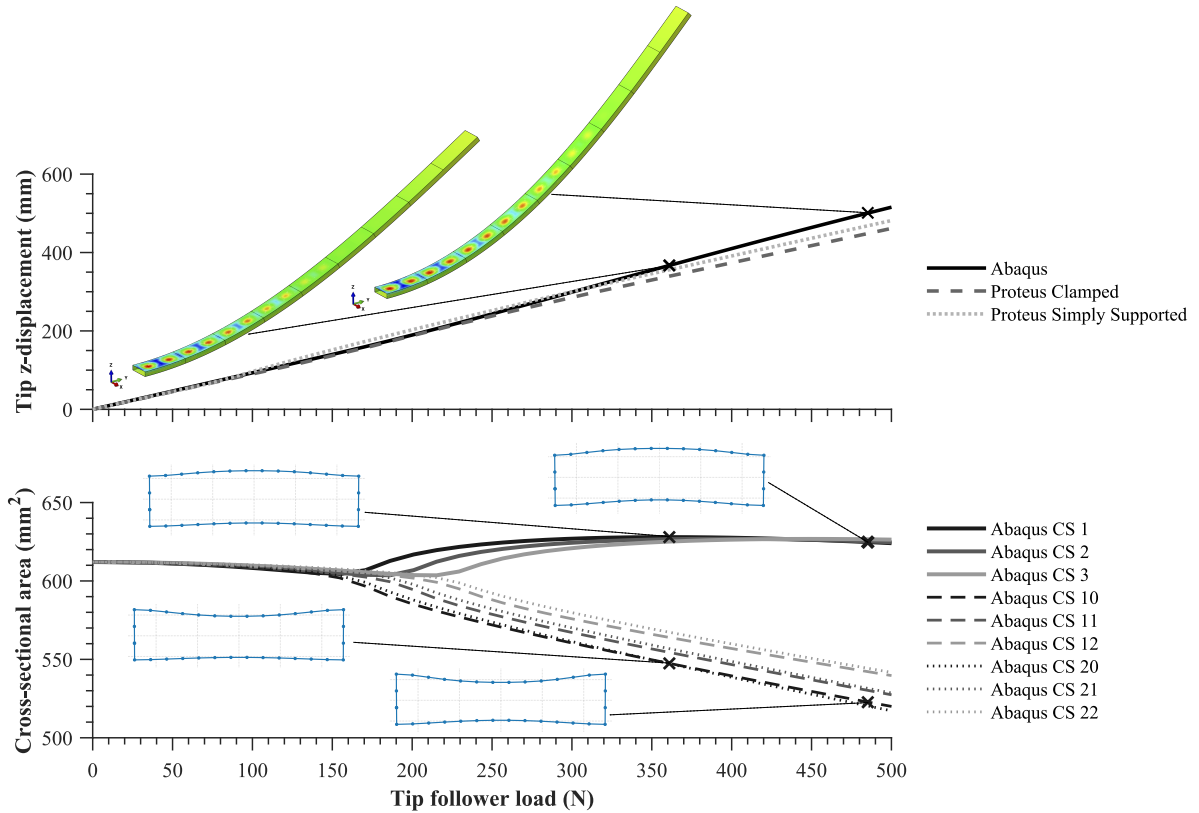


Figure 6: High-AR-t5 with follower force.

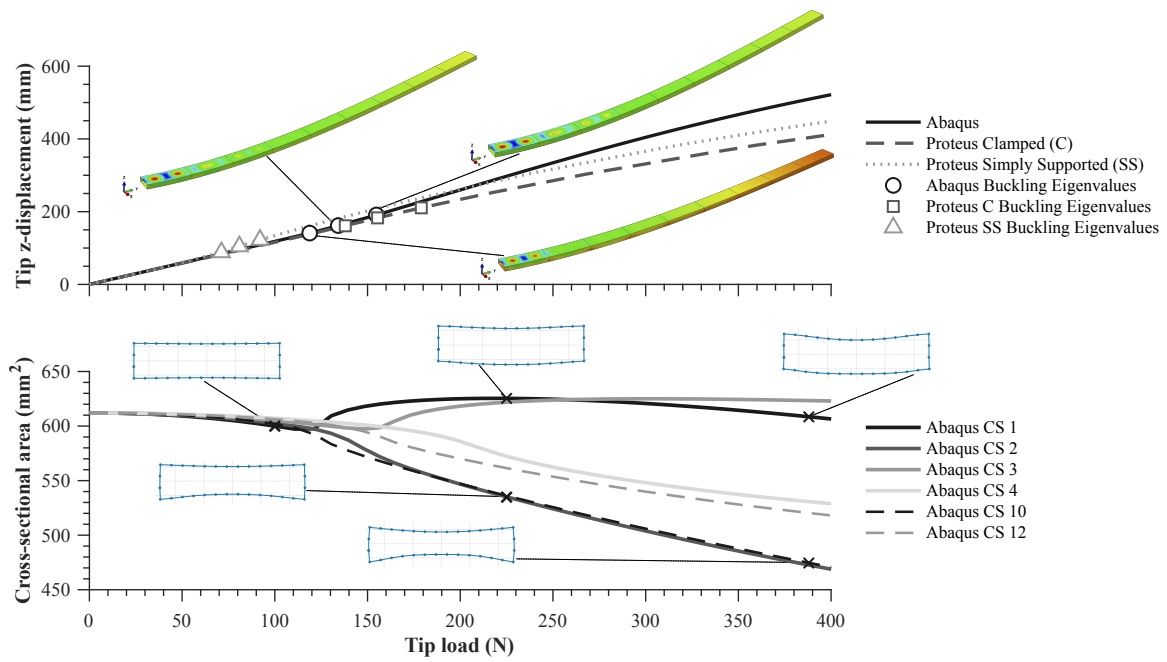


Figure 7: High-AR-t1 without follower force.

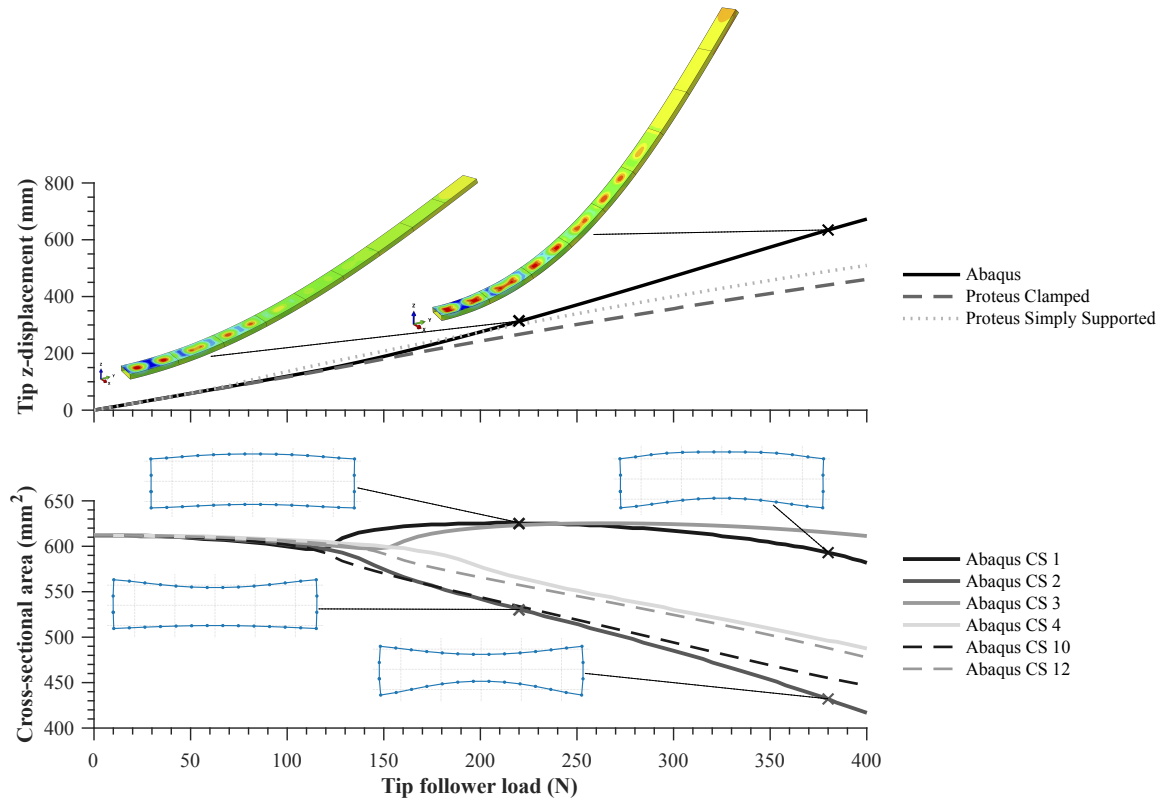


Figure 8: High-AR-t1 with follower force.

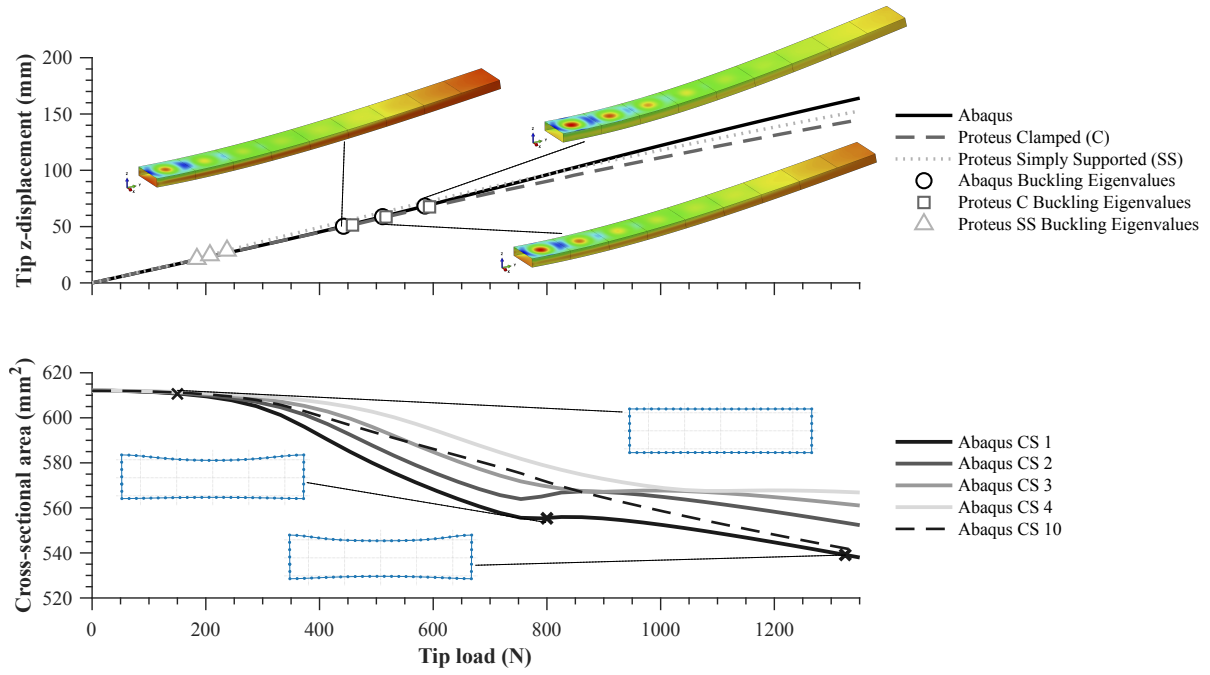


Figure 9: Low-AR-t5 without follower force.

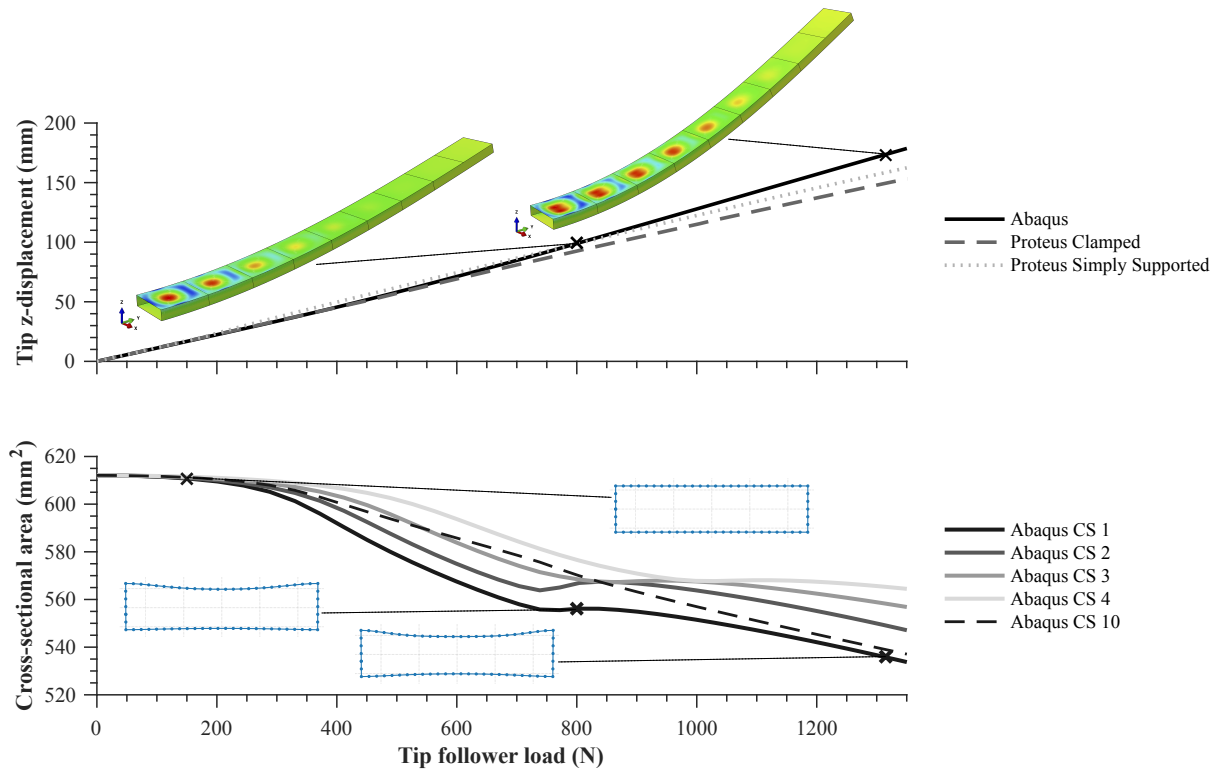


Figure 10: Low-AR-t5 with follower force.

Beam-like models for post-buckling of wingbox structures

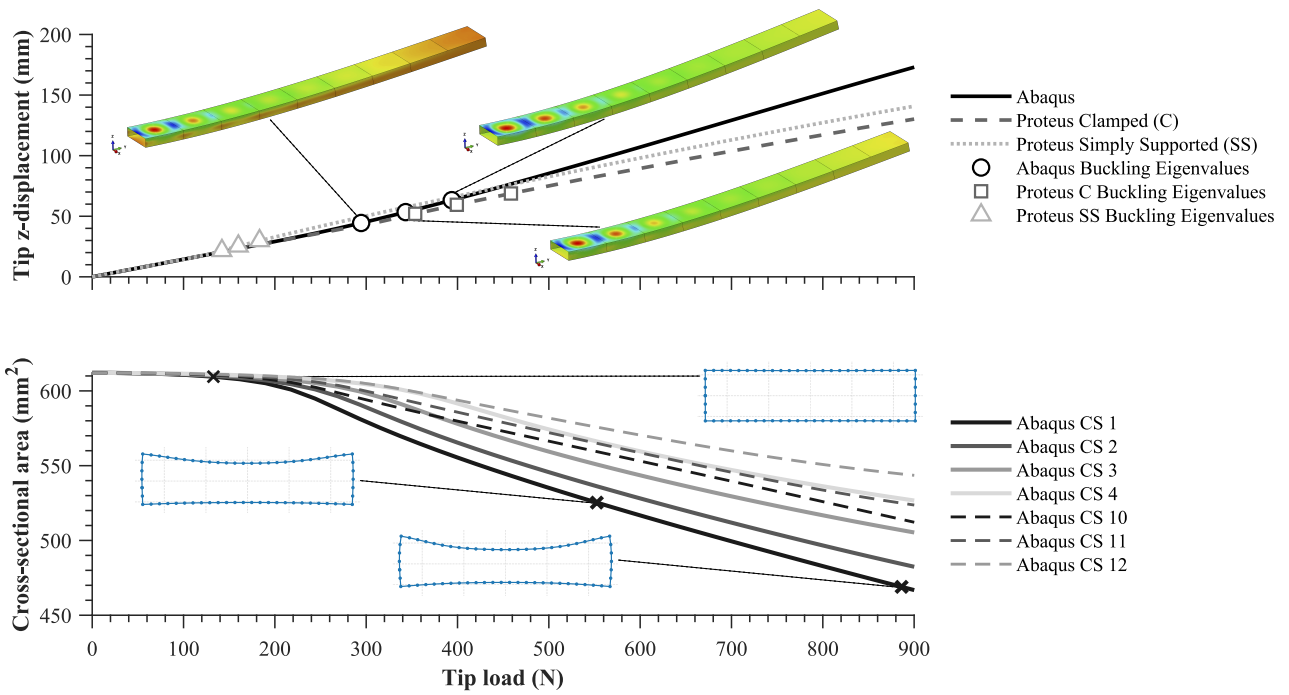


Figure 11: Low-AR-t1 without follower force.

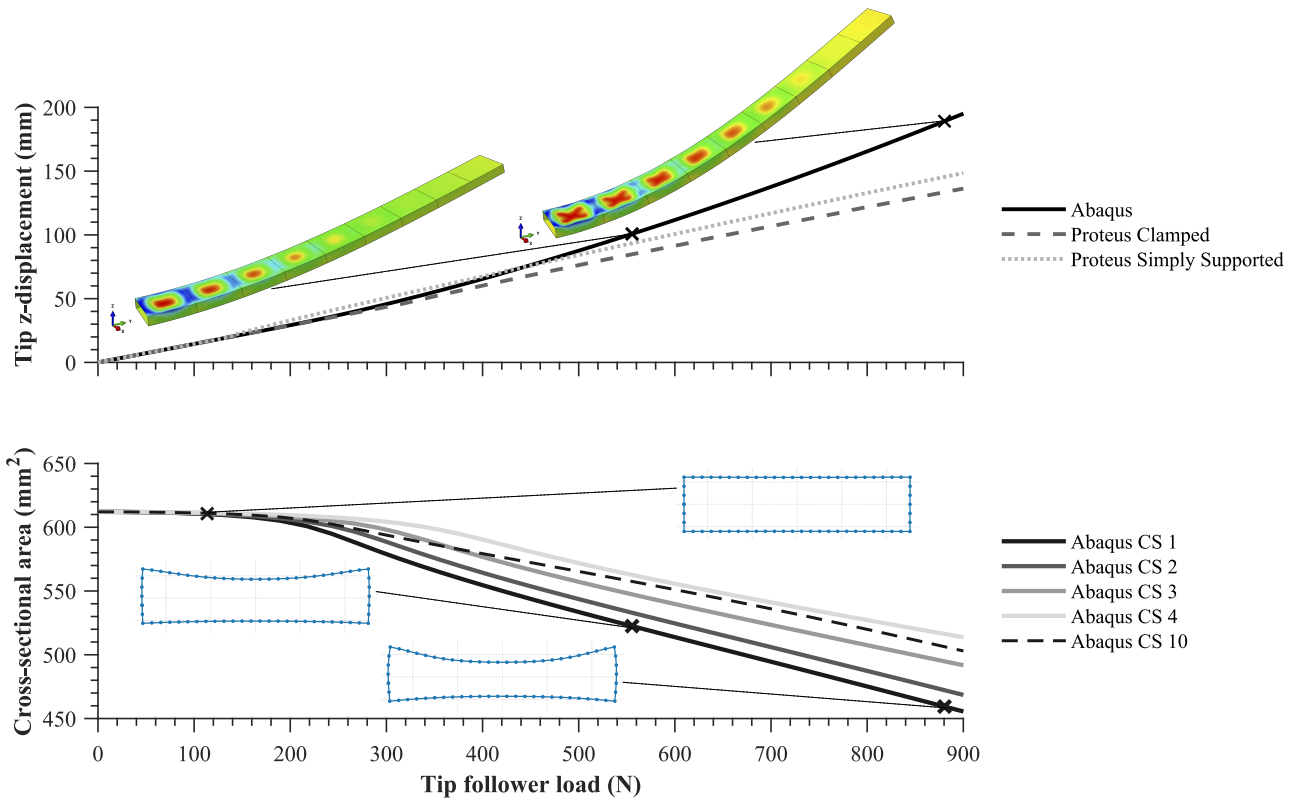


Figure 12: Low-AR-t1 with follower force.

3.3. Validity of Timoshenko's beam models and Brazier deformation

For all wingboxes, as shown in Figs. 5–12, the post-buckling prediction capability of the proposed beam models begins to deviate from Abaqus when the cross-sectional area starts to change, indicating that the prediction capability is driven by changes in cross-section shape that are not accounted for in the present formulation. Medeiros et al. [39] also observed that beam models become inaccurate when changes in cross-section are sufficiently high to produce changes in the overall bending stiffness of the beam.

In the high-aspect ratio results with a thicker spar, shown in Figs. 5 and 6, the deformation patterns between ribs show that buckling occurs with two half-waves within each bay. As a result, the first- (CS 10–CS 12) and third-quarter (CS 20–CS 22) cross-sections exhibit a reduction in enclosed area and a clear development of a sagging-type of deformation, where the skin panels deflect inward. In contrast, the mid-bay section (CS 1–CS 3) shows no noticeable sagging, and its enclosed area increases slightly.

The local reduction in enclosed area reflects inward deflection of the skins, which reduces the effective bending height of the section. This reduction produces a Brazier-type softening that is not captured by the current modelling approach. Consequently, Abaqus response becomes more compliant at higher load levels, leading to larger deflections in the finite element solution.

This behaviour is consistent with the findings of Jiang and Yu [40], who showed that rectangular thin-walled sections subjected to bending do not ovalize in the classical circular-tube sense but instead develop sagging deformation, which they identify as a manifestation of the generalised Brazier effect. Cecchini and Weaver [41] showed that thin-walled airfoil and box sections undergo inward crushing of their walls under bending, which reduces the effective second moment of area and produces a Brazier-type loss of stiffness [42].

Across the remaining cases, similar trends are observed with variations that reflect the changes in spar thickness, rib spacing, and panel aspect ratio. Reducing the spar thickness shifts the Abaqus buckling eigenvalues toward the simply supported prediction, although they generally remain closer to the clamped case as shown in Figs. 7, 11. The onset and magnitude of area reduction strongly influence the post-buckling deviation. Cases with larger area loss and clearer sagging deformation begin to diverge from the model earlier and show a greater mismatch. This effect is amplified under follower forces, where the increased bending moment accelerates the geometric changes and causes the Abaqus response to diverge sooner as shown in Figs. 8, 12. In addition, increasing the rib spacing makes the skin panels more susceptible to local deformation, which in turn increases the area reduction observed in the cross-sections and amplifies the discrepancy with the simplified model.

Beyond the onset of area reduction, the simply supported and clamped predictions no longer bracket the response. The Abaqus tip displacement exceeds even the simply supported case, so a design relying on these predictions would underestimate deflection. The two boundary conditions continue to bracket the linear buckling load across all cases considered, as shown in Table 3, so the onset of area reduction, rather than the boundary condition assumption, sets the effective limit of validity.

4. Conclusion

This work presented a computationally efficient approach for representing post-buckled regions within a one-dimensional beam model of a wingbox. By embedding the stiffness reduction of locally buckled panels into the cross-sectional stiffness matrix, the method provides a practical way to incorporate post-buckling effects within design and optimisation studies. The formulation combines a semi-analytical post-buckling model for individual panels with a cross-section modeller that maps the three-dimensional wingbox onto a one-dimensional beam representation.

Verification against Abaqus analyses showed that the methodology captures the onset and progression of buckling accurately, under both simply supported and clamped boundary conditions. The model reproduces the stiffness drop associated with panel buckling and follows the global post-buckling trend well. Deviations appear mainly when significant cross-sectional sagging develops – an effect not included in the present formulation – but the overall behaviour remains consistent with expectations for thin-walled wing structures. Within the preliminary design range, the method captures the essential post-buckling behaviour while remaining computationally efficient.

Future work will focus on applying the enhanced method within aeroelastic optimisation studies to enable a deeper exploration of structural efficiency beyond current design limits.

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Appendix A. Cross-section Modeller Formulation

An introduction to Saint–Venant’s principle is necessary for understanding the formulation of the cross-section modeller. A fundamental implication of this principle is that two statically equivalent load distributions produce the same stress state at points sufficiently far from the region where the loads are applied. This behaviour arises from the fact that the linear elastostatic problem is an elliptic boundary-value problem [43]. The elliptic nature of the equations implies the absence of characteristic curves; therefore, discontinuities cannot propagate through the domain. Any localised irregularity or discontinuity at the boundary is smoothed out, resulting in a continuous stress field throughout the interior. This means that only the local disturbances associated with how the load is applied decay with distance, whereas the global effect of the load persists throughout the structure. This spatial smoothing is analogous to dropping a small droplet of ink into still water: the concentration is initially high near the droplet but gradually spreads until a uniform, lower concentration is reached elsewhere.

Consider the prismatic beam shown in Fig. 13, which has cross-sectional properties constant along the z -direction. According to Saint–Venant’s principle, stresses resulting from statically equivalent loads become identical at cross-sections sufficiently far from the load-introduction region. The equilibrium equations of

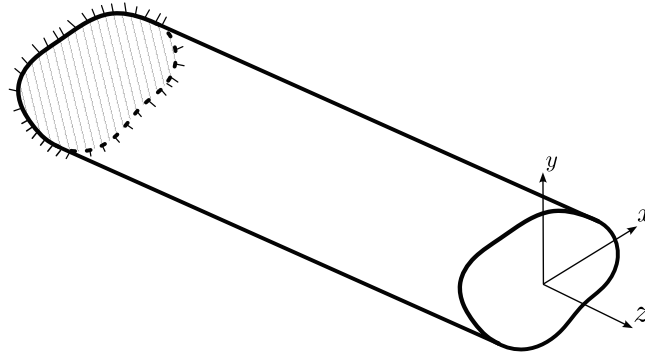


Figure 13: Prismatic beam with cross-sectional properties constant along the z -direction.

elasticity, in the absence of body forces, can be written in general tensor form as

$$\sigma_{ij,j} = 0 \quad (\text{A.1})$$

The general solution of the equilibrium equations can be expressed as a superposition of exponential modes of the form:

$$\sigma(z) = c_i e^{\lambda_i z} \quad (\text{A.2})$$

where σ is a vector containing relevant stress components, while c_i is a vector of constants and λ_i is, in general, a complex number given by:

$$\lambda_i = R_i + j \omega_i. \quad (\text{A.3})$$

The real part R_i governs the exponential growth or decay of each mode along z , whereas the imaginary part ω_i represents possible oscillatory behaviour as illustrated in Fig. 14.

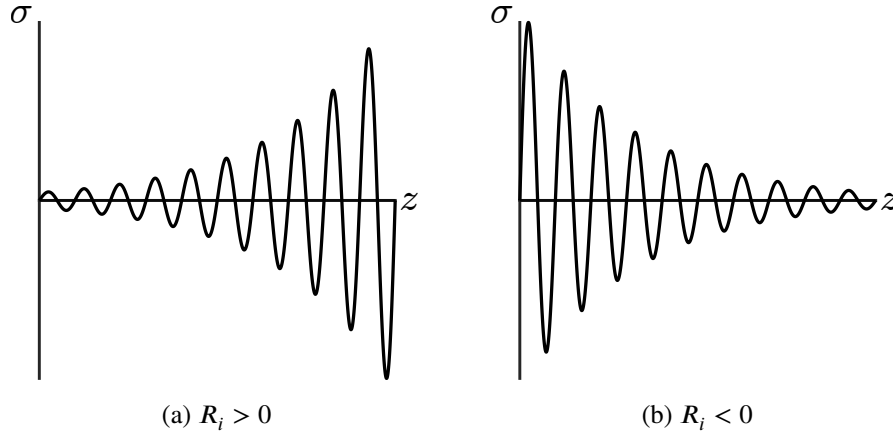


Figure 14: Different solution representations of Eq. A.2.

According to Saint–Venant’s principle, the growing modes ($R_i > 0$) are not physically admissible, as they would lead to an unbounded stress field away from the load introduction region. The decaying modes ($R_i < 0$) correspond to localised stress disturbances that die out with distance from the load-introduction zone. Of primary interest in the present analysis are the persistent modes, which neither grow nor decay exponentially and therefore correspond to $\lambda_i = 0$. These represent the six classical Saint–Venant stress states: one axial, two bending, one torsional, and two shear modes.

To derive the governing equations, the strain definitions are first introduced using the infinitesimal strain assumption as follows:

$$\begin{aligned}
 \epsilon_x &= u_{,x}, \\
 \epsilon_y &= v_{,y}, \\
 \epsilon_z &= w_{,z}, \\
 \gamma_{yz} &= v_{,z} + w_{,y}, \\
 \gamma_{xz} &= u_{,z} + w_{,x}, \\
 \gamma_{xy} &= u_{,y} + v_{,x}
 \end{aligned} \tag{A.4}$$

The strain–displacement relations can be organised in matrix form as:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \tag{A.5}$$

which can be written in a compact form as:

$$\epsilon = \mathbf{B}_0 \mathbf{q} + \mathbf{B}_1 \mathbf{q}' \tag{A.6}$$

where the prime denotes differentiation with respect to z .

The total strain energy is then defined as:

$$U = \int_V \frac{1}{2} \epsilon^\top \mathbf{C} \epsilon dV \tag{A.7}$$

where V represents the volume and $\mathbf{C}$ is the constitutive matrix. Per unit depth in the z direction, the integration becomes:

$$U = \int_A \frac{1}{2} \boldsymbol{\varepsilon}^\top \mathbf{C} \boldsymbol{\varepsilon} dA \quad (\text{A.8})$$

where A represents the area.

Substituting Eq. A.6 into Eq. A.8 yields:

$$U = \frac{1}{2} \int_A \left(\mathbf{q}^\top \mathbf{B}_0^\top \mathbf{C} \mathbf{B}_0 \mathbf{q} + 2 \mathbf{q}^\top \mathbf{B}_0^\top \mathbf{C} \mathbf{B}_1 \mathbf{q}' + \mathbf{q}'^\top \mathbf{B}_1^\top \mathbf{C} \mathbf{B}_1 \mathbf{q}' \right) dA \quad (\text{A.9})$$

which could be expressed more compactly as:

$$U = \frac{1}{2} \int_A \left(\mathbf{q}^\top \mathbf{K}_{00} \mathbf{q} + 2 \mathbf{q}^\top \mathbf{K}_{01} \mathbf{q}' + \mathbf{q}'^\top \mathbf{K}_{11} \mathbf{q}' \right) dA \quad (\text{A.10})$$

where

$$\mathbf{K}_{00} = \mathbf{B}_0^\top \mathbf{C} \mathbf{B}_0, \quad \mathbf{K}_{01} = \mathbf{B}_0^\top \mathbf{C} \mathbf{B}_1, \quad \mathbf{K}_{11} = \mathbf{B}_1^\top \mathbf{C} \mathbf{B}_1 \quad (\text{A.11})$$

The Lagrangian functional is generally defined as follows:

$$L = U - W - T \quad (\text{A.12})$$

where W is the work done by external forces and T is the kinetic energy.

For the present static and unloaded cross-sectional formulation, neither the kinetic nor the external potential terms are considered. Consequently, the Lagrangian density is defined to be:

$$\mathcal{L} = \frac{1}{2} \left(\mathbf{q}^\top \mathbf{K}_{00} \mathbf{q} + 2 \mathbf{q}^\top \mathbf{K}_{01} \mathbf{q}' + \mathbf{q}'^\top \mathbf{K}_{11} \mathbf{q}' \right) \quad (\text{A.13})$$

To set up the Euler-Lagrange equations, the generalised momenta are defined as follows:

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}'} = \mathbf{K}_{10} \mathbf{q} + \mathbf{K}_{11} \mathbf{q}' \quad (\text{A.14})$$

where $\mathbf{K}_{10} = \mathbf{K}_{01}^\top$.

The Euler-Lagrange equation states that:

$$\frac{d\mathbf{p}}{dz} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \quad (\text{A.15})$$

leading to:

$$\mathbf{p}' = \mathbf{K}_{00} \mathbf{q} + \mathbf{K}_{01} \mathbf{q}' \quad (\text{A.16})$$

To obtain $\mathbf{q}'$ in terms of $\mathbf{q}$ and $\mathbf{p}$, Eq. A.14 is rearranged:

$$\mathbf{q}' = -\mathbf{K}_{11}^{-1} \mathbf{K}_{10} \mathbf{q} + \mathbf{K}_{11}^{-1} \mathbf{p} \quad (\text{A.17})$$

Eqs. A.17, A.16 can be expressed in matrix form as follows:

$$\begin{bmatrix} \mathbf{q}' \\ \mathbf{p}' \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{11}^{-1} \mathbf{K}_{10} & \mathbf{K}_{11}^{-1} \\ \mathbf{K}_{00} - \mathbf{K}_{01} \mathbf{K}_{11}^{-1} \mathbf{K}_{10} & \mathbf{K}_{01} \mathbf{K}_{11}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} \quad (\text{A.18})$$

This system can be represented as:

$$\mathbf{x}' = \mathcal{A} \mathbf{x} \quad (\text{A.19})$$

Before proceeding, the reader is referred to [Appendix B](#), where the algebraic relations associated with rigid-body motion—such as the null-space properties of $\mathbf{K}_{00}$ and the coupling relations among $\mathbf{K}_{00}$, $\mathbf{K}_{01}$, $\mathbf{K}_{10}$, and $\mathbf{K}_{11}$ —are derived. These relations are not needed conceptually at this point, but they will be used extensively to simplify the expressions that follow.

Persistent solutions come from eigenvectors corresponding to zero eigenvalues. Looking for solutions in the null space of $\mathcal{A}$, the following relation is considered:

$$\begin{bmatrix} -\mathbf{K}_{11}^{-1}\mathbf{K}_{10} & \mathbf{K}_{11}^{-1} \\ \mathbf{K}_{00} - \mathbf{K}_{01}\mathbf{K}_{11}^{-1}\mathbf{K}_{10} & \mathbf{K}_{01}\mathbf{K}_{11}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad (\text{A.20})$$

where $\mathbf{r}$ and $\mathbf{s}$ are the unknown vectors to solve for. The previous system can be rewritten as follows:

$$\mathcal{A} \boldsymbol{\phi}_i^0 = \mathbf{0} \quad (\text{A.21})$$

where $i = 1, 2, \dots$

From the results obtained in [Appendix B](#), it is clear that $\mathbf{r}$ lies in the null space of $\mathbf{K}_{00}$ and $\mathbf{K}_{10}$ when it is equal to any of the following vectors: $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{r}_3\}$. For the four cases, solving for $\mathbf{s}$ gives the following eigenvectors $\boldsymbol{\phi}_i^0$:

$$\begin{bmatrix} \mathbf{t}_1 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{t}_2 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{t}_3 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{r}_3 \\ \mathbf{0} \end{bmatrix} \quad (\text{A.22})$$

These four eigenvectors correspond to uniform cross-sectional displacements accompanied by zero stress resultants. The eigenvalue problem of Eq. A.21 therefore yields four eigenvectors associated with the zero eigenvalue, although the algebraic multiplicity of this eigenvalue is known to be higher. This discrepancy shows that the eigenvalue is defective: the four eigenvectors do not span the full invariant subspace associated with $\lambda = 0$. This is consistent with the fact that the system matrix $\mathcal{A}$ is non-normal, meaning that $\mathcal{A}\mathcal{A}^\top \neq \mathcal{A}^\top\mathcal{A}$, and non-normal matrices may not admit a full basis of linearly independent eigenvectors for repeated eigenvalues. As a result, the eigenvectors obtained from Eq. A.21 must be supplemented by generalised eigenvectors in order to span the minimal invariant subspace associated with the zero eigenvalue.

In the present formulation, the state vector $\mathbf{x} = [\mathbf{q}; \mathbf{p}]$ contains both displacement components and stress resultants, as later shown in Eq. A.69. The eigenvectors obtained above represent the lowest-order persistent modes, characterised by uniform displacements and zero stress resultants. The generalised eigenvectors introduce higher-order persistent patterns: the first set exhibits linear variations in the displacement field accompanied by uniform stress resultants, while the next set introduces quadratic displacement patterns coupled with linearly varying stress resultants. Taken together, the eigenvectors and their generalised counterparts span the complete invariant subspace associated with the persistent Saint–Venant solutions in the displacement–stress formulation. The first set of generalised eigenvectors is obtained from the relation:

$$\mathcal{A}\mathbf{m}_i = \boldsymbol{\phi}_i^0 \quad (\text{A.23})$$

where $\mathbf{m}_i$ are the vectors in the first set of generalised eigenvectors, and $\boldsymbol{\phi}_i^0$ are the obtained eigenvectors in the null space of $\mathcal{A}$.

To obtain the first vector, consider that:

$$\mathbf{m}_1 = \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix}, \quad \boldsymbol{\phi}_1^0 = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{0} \end{bmatrix} \quad (\text{A.24})$$

The first equation of this system of equations is:

$$-\mathbf{K}_{11}^{-1}\mathbf{K}_{10}\mathbf{r} + \mathbf{K}_{11}^{-1}\mathbf{s} = \mathbf{t}_1 \quad (\text{A.25})$$

Solving for $\mathbf{s}$ gives:

$$\mathbf{s} = \mathbf{K}_{10}\mathbf{r} + \mathbf{K}_{11}\mathbf{t}_1 \quad (\text{A.26})$$

Substituting in the second equation of the system leads to:

$$(\mathbf{K}_{00} - \mathbf{K}_{01}\mathbf{K}_{11}^{-1}\mathbf{K}_{10})\mathbf{r} + \mathbf{K}_{01}\mathbf{K}_{11}^{-1}(\mathbf{K}_{10}\mathbf{r} + \mathbf{K}_{11}\mathbf{t}_1) = \mathbf{0} \quad (\text{A.27})$$

After simplification and using the relations of Eq. B.14, the following relation is obtained:

$$\mathbf{K}_{00}\mathbf{r} = \mathbf{K}_{00}\mathbf{r}_2 \quad (\text{A.28})$$

The solvability of this system is checked by left-multiplying both sides by the transpose of the vectors in the null space of $\mathbf{K}_{00}$. It is very clear in that case that all relations obtain zero, given the symmetry of $\mathbf{K}_{00}$, for example:

$$\mathbf{t}_1^\top \mathbf{K}_{00}\mathbf{r} = \mathbf{t}_1^\top \mathbf{K}_{00}\mathbf{r}_2 \rightarrow (\mathbf{K}_{00}\mathbf{t}_1)^\top \mathbf{r} = (\mathbf{K}_{00}\mathbf{t}_1)^\top \mathbf{r}_2 = \mathbf{0} \quad (\text{A.29})$$

That means the system is solvable and the current relation is obtained:

$$\mathbf{r} = \mathbf{r}_2 \quad (\text{A.30})$$

Substituting Eq. A.30 into Eq. A.26, gives:

$$\mathbf{s} = \mathbf{K}_{10}\mathbf{r}_2 + \mathbf{K}_{11}\mathbf{t}_1 \quad (\text{A.31})$$

By investigating Eq. B.15, the equations simplify to:

$$\mathbf{s} = \mathbf{0} \rightarrow \mathbf{m}_1 = \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{0} \end{bmatrix} \quad (\text{A.32})$$

Following the same procedure, but using the rest of the vectors (ϕ_i^0) in the null space of $\mathcal{A}$. The following generalised eigenvectors are obtained:

$$\mathbf{m}_2 = \begin{bmatrix} -\mathbf{r}_1 \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{m}_3 = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{K}_{10}\mathbf{w}_1 + \mathbf{K}_{11}\mathbf{t}_3 \end{bmatrix}, \quad \mathbf{m}_4 = \begin{bmatrix} \mathbf{w}_2 \\ \mathbf{K}_{10}\mathbf{w}_2 + \mathbf{K}_{11}\mathbf{r}_3 \end{bmatrix} \quad (\text{A.33})$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are the solutions of the systems of equations:

$$\mathbf{K}_{00}\mathbf{w}_1 = -\mathbf{K}_{01}\mathbf{t}_3, \quad \mathbf{K}_{00}\mathbf{w}_2 = -\mathbf{K}_{01}\mathbf{r}_3 \quad (\text{A.34})$$

The next set of generalised eigenvectors is obtained by solving the following equation:

$$\mathcal{A}\mathbf{d}_i = \mathbf{m}_i \quad (\text{A.35})$$

To obtain the first vector in this set:

$$\mathbf{d}_1 = \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix}, \quad \mathbf{m}_1 = \begin{bmatrix} \mathbf{r}_2 \\ \mathbf{0} \end{bmatrix} \quad (\text{A.36})$$

The first equation of the system will be:

$$-\mathbf{K}_{11}^{-1}\mathbf{K}_{10}\mathbf{r} + \mathbf{K}_{11}^{-1}\mathbf{s} = \mathbf{r}_2 \quad (\text{A.37})$$

Solving for $\mathbf{s}$ gives:

$$\mathbf{s} = \mathbf{K}_{10}\mathbf{r} + \mathbf{K}_{11}\mathbf{r}_2 \quad (\text{A.38})$$

Substituting $\mathbf{s}$ into the second equation of the system yields:

$$(\mathbf{K}_{00} - \mathbf{K}_{01}\mathbf{K}_{11}^{-1}\mathbf{K}_{10})\mathbf{r} + \mathbf{K}_{01}\mathbf{K}_{11}^{-1}(\mathbf{K}_{10}\mathbf{r} + \mathbf{K}_{11}\mathbf{r}_2) = \mathbf{0} \quad (\text{A.39})$$

which simplifies to:

$$\mathbf{K}_{00}\mathbf{r} = -\mathbf{K}_{01}\mathbf{r}_2 \quad (\text{A.40})$$

The solvability condition is checked in the same procedure as done in Eq. A.29. It is clear that the solvability condition is achieved for all the vectors in the null space of $\mathbf{K}_{00}$, with the solution of the system assumed to be:

$$\mathbf{r} = \mathbf{w}_3 \quad (\text{A.41})$$

leading to:

$$\mathbf{d}_1 = \begin{bmatrix} \mathbf{w}_3 \\ \mathbf{K}_{10}\mathbf{w}_3 + \mathbf{K}_{11}\mathbf{r}_2 \end{bmatrix} \quad (\text{A.42})$$

Following the same procedure, the second generalised eigenvector in this set is obtained:

$$\mathbf{d}_2 = \begin{bmatrix} \mathbf{w}_4 \\ \mathbf{K}_{10}\mathbf{w}_4 - \mathbf{K}_{11}\mathbf{r}_1 \end{bmatrix} \quad (\text{A.43})$$

where $\mathbf{w}_4$ is the solution of the system:

$$\mathbf{K}_{00}\mathbf{r} = \mathbf{K}_{01}\mathbf{r}_1 \quad (\text{A.44})$$

For the rest of the vectors, the solvability conditions are not met for the following systems of equations:

$$\begin{aligned} \mathbf{K}_{00}\mathbf{r} + \mathbf{K}_{01}\mathbf{w}_1 &= \mathbf{K}_{10}\mathbf{w}_1 + \mathbf{K}_{11}\mathbf{t}_3, \\ (\mathbf{K}_{00} - \mathbf{K}_{01}\mathbf{K}_{11}^{-1}\mathbf{K}_{10})\mathbf{r} + \mathbf{K}_{01}\mathbf{K}_{11}^{-1}(\mathbf{K}_{10}\mathbf{r} + \mathbf{K}_{11}\mathbf{w}_1) &= \mathbf{K}_{10}\mathbf{w}_2 + \mathbf{K}_{11}\mathbf{r}_3 \end{aligned} \quad (\text{A.45})$$

Theoretically, if there exists another set of generalised eigenvectors, it will follow the relation:

$$\mathcal{A}\mathbf{h}_i = \mathbf{d}_i \quad (\text{A.46})$$

But again, solvability conditions are not achieved, which means no more generalised eigenvectors can be obtained for the system. Although no additional generalised eigenvectors can be obtained from the system, the subspace associated with the zero eigenvalue is not restricted to the specific vectors already derived. A linear combination of the existing eigenvectors and generalised eigenvectors still lies in the same invariant subspace and therefore represents an admissible persistent solution. For this reason, additional generalised eigenvectors are constructed as suitable linear combinations of the previously obtained vectors. Rewriting the generalised eigenvectors from Eqs. A.33, A.42 and A.43 in this form:

$$\mathbf{m}_3 = \begin{bmatrix} \mathbf{w}_1 \\ \chi_1 \end{bmatrix}, \quad \mathbf{m}_4 = \begin{bmatrix} \mathbf{w}_2 \\ \chi_2 \end{bmatrix}, \quad \mathbf{d}_1 = \begin{bmatrix} \mathbf{w}_3 \\ \chi_3 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} \mathbf{w}_4 \\ \chi_4 \end{bmatrix} \quad (\text{A.47})$$

A new set of generalised eigenvectors could be obtained by using linear combinations of previously obtained generalised eigenvectors as follows:

$$\begin{bmatrix} -\mathbf{K}_{11}^{-1}\mathbf{K}_{10} & \mathbf{K}_{11}^{-1} \\ \mathbf{K}_{00} - \mathbf{K}_{01}\mathbf{K}_{11}^{-1}\mathbf{K}_{10} & \mathbf{K}_{01}\mathbf{K}_{11}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_3 + \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 \\ \chi_3 + \alpha_1\chi_1 + \alpha_2\chi_2 \end{bmatrix} \quad (\text{A.48})$$

The system of equations is simplified to the following form:

$$\mathbf{K}_{00}\mathbf{r} = \mathbf{K}_{01}(\mathbf{w}_3 + \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2) + (\chi_3 + \alpha_1\chi_1 + \alpha_2\chi_2) \quad (\text{A.49})$$

To check the solvability of Eq. A.49, the equation is left-multiplied by $\mathbf{t}_1^\top$, $\mathbf{t}_2^\top$, $\mathbf{t}_3^\top$, and $\mathbf{r}_3^\top$. The first condition to check is:

$$\mathbf{t}_1^\top \mathbf{K}_{00}\mathbf{r} = \mathbf{t}_1^\top \mathbf{K}_{01}(\mathbf{w}_3 + \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2) + \mathbf{t}_1^\top (\chi_3 + \alpha_1\chi_1 + \alpha_2\chi_2) \quad (\text{A.50})$$

The term $t_1^T \chi_1$, is simplified as follows:

$$t_1^T \chi_1 = (\mathbf{K}_{01} t_1)^T w_1 + (\mathbf{K}_{11} t_1)^T t_3 \quad (\text{A.51})$$

From Eq. B.14, the expression could be further simplified:

$$t_1^T \chi_1 = (-\mathbf{K}_{00} r_2)^T w_1 + (\mathbf{K}_{11} t_1)^T t_3 = -r_2^T \mathbf{K}_{00} w_1 + (\mathbf{K}_{11} t_1)^T t_3 \quad (\text{A.52})$$

Using Eq. A.34, further tailoring is possible:

$$t_1^T \chi_1 = r_2^T \mathbf{K}_{01} t_3 + (\mathbf{K}_{11} t_1)^T t_3 = (\mathbf{K}_{10} r_2)^T t_3 + (\mathbf{K}_{11} t_1)^T t_3 \quad (\text{A.53})$$

Finally, using Eq. B.15:

$$t_1^T \chi_1 = (-\mathbf{K}_{11} t_1)^T t_3 + (\mathbf{K}_{11} t_1)^T t_3 = 0 \quad (\text{A.54})$$

The same procedure can be followed to prove the rest of the relations:

$$t_1^T \chi_1 = t_1^T \chi_2 = t_1^T \chi_3 = t_1^T \chi_4 = 0 \quad (\text{A.55})$$

Also, it could be proven that:

$$t_2^T \chi_1 = t_2^T \chi_2 = t_2^T \chi_3 = t_2^T \chi_4 = 0 \quad (\text{A.56})$$

But for the other vectors:

$$\begin{aligned} t_3^T \chi_1 \neq 0, \quad t_3^T \chi_2 \neq 0, \quad t_3^T \chi_3 \neq 0, \quad t_3^T \chi_4 \neq 0 \\ r_3^T \chi_1 \neq 0, \quad r_3^T \chi_2 \neq 0, \quad r_3^T \chi_3 \neq 0, \quad r_3^T \chi_4 \neq 0 \end{aligned} \quad (\text{A.57})$$

The previous results show that Eq. A.50 equals zero. Similarly, left multiplying Eq. A.49 by t_2^T leads to zero. The other two solvability conditions are then:

$$\begin{aligned} t_3^T \chi_3 + \alpha_1 t_3^T \chi_1 + \alpha_2 t_3^T \chi_2 &= 0 \\ r_3^T \chi_3 + \alpha_1 r_3^T \chi_1 + \alpha_2 r_3^T \chi_2 &= 0 \end{aligned} \quad (\text{A.58})$$

which could be represented in matrix form as:

$$\begin{bmatrix} t_3^T \chi_1 & t_3^T \chi_2 \\ r_3^T \chi_1 & r_3^T \chi_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -t_3^T \chi_3 \\ -r_3^T \chi_3 \end{bmatrix} \quad (\text{A.59})$$

Given that condition, a solution to the system of Eq. A.48 is:

$$h_1 = \begin{bmatrix} \mathbf{K}_{10} w_5 + \mathbf{K}_{11} (w_3 + \alpha_1 w_1 + \alpha_2 w_2) \\ w_5 \end{bmatrix} = \begin{bmatrix} w_5 \\ \chi_5 \end{bmatrix} \quad (\text{A.60})$$

where w_5 is the solution to the system of Eq. A.49

The second generalised eigenvector is obtained by solving the system:

$$\begin{bmatrix} -\mathbf{K}_{11}^{-1} \mathbf{K}_{10} & \mathbf{K}_{11}^{-1} \\ \mathbf{K}_{00} - \mathbf{K}_{01} \mathbf{K}_{11}^{-1} \mathbf{K}_{10} & \mathbf{K}_{01} \mathbf{K}_{11}^{-1} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} w_4 + \alpha_3 w_1 + \alpha_4 w_2 \\ \chi_4 + \alpha_3 \chi_1 + \alpha_4 \chi_2 \end{bmatrix} \quad (\text{A.61})$$

The system of equations is simplified to the following form:

$$\mathbf{K}_{00} r = \mathbf{K}_{01} (w_4 + \alpha_3 w_1 + \alpha_4 w_2) + (\chi_4 + \alpha_3 \chi_1 + \alpha_4 \chi_2) \quad (\text{A.62})$$

Following the same procedure as before, the solvability conditions require that:

$$\begin{bmatrix} t_3^T \chi_1 & t_3^T \chi_2 \\ r_3^T \chi_1 & r_3^T \chi_2 \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} -t_3^T \chi_4 \\ -r_3^T \chi_4 \end{bmatrix} \quad (\text{A.63})$$

and the solution to the system of Eq. A.61 is:

$$h_2 = \begin{bmatrix} w_6 \\ \chi_6 \end{bmatrix} = \begin{bmatrix} w_6 \\ \chi_6 \end{bmatrix} \quad (\text{A.64})$$

where w_6 is the solution to the system of Eq. A.62.

All eigenvectors and generalised eigenvectors required for the present formulation have now been derived. To describe persistent solutions, x is defined as a linear combination of predefined eigenvectors and generalised eigenvectors as follows:

$$x = c_1 \phi_1^0 + c_2 \phi_2^0 + c_3 \phi_3^0 + c_4 \phi_4^0 + c_5 m_1 + c_6 m_2 + c_7 m_3 + c_8 m_4 + c_9 d_1 + c_{10} d_2 + c_{11} h_1 + c_{12} h_2 \quad (\text{A.65})$$

Substituting Eq. A.65 into Eq. A.19:

$$\begin{aligned} & c'_1 \phi_1^0 + c'_2 \phi_2^0 + c'_3 \phi_3^0 + c'_4 \phi_4^0 + c'_5 m_1 + c'_6 m_2 + \\ & c'_7 m_3 + c'_8 m_4 + c'_9 d_1 + c'_{10} d_2 + c'_{11} h_1 + c'_{12} h_2 \\ = & A(c_1 \phi_1^0 + c_2 \phi_2^0 + c_3 \phi_3^0 + c_4 \phi_4^0 + c_5 m_1 + c_6 m_2 + \\ & c_7 m_3 + c_8 m_4 + c_9 d_1 + c_{10} d_2 + c_{11} h_1 + c_{12} h_2) \end{aligned} \quad (\text{A.66})$$

Substituting Eqs. A.21, A.23, A.35, A.48, and A.61 into Eq. A.66:

$$\begin{aligned} & c'_1 \phi_1^0 + c'_2 \phi_2^0 + c'_3 \phi_3^0 + c'_4 \phi_4^0 + c'_5 m_1 + c'_6 m_2 + \\ & c'_7 m_3 + c'_8 m_4 + c'_9 d_1 + c'_{10} d_2 + c'_{11} h_1 + c'_{12} h_2 \\ = & c_6 \phi_2^0 + c_7 \phi_3^0 + c_8 \phi_4^0 + c_9 m_1 + c_{10} m_2 + \\ & c_{11} (d_1 + \alpha_1 m_3 + \alpha_2 m_4) + c_{12} (d_2 + \alpha_3 m_3 + \alpha_4 m_4) \end{aligned} \quad (\text{A.67})$$

From the previous equation, it could be easily proven:

$$\begin{aligned} c'_1 &= c_5, \quad c'_2 = c_6, \quad c'_3 = c_7, \quad c'_4 = c_8, \quad c'_5 = c_9, \\ c'_6 &= c_{10}, \quad c'_7 = \alpha_1 c_{11} + \alpha_3 c_{12}, \quad c'_8 = \alpha_2 c_{11} + \alpha_4 c_{12}, \\ c'_9 &= c_{11}, \quad c'_{10} = c_{12}, \quad c'_{11} = 0, \quad c'_{12} = 0 \end{aligned} \quad (\text{A.68})$$

The tractions over the cross-section are now defined to derive the cross-sectional stiffness. The generalised momenta expression could be expanded by substituting Eq. A.11 into Eq. A.14:

$$p = B_1^T C (B_0 q + B_1 q') = B_1^T \sigma = \begin{bmatrix} \tau_{zx} \\ \tau_{zy} \\ \sigma_z \end{bmatrix} \quad (\text{A.69})$$

which gives the tractions over the cross-section. For simplicity, the forces and moments over the cross-section will be named f_1, f_2, f_3, M_1, M_2 , and M_3 and are defined as follows:

$$\begin{aligned} f_1 &= t_1^T p, \quad f_2 = t_2^T p, \quad f_3 = t_3^T p, \\ M_1 &= r_1^T p, \quad M_2 = r_2^T p, \quad M_3 = r_3^T p, \end{aligned} \quad (\text{A.70})$$

Given Eq. A.65, the previous equations can be represented in matrix form as follows:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} t_1^\top \chi_1 & t_1^\top \chi_2 & t_1^\top \chi_3 & t_1^\top \chi_4 & t_1^\top \chi_5 & t_1^\top \chi_6 \\ t_2^\top \chi_1 & t_2^\top \chi_2 & t_2^\top \chi_3 & t_2^\top \chi_4 & t_2^\top \chi_5 & t_2^\top \chi_6 \\ t_3^\top \chi_1 & t_3^\top \chi_2 & t_3^\top \chi_3 & t_3^\top \chi_4 & t_3^\top \chi_5 & t_3^\top \chi_6 \\ r_1^\top \chi_1 & r_1^\top \chi_2 & r_1^\top \chi_3 & r_1^\top \chi_4 & r_1^\top \chi_5 & r_1^\top \chi_6 \\ r_2^\top \chi_1 & r_2^\top \chi_2 & r_2^\top \chi_3 & r_2^\top \chi_4 & r_2^\top \chi_5 & r_2^\top \chi_6 \\ r_3^\top \chi_1 & r_3^\top \chi_2 & r_3^\top \chi_3 & r_3^\top \chi_4 & r_3^\top \chi_5 & r_3^\top \chi_6 \end{bmatrix} \begin{bmatrix} c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \end{bmatrix} \quad (\text{A.71})$$

which can be represented in a compact form as follows:

$$f = \Phi_1 c \quad (\text{A.72})$$

Strain is defined in Eq. A.6 in terms of q and q' . From Eq. A.65, q can be expanded as follows:

$$q = c_1 t_1 + c_2 t_2 + c_3 t_3 + c_4 r_3 + c_5 r_1 + c_6 r_2 + c_7 w_1 + c_8 w_2 + c_9 w_3 + c_{10} w_4 + c_{11} w_5 + c_{12} w_6 \quad (\text{A.73})$$

Similarly, q' is expanded and using Eq. A.68, strain is expressed as follows:

$$\begin{aligned} \varepsilon = & c_7 (B_0 w_1 + B_1 t_3) + c_8 (B_0 w_2 + B_1 r_3) + c_9 (B_0 w_3 + B_1 r_2) + \\ & c_{10} (B_0 w_4 - B_1 r_1) + c_{11} (B_0 w_5 + B_1 (w_3 + \alpha_1 w_1 + \alpha_2 w_2)) + \\ & c_{12} (B_0 w_6 + B_1 (w_4 + \alpha_3 w_1 + \alpha_4 w_2)) \end{aligned} \quad (\text{A.74})$$

which can be represented in a more compact form as:

$$\varepsilon = c_7 \psi_1 + c_8 \psi_2 + c_9 \psi_3 + c_{10} \psi_4 + c_{11} \psi_5 + c_{12} \psi_6 = \Phi_2 c \quad (\text{A.75})$$

Substituting Eq. A.75 into Eq. A.8:

$$U = \frac{1}{2} \int_A c^\top \Phi_2^\top C \Phi_2 c \, dA = \frac{1}{2} \int_A f^\top (\Phi_1^{-1})^\top \Phi_2^\top C \Phi_2 \Phi_1^{-1} f \, dA \quad (\text{A.76})$$

Further simplifications can be made by assuming:

$$\Phi_3 = \Phi_2^\top C \Phi_2 \quad (\text{A.77})$$

The elements of the matrix can be simplified. For example, the first term reads:

$$\begin{aligned} \psi_1^\top C \psi_1 = & (w_1^\top B_0^\top + t_3^\top B_1^\top) C (B_0 w_1 + B_1 t_3) \\ = & w_1^\top B_0^\top C B_0 w_1 + w_1^\top B_0^\top C B_1 t_3 + t_3^\top B_1^\top C B_0 w_1 + t_3^\top B_1^\top C B_1 t_3 \\ = & w_1^\top K_{00} w_1 + w_1^\top K_{01} t_3 + t_3^\top K_{10} w_1 + t_3^\top K_{11} t_3 \\ = & w_1^\top K_{00} w_1 + 2w_1^\top K_{01} t_3 + t_3^\top K_{11} t_3 \end{aligned} \quad (\text{A.78})$$

Substituting Eq. A.34 into Eq. A.78:

$$\begin{aligned} \psi_1^\top C \psi_1 = & t_3^\top K_{01} w_1 + t_3^\top K_{11} t_3 = t_3^\top (K_{01} w_1 + K_{11} t_3) \\ = & t_3^\top \chi_1 \end{aligned} \quad (\text{A.79})$$

Similarly, the other terms of the matrix can be simplified, leading to the following:

$$\Phi_3 = \begin{bmatrix} t_3^\top \chi_1 & t_3^\top \chi_2 & t_3^\top \chi_3 & t_3^\top \chi_4 & t_3^\top \chi_5 + w_1^\top \eta_1 & t_3^\top \chi_6 + w_1^\top \eta_2 \\ r_3^\top \chi_1 & r_3^\top \chi_2 & r_3^\top \chi_3 & r_3^\top \chi_4 & r_3^\top \chi_5 + w_2^\top \eta_1 & r_3^\top \chi_6 + w_2^\top \eta_2 \\ r_2^\top \chi_1 & r_2^\top \chi_2 & r_2^\top \chi_3 & r_2^\top \chi_4 & r_2^\top \chi_5 + w_3^\top \eta_1 & r_2^\top \chi_6 + w_3^\top \eta_2 \\ -r_1^\top \chi_1 & -r_1^\top \chi_2 & -r_1^\top \chi_3 & -r_1^\top \chi_4 & -r_1^\top \chi_5 + w_4^\top \eta_1 & -r_1^\top \chi_6 + w_4^\top \eta_2 \\ \beta_1^\top \chi_1 & \beta_1^\top \chi_2 & \beta_1^\top \chi_3 & \beta_1^\top \chi_4 & \beta_1^\top \chi_5 + w_5^\top \eta_1 & \beta_1^\top \chi_6 + w_5^\top \eta_2 \\ \beta_2^\top \chi_1 & \beta_2^\top \chi_2 & \beta_2^\top \chi_3 & \beta_2^\top \chi_4 & \beta_2^\top \chi_5 + w_6^\top \eta_1 & \beta_2^\top \chi_6 + w_6^\top \eta_2 \end{bmatrix} \quad (\text{A.80})$$

where

$$\begin{aligned}
 \boldsymbol{\eta}_1 &= \alpha_1 \chi_1 + \alpha_2 \chi_2 + \chi_3, \\
 \boldsymbol{\eta}_2 &= \alpha_3 \chi_1 + \alpha_4 \chi_2 + \chi_4, \\
 \boldsymbol{\beta}_1 &= \alpha_1 \boldsymbol{w}_1 + \alpha_2 \boldsymbol{w}_2 + \boldsymbol{w}_3, \\
 \boldsymbol{\beta}_2 &= \alpha_3 \boldsymbol{w}_1 + \alpha_4 \boldsymbol{w}_2 + \boldsymbol{w}_4
 \end{aligned} \tag{A.81}$$

The compliance matrix is then defined as:

$$\mathbf{S} = (\boldsymbol{\Phi}_1^{-1})^\top \boldsymbol{\Phi}_3 \boldsymbol{\Phi}_1^{-1} \tag{A.82}$$

The 6×6 Timoshenko stiffness matrix of the cross-section is defined as:

$$\mathbf{K}_{cs} = \mathbf{S}^{-1} = \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_3^{-1} \boldsymbol{\Phi}_1^\top \tag{A.83}$$

Appendix B. Rigid body motion relations

Starting with the rigid body translation by considering a unit translation ($u = 1$) in the x direction:

$$\mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{t}_1, \quad \mathbf{q}' = \mathbf{0} \quad (\text{B.1})$$

The rigid body translation does not cause any deformation, so the strain will be zero ($\epsilon = \mathbf{0}$):

$$\mathbf{B}_0 \mathbf{t}_1 = \begin{bmatrix} \frac{\partial 1}{\partial x} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \quad (\text{B.2})$$

Similarly, considering a unit translation ($v = 1$) in the y direction.

$$\mathbf{t}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{B}_0 \mathbf{t}_2 = \mathbf{0} \quad (\text{B.3})$$

Finally, considering a unit translation ($w = 1$) in the z direction.

$$\mathbf{t}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{B}_0 \mathbf{t}_3 = \mathbf{0} \quad (\text{B.4})$$

Now for the rigid rotations, consider that around the z axis:

$$\mathbf{q} = \begin{bmatrix} -y \\ x \\ 0 \end{bmatrix} = \mathbf{r}_3, \quad \mathbf{q}' = \mathbf{0} \quad (\text{B.5})$$

where the positive velocity in the x direction due to rotation around the z axis is proportional to the distance in the negative y direction.

The strains are then defined as follows:

$$\mathbf{B}_0 \mathbf{r}_3 = \begin{bmatrix} \frac{\partial(-y)}{\partial x} \\ \frac{\partial x}{\partial x} \\ \frac{\partial 0}{\partial y} \\ 0 \\ \frac{\partial 0}{\partial y} \\ \frac{\partial 0}{\partial x} \\ \frac{\partial(-y)}{\partial y} + \frac{\partial x}{\partial x} \end{bmatrix} = \mathbf{0} \quad (\text{B.6})$$

The rigid rotation around the x axis is defined as follows:

$$\mathbf{q} = \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ y \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \mathbf{r}_1 - z \mathbf{t}_2 \quad (\text{B.7})$$

which leads to:

$$\mathbf{q}' = -\mathbf{t}_2 \quad (\text{B.8})$$

Substituting Eqs. B.7, B.8 into Eq. A.6 leads to the following relation:

$$\mathbf{B}_0 \mathbf{r}_1 = \mathbf{B}_1 t_2 \quad (\text{B.9})$$

Finally, looking at the rotation about the y axis:

$$\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ -x \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{r}_2 + z \mathbf{t}_1 \quad (\text{B.10})$$

which similarly results in the following:

$$\mathbf{B}_0 \mathbf{r}_2 = -\mathbf{B}_1 t_1 \quad (\text{B.11})$$

From Eqs. B.2, B.3, B.4 and B.6, it is concluded that $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{r}_3\}$ are in the null space of $\mathbf{B}_0$. From Eq. A.11, the following relations can be drawn:

$$\mathbf{K}_{00} \mathbf{t}_1 = \mathbf{K}_{00} \mathbf{t}_2 = \mathbf{K}_{00} \mathbf{t}_3 = \mathbf{K}_{00} \mathbf{r}_3 = \mathbf{0} \quad (\text{B.12})$$

Consequently, $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{r}_3\}$ are in the null space of $\mathbf{K}_{00}$, which means that $\mathbf{K}_{00}$ is singular.

Similarly,

$$\mathbf{K}_{10} \mathbf{t}_1 = \mathbf{K}_{10} \mathbf{t}_2 = \mathbf{K}_{10} \mathbf{t}_3 = \mathbf{K}_{10} \mathbf{r}_3 = \mathbf{0} \quad (\text{B.13})$$

Similarly, from Eqs. B.9, B.11, the following relations can be drawn:

$$\mathbf{K}_{01} \mathbf{t}_1 = -\mathbf{K}_{00} \mathbf{r}_2, \quad \mathbf{K}_{01} \mathbf{t}_2 = \mathbf{K}_{00} \mathbf{r}_1 \quad (\text{B.14})$$

Also,

$$\mathbf{K}_{11} \mathbf{t}_1 = -\mathbf{K}_{10} \mathbf{r}_2, \quad \mathbf{K}_{11} \mathbf{t}_2 = \mathbf{K}_{10} \mathbf{r}_1 \quad (\text{B.15})$$